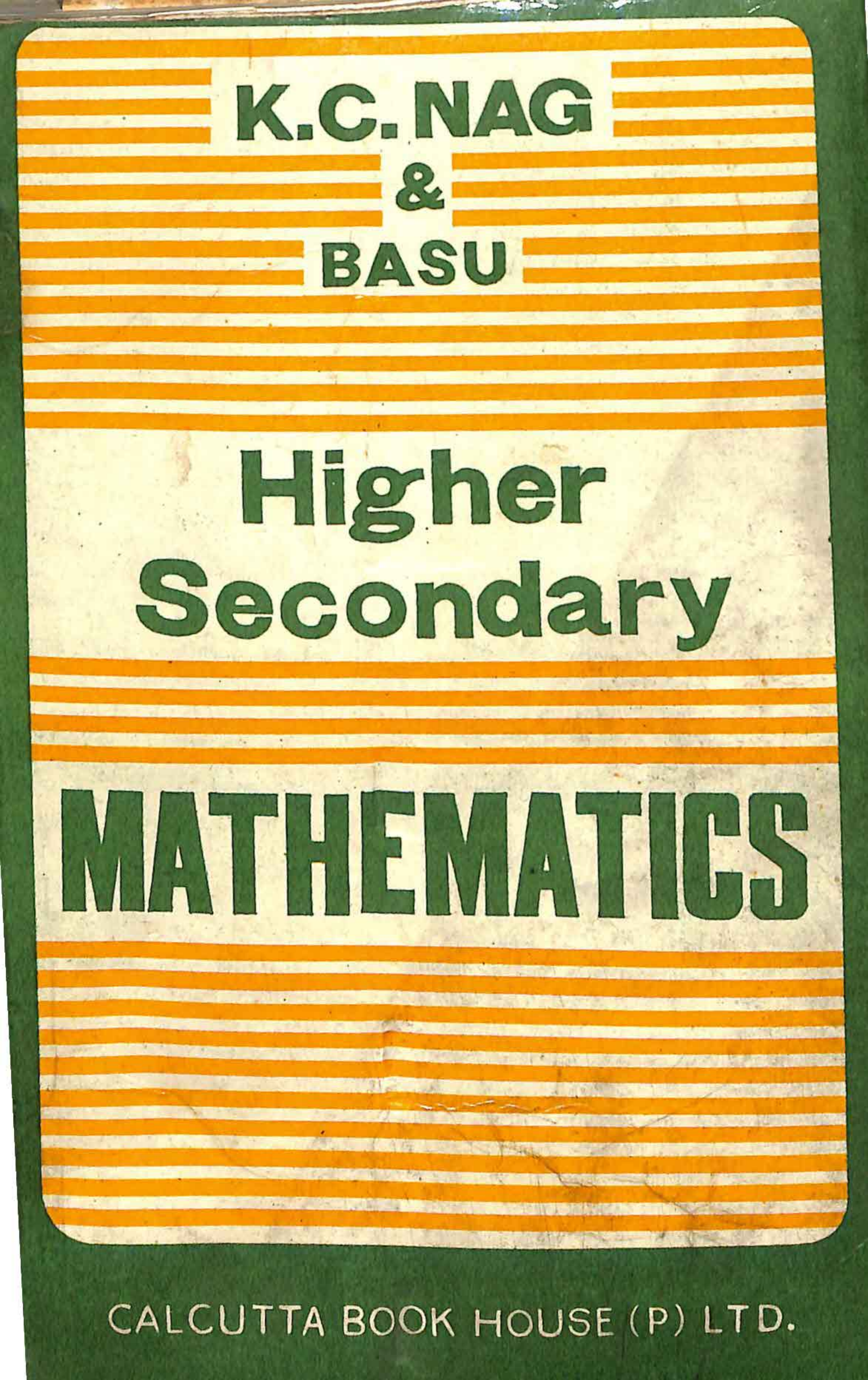

K.C. NAG
&
BASU

Higher Secondary

MATHEMATICS

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of West Bengal Council of Higher Secondary Education.

HIGHER SECONDARY
MATHEMATICS

(SECOND PAPER)

[● Differential Calculus ● Integral Calculus
● Differential Equations ● Application of Calculus]

Keshab Chandra Nag

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PREFACE

This book has been written according to the new syllabus for second paper of Mathematics for Higher Secondary Courses. The change of syllabus necessitated thorough revision of our former publication. The Portions of "*Differential Calculus*" and "*Application of Calculus*" have been completely rewritten. I have endeavoured to preserve all the speacial features of the books written by my most revered teacher Late Keshab Chandra Nag. The worked out examples, which are luxuriously many, have been taken amongst others from the question papers of H. S., West Bengal Joint Entrance Examinations and Joint Entrance Examinations for admission to the I. I. T's. At the end of the Book a set of sample objective and short answer type questions have been added.

I must gratefully acknowledge the sincere efforts and co-operation of the publishers, their staff and others concerned for the publication of this Book.

We trust, this edition will receive the same appreciation of the esteemed teachers and the beloved students as the First Paper of this Book. Finally, the late publication of this edition in regreted.

Golf Green
The 15th September, 1989

Keshab Basu

SYLLABUS FOR MATHEMATICS

SECOND PAPER

Marks—100

1. Differential Calculus :

1·1. Function of a single variable including trigonometrical functions : Sets of real numbers, intervals, variable, constant, function, Geometrical representations of functions x, $ax+b$, $|x|$, x^2, x^3, examples of rational functions, trigonometrical functions, $y=e^x$ (the concept of the number e may by used to define e^x instead of the series (e^x) function of a function, idea of a function bounded in an interval. Increasing and decreasing functions.

1·2. Limit and continuity (Geometrical and intuitive approach only, no formal ($\epsilon-\delta$) definition), Algebra of limits (statements only, no proof), Proof of lim x^n (n positive or negative integer) as $x \to a$, $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a}$ (n positive integer). Simple applications of above limits and the following limits which may be assumed :

$$\lim_{x\to 0}\frac{\sin x}{x},\ \lim_{x\to 0}\frac{e^x-1}{x},\ \lim_{x\to 0}\frac{\log(1+x)}{x},\ \lim_{x\to a}\frac{x^n-a^n}{x-a} (n \text{ rational})$$

limit of a function of a function (statement only), Geometrical idea of continuity of a function at a point.

1·3. Inverse functions, functions of functions. Continuous and strictly increasing or decreasing functions in an interval ; $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, logarithmic function.

1·4. Derivatives of functions ; Differential co-efficient, Differential co-efficients of C (a constant), x^n (n rational), $\sin x$, $\cos x$, e^x, $\log x$ from first principle. Rules of differentiation of sum, product and quotient of two functions. Rule of differentiation of function of a function (statement only). Differential co-efficients of $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$, a^x, $\sin^{-1}x$, $\cos^{-1}x$, etc. (not from first principle). Differentiation of implicit functions, functions in parametric form. Derivation by taking logarithm of the functions.

1·5. Second order derivative of a function.

2. Integral Calculus :

2·1. Indefinite integrals: Integration as the inverse of differentiation.

Primitive, indefinite integral, integrals of x^n, $\sin mx$, $\cos mx$, $\sec mx$, $\operatorname{cosec} mx$, $\sec x \tan x$, $\operatorname{cosec} x \cot x$, $\frac{1}{x}$, e^{mx} etc. Integral of the sum of two functions.

2·2. Integration by simple substitution : Integration of functions of the form $\frac{f'(x)}{f(x)}$

or, $\{f(x)\}^2 f'(x)$, standard integrals of the form $\int \frac{dx}{x^2 \pm a^2}$, $\int \frac{dx}{\sqrt{x^2 \pm a^2}}$, $\int \frac{dx}{\sqrt{a^2 - x^2}}$ and direct applications to integrals dependent on them.

2·3. Integration by parts : Rule of integration by parts. Application in simple cases. Standard integrals of the form $\int \sqrt{x^2 \pm a^2}\, dx$, $\sqrt{a^2 - x^2}\, dx$, $\int e^{ax} \sin bx$, $\int e^{ax} \cos bx$.

Integration of $\frac{ax+b}{(x-\alpha)(x-\beta)}$ (a may be zero or b may be zero)

2·4. Definite integral as the limit of a sum. Definite integrals of x, x^2 and of a constant from above definition. Fundamental theorem connecting primitive with definite integral (statement only, no proof).

Elementary properties of definite integrals (statement only) such as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^a f(x)\,dx + \int_a^b g(x)\,dx, \quad \int_a^a f(x)\,dx = 0.$$

Evaluation of definite integrals.

3. Differential equations :

Solution of first order and first degree differential equations of the forms $\frac{dy}{dx} = f(x)\,g(y)$, $\frac{dy}{dx} = \frac{ax+by}{cx+dy}$

Solutions of linear second order differential equations with constant coefficients of the form $\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0$

(i) auxiliary equations involving complex roots to be avoided.

(ii) Cases of solutions involving $\sin x$, $\cos x$, $e^{ax} \sin bx$, $e^{ax} \cos bx$ are to be verified indirectly.

(iii) Use of initial conditions in all cases.

4. Application of Calculus :

4·1. Interpretation of differential coefficient as rate measurer.

4·2. Geometrical Interpretation of Differential coefficient. Slope of a tangent, Equations of tangent and normals to curves of the form $y=f(x)$ at the point (x_1, y_1). Application to circle, parabola, ellipse and hyperbola. Condition that the straight line $y=mx+c$ may be tangent to a circle or to a conic. Number of tangents and normals. Length of tangent to a circle from a given point.

4·3. Idea of Maxima and Minima of $y=f(x)$ at a point, where $\frac{d^2y}{dx^2}\neq 0$ (Statement only). Application to algebraic functions, $\sin x$ $\cos x$.

4·4. Determination of areas in simple cases : Interpretation of a definite integral as an area, calculation or areas bounded by known curves (on one side of any axis), that axis and two ordinates or two abscissa as the case may be.

4·5. Expressions for velocity and acceleration of a particle in terms of derivatives :

Velocity $=\frac{ds}{dt}$; Acceleration $=\frac{dv}{dt}, \frac{d^2s}{dt^2}, v\frac{dv}{ds}$;

where s represents the displacement.

With the above expressions for velocity and acceleration to establish the formulæ $s=vt$ (v constant velocity), $v=u+ft$, $s=ut+\frac{1}{2}ft^2$, $v^2=u^2+2fs$.

Simple applications.

Vertical motion under gravity, greatest height and time to reach the greatest height and total time of flight (use of initial conditions to the differential equations).

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Trigonometry

I. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\cot(A \pm B) = \frac{\cot B \cot A \mp 1}{\cot B \pm \cot A}.$$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

II. $2 \sin A \cos B = \sin(A+B) + \sin(A-B).$

$2 \cos A \sin B = \sin(A+B) - \sin(A-B).$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B).$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B).$

III. $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}.$

$\sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}.$

$\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}.$

$\cos C - \cos D = 2 \sin \dfrac{C+D}{2} \sin \dfrac{D-C}{2}.$

IV. $\sin(A+B) \sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$

$\cos(A+B) \cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$

V. $\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$; $\cot 2A = \dfrac{\cot^2 A - 1}{2 \cot A}.$

$\left.\begin{matrix} 1 + \cos 2A = 2 \cos^2 A \\ 1 - \cos 2A = 2 \sin^2 A \end{matrix}\right\}$; $\tan^2 A = \dfrac{1 - \cos 2A}{1 + \cos 2A}.$

$\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$; $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}.$

VI. $\sin 3A = 3 \sin A - 4 \sin^3 A.$

$\cos 3A = 4 \cos^3 A - 3 \cos A.$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

$$\cot 3A = \frac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}.$$

VII. $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$

$$\cos\theta=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}=2\cos^2\frac{\theta}{2}-1=1-2\sin^2\frac{\theta}{2}.$$

$$\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}},\qquad \cot\theta=\frac{\cot^2\frac{\theta}{2}-1}{2\cot\frac{\theta}{2}}$$

$$\sin\theta=\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}},\qquad \cos\theta=\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}},$$

$$\left.\begin{aligned}1+\cos\theta&=2\cos^2\frac{\theta}{2}\\ 1-\cos\theta&=2\sin^2\frac{\theta}{2}\end{aligned}\right\}\qquad \tan^2\frac{\theta}{2}=\frac{1-\cos\theta}{1+\cos\theta}$$

VIII. If $A+B+C=\pi$, then

(*a*) $\sin A+\sin B+\sin C=4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C.$

(*b*) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C.$

(*c*) $\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C-1$

(*d*) $\cos A+\cos B+\cos C=1+4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C.$

(*e*) $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C.$

(*f*) $\tan A+\tan B+\tan C=\tan A\tan B\tan C.$

(*g*) $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C.$

(*h*) $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C.$

(*i*) $\cot B\cot C+\cot C\cot A+\cot A\cot B=1.$

(*j*) $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{B+C}{4}\sin\frac{C+A}{4}\sin\frac{A+B}{4}$

(*k*) $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}$

(*l*) $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1.$

(*m*) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$

IX. $\left.\begin{array}{l}\text{when } \sin\theta=\sin\alpha,\\ \text{then } \theta=n\pi+(-1)^n\alpha,\end{array}\right\}$ $[n=0, \pm1, \pm2, \cdots\cdots]$

when $\cos\theta=\cos\alpha$, then $\theta=2n\pi\pm\alpha$, [„]

„ $\tan\theta=\tan\alpha$, „ $\theta=n\pi+\alpha$, [„]

when $\sin\theta=0$ or, $\tan\theta=0$, then $\theta=n\pi$ [„]

„ $\cos\theta=0$ or, $\cot\theta=0$, „ $\theta=(2n+1)\frac{\pi}{2}$ [„]

when $\sin\theta=1$, then $\theta=(4m+1)\frac{\pi}{2}$ [Value of m „]

„ $\sin\theta=-1$ „ $\theta=(4m-1)\frac{\pi}{2}$ [„ „]

„ $\cos\theta=1$ „ $\theta=2m\pi$, [„ „]

„ $\cos\theta=-1$, „ $\theta=(2m+1)\pi$ [„ „]

X. $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$;

$\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$

$\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}$

$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$.

$\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}$.

$\cot^{-1}x+\cot^{-1}y=\cot^{-1}\frac{xy-1}{y+x}$.

$\cot^{-1}x-\cot^{-1}y=\cot^{-1}\frac{xy+1}{y-x}$.

$2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$.

$2\cos^{-1}x=\cos^{-1}(2x^2-1)$.

$2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}$

$\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\}$

$\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\{xy\mp\sqrt{1-x^2}\sqrt{1-y^2}\}$.

XI. $\log_a(m\times n)=\log_a m+\log_a n$.

$\log_a\frac{m}{n}=\log_a m-\log_a n$; $\log_a(m)^n=n\log_a m$

$\log_a m=\log_b m\times\log_a b$; $\log_a 1=0$; $\log_a a=1$.

XII. $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R.$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc};\ \cos B=\frac{c^2+a^2-b^2}{2ca};$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}.$$

$a=b\cos C+c\cos B,\ b=c\cos A+a\cos C,$
$c=a\cos B+b\cos A.$

In any triangle if $2s=a+b+c$

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}};\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}};\ \cot\frac{A}{2}=\frac{s(s-a)}{\Delta}.$$

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2\Delta}{bc}$$

$$\sin B=\frac{2\Delta}{ca};\ \sin C=\frac{2\Delta}{ab}.$$

$$\Delta=\tfrac{1}{2}bc\sin A=\tfrac{1}{2}ca\sin B=\tfrac{1}{2}ab\sin C.$$

$$=\sqrt{s(s-a)(s-b)(s-c)}=\frac{abc}{4R}$$

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}.$$

$$r=\frac{\Delta}{s}$$

$$=4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$$

$$=(s-a)\tan\tfrac{1}{2}A=(s-b)\tan\tfrac{1}{2}B=(s-c)\tan\tfrac{1}{2}C.$$

$$r_1=\frac{\Delta}{s-a}=4R\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C=s\tan\tfrac{1}{2}A.$$

$$r_2=\frac{\Delta}{s-b}=4R\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C=s\tan\tfrac{1}{2}B.$$

$$r_3=\frac{\Delta}{s-c}=4R\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\sin\tfrac{1}{2}C=s\tan\tfrac{1}{2}C.$$

Some Important Results

$\sqrt{2}=1{\cdot}4142135\ldots\ldots$
$\sqrt{3}=1{\cdot}7320508\ldots\ldots$
$\sqrt{5}=2{\cdot}2360679\ldots\ldots$
$\sqrt{6}=2{\cdot}4494897\ldots\ldots$
$\sqrt{7}=2{\cdot}6457513\ldots\ldots$
$\sqrt{8}=2{\cdot}8284271\ldots\ldots$
$\sqrt{10}=3{\cdot}1622776\ldots\ldots$

$\pi=3{\cdot}14159265\ldots\ldots$
$\frac{1}{\pi}=0{\cdot}31830989$
$\log 2={\cdot}30103$
$\log 3={\cdot}47712$
$\log 5={\cdot}69897$
$\log 7={\cdot}84510.$

DIFFERENTIAL CALCULUS

I. If $f(x)$ and $g(x)$. be two functions of x and $\lim\limits_{x\to a} f(x)=l$, and $\lim\limits_{x\to a} g(x)=m$ (l and m are finite quantities), then

(i) $\lim\limits_{x\to a} \{k\, f(x)\}=k \lim\limits_{x\to a} f(x)$, where k is a constant ;

(ii) $\lim\limits_{x\to a} \{f(x)\pm g(x)\}=\lim\limits_{x\to a} \{(x)\}\pm\lim\limits_{x\to a} g(x)$;

(iii) $\lim\limits_{x\to a} \{f(x).\, g(x)\}=\lim\limits_{x\to a} f(x).\lim\limits_{x\to a} g(x)$

(iv) $\lim\limits_{x\to a} \left\{\dfrac{f(x)}{g(x)}\right\}=\dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ if $\lim\limits_{x\to a} g(x)\neq 0$.

II. (i) $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a}=na^{n-1}$ where n is any constant ;

(ii) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}=1$, where x is in radians ;

(iii) $\lim\limits_{x\to 0} \dfrac{e^x-1}{x}=1$ (iv) $\operatorname{lin}\limits_{x\to 0} \dfrac{\log_e(1+x)}{x}=1$;

(v) $\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x}\right)=0$;

III. A function $f(x)$ is continuous at a point $x=a$ in its domain of definition if and only if $\lim\limits_{x\to a} f(x)=f(a)$;

$$\text{or, } \lim_{h\to 0} f(a+h)=f(a).$$

IV. (*a*) A function $f(x)$ is differentiable at $x=a$;

if $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists and this limit is denoted as $f'(a)$;

(*b*) (i) If $f(x)=x^n$, then $f'(x)=nx^{n-1}$ where n is a constant.

(ii) If $y=e^x$, then $\dfrac{dy}{dx}=e^x$;

(iii) If $y=a^x$, then $\dfrac{dy}{dx}=a^x\log_e{}^a$;

If $y=\log x$, $\dfrac{dy}{dx}=\dfrac{1}{x}$; $y=\sin x$, $\dfrac{dy}{dx}=\cos x$;

$y=\cos x,\ \dfrac{dy}{dx}=-\sin x$; $y=\tan x,\ \dfrac{dy}{dx}=\sec^2 x$;

$y=\cot x,\ \dfrac{dy}{dx}=-\operatorname{cosec}^2 x$;

$y=\sec x,\ \dfrac{dy}{dx}=\sec x \tan x$;

$y=\operatorname{cosec} x,\ \dfrac{dy}{dx}=-\operatorname{cosec} x.\cot x$;

$y=\sin^{-1}x,\ \dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-x^2}}$;

$y=\cos^{-1}x,\ \dfrac{dy}{dx}=-\dfrac{1}{\sqrt{1-x^2}}$;

$y=\tan^{-1}x,\ \dfrac{dy}{dx}=\dfrac{1}{1+x^2}$;

$y=\cot^{-1}x,\ \dfrac{dy}{dx}=-\dfrac{1}{1+x^2}$;

$y=\sec^{-1}x,\ \dfrac{dy}{dx}=\dfrac{1}{x\sqrt{x^2-1}}$;

$y=\operatorname{cosec}^{-1}x,\ \dfrac{dy}{dx}=-\dfrac{1}{x\sqrt{x^2-1}}$.

V. If u and v be two differentiable functions of x, then

(i) $\dfrac{d}{dx}\{ku\}=k\dfrac{du}{dx}$ where k is a constant ;

(ii) $\dfrac{d}{dx}\{u\pm v\}=\dfrac{du}{dx}\pm\dfrac{dv}{dx}$;

(iii) $\dfrac{d}{dx}\{uv\}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$;

(iv) $\dfrac{d}{dx}\left\{\dfrac{u}{v}\right\}=\dfrac{v.\dfrac{du}{dx}-u.\dfrac{dv}{dx}}{v^2}$.

VI. (i) If $y=f\{g(x)\}$, then

$$\frac{dy}{dx}=f'\{g(x)\}.\,g'(x), \text{ or, } \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx} \text{ where } z=g(x) ;$$

(ii) If $x=f(t)$, $y=\phi(t)$, where t is a parameter,

then $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$; (iii) $\dfrac{dy}{dx}=\dfrac{1}{\frac{dx}{dy}}$.

INTEGRAL CALCULUS

1. (i) $\int x^n dx = \frac{x^{n+1}}{n+1} (n \neq 1)$. (ii) $\int dx = x$.

(iii) $\int (a+x)^n dx = \frac{(a+x)^{n+1}}{n+1}$, (iv) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$.

2. $\int \frac{1}{x} dx = \log x$.

3. $\int Ku\,dx = K \int u\,dx$, K constant and u is a function of x.

4. $\int \{u_1 \pm u_2 \pm \cdots\cdots \pm u_n\} dx = \int u_1 dx \pm \int u_2 dx \pm \cdots\cdots \pm \int u_n dx$
$u_1, u_2, \cdots\cdots u_n$ constants or functions of x.

5. (i) $\int e^{mx} dx = \frac{e^{mx}}{m}$, (ii) $\int e^x dx = e^x$, (iii) $\int a^x dx = \frac{a^x}{\log_e a} (a > 0)$.

6. (*a*) $\int \sin mx\,dx = \frac{-\cos mx}{m}$. (*g*) $\int \sin x\,dx = -\cos x$.

(*b*) $\int \cos mx\,dx = \frac{\sin mx}{m}$. (*h*) $\int \cos x\,dx = \sin x$.

(*c*) $\int \sec^2 mx\,dx = \frac{\tan mx}{m}$. (*i*) $\int \sec^2 x\,dx = \tan x$.

(*d*) $\int \operatorname{cosec}^2 mx\,dx = \frac{-\cot mx}{m}$. (*j*) $\int \operatorname{cosec}^2 x\,dx = -\cot x$.

(*e*) $\int \sec mx \tan mx\,dx = \frac{\sec mx}{m}$. (*k*) $\int \sec x \tan x\,dx = \sec x$.

(*f*) $\int \operatorname{cosec} mx \cot mx\,dx = \frac{-\operatorname{cosec} mx}{m}$.

(*l*) $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$.

7. (*a* $\int f(x) dx = \int f[g(z)] g'(z) dz$, where $x = g(z)$.

(*b*) $\int \{f(x)\}^n f'(x)\,dx = \frac{1}{n+1} \{f(x)\}^{n+1}$, $n \neq -1$,

(*c*) $\int \frac{f'(x)}{f(x)} dx = \log f(x)$.

8. (*a*) $\int \tan x\,dx = \log \sec x$., (*b*) $\int \cot x\,dx = \log \sin x$.

(*c*) $\int \operatorname{cosec} x\,dx = \log \tan \frac{x}{2} = \log (\operatorname{cosec} x - \cot x)$.

(*d*) $\int \sec x\,dx = \log \tan \left(\frac{\pi}{4} + \frac{1}{2} x\right) = \log(\sec x + \tan x)$.

9. (*a*) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$.

(*b*) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} (x > a)$.

(*c*) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} (x < a)$. (*d*) $\int \frac{dx}{1+x^2} = \tan^{-1} x$.

10. (*a*) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$.

(*b*) $\int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$. (*c*) $\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x$.

11. For dertermination of

(i) $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$, put $cx+d=z^2$.

(ii) $\int \frac{dx}{(px+q)\sqrt{(ax^2+bx+c)}}$, put $px+q=\frac{1}{z}$.

12. Integral of the product of two functions = First function × integral of the second function − integral of {the derivative of the first function × integral of the second function.}

i. e., $\int uv dx=u\int v dx-\int\left\{\frac{du}{dx}.\int v dx\right\}dx$.

13. $\int e^x[f(x)+f'(x)]dx=e^x f(x)$.

14. (i) $\int e^{ax}\cos bx dx=\frac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}$

$$=\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right).$$

(ii) $\int e^{ax}\sin bx dx=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$

$$=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)$$

15. (i) $\int\sqrt{x^2+a^2}\,dx=\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\log(x+\sqrt{x^2+a^2})$.

(ii) $\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log(x+\sqrt{x^2-a^2})$.

(iii) $\int\sqrt{a^2-x^2}=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$.

16. (i) $\int_a^b f(x)dx=F(b)-F(a)$ if $\frac{d}{dx}F(x)=f(x)$.

(ii) $\int_a^b f(x)dx=\underset{h\to 0}{\text{Lt}}\,h[f(a)+f(a+h)+f(a+2h)+\ldots\ldots$

$+f(a+\overline{n-1}h)]$ where $nh=b-a$.

(iii) $\int_a^b f(x)dx=\int_a^b f(z)dz$.

(iv) $\int_a^b f(x)dx=-\int_b^a f(x)dx$.

(v) $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx\ ;\ (a<c<b).$

(vi) $\int_0^a f(x)dx=\int_0^a f(a-x)dx.$

(vii) $\int_0^a f(x)dx=2\int_0^{\frac{a}{2}} f(x)dx$ if $f(a-x)=f(x)$

$=0$ if $f(a-x)=-f(x)$

(viii) $\int_{-a}^a f(x)dx=2\int_0^a f(x)dx$ if $f(-x)=f(x)$

$=0$ if $f(-x)=-f(x).$

DIFFERENTIAL EQUATION

I. The order of a differential equation is the order of the highest order derivative present in the equation.

The degree of a differential equation is the degree of the highest order derivative present in the equation.

II. Rules of substitution of variables for separation of variables of a differential equation.

(*a*) In equations of the form $\frac{dy}{dx}=f(ax+by+c)$, put $ax+by+c=z$.

(*b*) In Homogeneous differential equations in x and y put $y=vx$.

(*c*) To solve equations of the form

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\left(\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\right),$$

put $x=x'+h$; $y=y'+k$ to make the equation homogeneous where h and k satisfy $a_1h+b_1k+c_1=0$, $a_2h+b_2k+c_2=0$.

(*d*) for $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\left(\frac{a_1}{a_2}=\frac{b_1}{b_2}\right)$,

put $a_1x+b_1y=z$.

III. The auxiliary equation of the differential equation

(*d*) $\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0\ (p_1,\ p_2$ constants) is $m^2+p_1m+p_2=0$.

If the roots of the auxiliary equation be (i) real and unequal α, β (say), then the general solution of the differential equation is

$$y=c_1e^{\alpha x}+c_2e^{\beta x}$$

(ii) real and cqual α, α (say) then the general solution of the differen tial equation is $y=(c_1+c_2x)e^{\alpha x}$.

(iii) Imaginary $\alpha \pm i\beta$ (say), then the general solution of the differential equation is $y = e^{\alpha x}\,(A \cos \beta x + B \sin \beta x)$

Application of Calculus

I. (*a*) $\dfrac{dy}{dx}$ is the instantaneous rate of change of y with respect to x.

(*b*) Geometrical meaning of $\dfrac{dy}{dx}$:

$\dfrac{dy}{dx}$ (when it exists) is the gradient of the tangent to the curve $y = f(x)$ at the point (x, y).

(*c*) If $f'(a) > 0$, then the function $f(x)$ is increasing at $x = a$. If $f'(a) < 0$, then the function $y = f(x)$ is decreasing at $x = a$.

(*d*) $dy = \dfrac{dy}{dx} \cdot dx$ or if $y = f(x)$, then $dy = f'(x)dx$

$d(y \pm z) = dy \pm dz$

$d(xy) \Rightarrow ydx + xdy$;

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

(*e*) If $|h|$ be very small, then $f(x+h) = f(x) + hf'(x)$

II. (*a*) The equation of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) of it is $y - y_1 = \left[\dfrac{dy}{dx}\right]_{(x_1,\, y_1)} (x - x_1)$

The equation of the tangent to the

(i) circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) of it is $xx_1 + yy_1 = a^2$.

(ii) parabola $y^2 = 4ax$ at the point (x_1, y_1) of it is $yy_1 = 2a(x + x_1)$

(iii) ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point (x_1, y_1) of it is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(iv) hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point (x_1, y_1) of it is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(v) rectangular hyperbola $xy = a^2$ is $xy_1 + x_1 y = 2a^2$

(*b*) Equation of the normal to the curve $y = f(x)$ at the point (x_1, y_1) of it is $y - y_1 = -\left[\dfrac{1}{\dfrac{dy}{dx}}\right]_{(x_1,\, y_1)} (x - x_1)$

The equation of the normal to the

(i) circle $x^2+y^2=a^2$ at the point (x_1, y_1) of it is $\frac{x}{x_1}=\frac{y}{y_1}$.

(ii) parabola $y^2=4ax$ at the point (x_1, y_1) of it is

$$y-y_1=-\frac{y_1}{2a}(x-x_1)$$

(iii) ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) of it is $\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}$

(iv) hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) of it is

$$\frac{x-x_1}{a^2}=-\frac{y-y_1}{b^2}.$$

(v) rectangular hyperbola $xy=a^2$
at the point (x_1, y_1) of it is $xx_1-yy_1=x_1^2-y_1^2$,

(*c*) The condition that the straight line $y=mx+c$ will touch.

(i) the circle $x^2+y^2=a^2$ is $c^2=a^2(1+m^2)$.

(ii) the parabola $y^2=4ax$ is $c=\frac{a}{m}$.

(iii) the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$.

(iv) the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $c^2=a^2m^2-b^2$.

(*d*) The equation of the chord of contact of the tangents from the external point (x_1, y_1).

(i) of the circle $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$.

(ii) of the parabola $y^2=4ax$ is $yy_1=2a(x+x_1)$.

(iii) of the ellipse $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(iv) of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(v) of the rectangular hyperbola $xy=a^2$ is $xy_1+x_1y=2a^2$.

(*e*) (i) length of the tangent at the point (x, y) of the curve

$$y=f(x) \text{ is } \frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}.$$

(ii) length of the normal at the point (x, y) of ;he curve $y=f(x)$ is

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

(iii) length of the sub-tangent at the point (x, y) of the curve

$y=f(x)$ is $\dfrac{y}{\frac{dy}{dx}}$.

(iv) length of the sub-normal at the point (x, y).

of the curve $y=f(x)$ is $y\dfrac{dy}{dx}$.

III. Anccessary condition for the existence of maxima or minima of a function $f(x)$ at a point $x=a$ in the domain of definition of a function $y=f(x)$ is $\left[\dfrac{dy}{dx}\right]_{x=a}=f'(a)=0$.

If $f''(a)>0$, then $f(x)$ is minimum at $x=a$.

$f''(a)<0$, then $f(x)$ is maximum at $x=a$.

IV. Area of the region bounded by the curve $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$ is $\int_a^b y\,dx=\int_a^b f(x)\,dx$ sq. units.

V. (*a*) If s be the distance at time t after start of a particle moving along a straight line from a fixed point of the straight line then velocity at time t is $v=\dfrac{ds}{dt}$.

acceleration at time t is $f=\dfrac{d^2s}{dt^2}=\dfrac{dv}{dt}=v\dfrac{dv}{ds}$.

(*b*) Equations of motion of a particle moving along a straight line with uniform acceleration are

(i) $v=u+ft$: (ii) $s=ut+\frac{1}{2}ft^2$;

(iii) $v^2=u^2+2fs$; (iv) $S_n=u+\frac{1}{2}f(2n-1)$.

Where, u=initial velocity, v=velocity at time t after start f=uniform acceleration.

S=displacement in time t.

S_n=displacement in the nth second of motion.

(*c*) If a particle is projected vertically upwards with velocity u under gravity, then

(i) The greatest height attained$=\dfrac{u^2}{2g}$.

(ii) Total time of fiight$=\dfrac{2u}{g}$.

(iii) Time of ascent=time of descent$=\dfrac{u}{g}$.

HISTORICAL NOTES

Calculus is an important branch of Mathematics. In different branches of science such as Geometry, Mechanics, Physics, Economics, Psychology etc. we find immense application of calculus.

In the works of **Zeno** (495—435 B. C.), **Eudoxus** (408—355 B.C.) and **Archemedes** (287—212 B.C.) in ancient greece and those of Arya Bhatta and Manjula we find use of Mathematical concepts and methods, used in modern Calculus.

In the early seventeenth century the famous Philosopher and Mathematician **Rene Descartes** introduced use of variable co-ordinates of a point in Geometry. The introduction of the concepts of variable co-ordinates is epoch making in the History of Human Knowledge. It may be said that this new concept opened magic casements before the western mathematicians and inspired them to use this new concept in different areas of Mathematics and other disciplines.

In 1612 A.D. the Astronomer Kepler (1571—1630) was successful in measuring the volume of a wine cask by dividing the cask into infinitely many thin discs each of infinitely small thickness. This process was already used by kepler in measuring the area of an ellipse. These methods of Kepler are actually methods used in modern Integral Calculus for measurment of areas and volumes. **Fermate** (1608—1665) generalised these methods.

Though Newton and Leibnitz are regarded as the discoverer of Calculus in the present form, it was Fermate who in 1629 first used the methods of Differential Calculus by showing the processes of drawing tangents to a curve.

Fermate also discovered the concepts of Maxima and Minima. Other Mathematicians now followed the method of Fermate. Mathematicians like Pascal (1623—1662), Roberval (1602—1675), Haygens (1629—1695) made successful attempts for drawing tangents of curves and determination of areas enclosed by different curves. Dr. Isaac Barrow (1642—1727) was one of these Mathematicians. 'In 1629 his "Lectiones opticae Geometricae" was published. In this treatise Barrow used methods similar to those of Differential Calculus.

The famous British Mathematician Sir Isaac Newton was a pupil of Dr. Barrow. We have already said that both Newton and the German Mathematician G. W. Leibnitz are regarded as discoverers of Calculus. For several years great dispute and indignation were aroused as followers of both Newton and Leibnitz demanded their idols as the only and actual discoverer of Calculus. But we are not entering into those disputes. Now a days both Newton and Leibnitz are regarded as independent discoverers of Calculus.

After Newton and Leibnitz great enthusiasm was found amongst the European Mathematicians in solving different Mathematical problems and discovering new formulas and concepts. Amongst these Mathematicans names of Bernoulis and Euler require special mention. In 1734 Euler introduced the process of partial differentiation. Amongst other contemporary Mathematician names of Abel, D'Alembert, Lagrange, Laplace, Gauss, Weirstrass, Riemann, Cauchy, Maclaurin and Cantor are worth mentioning.

DIFFERENTIAL CALCULUS

CHAPTER ONE

Real Numbers

§ 1·1. Natural Numbers. The real number system starts with the natural numbers 1, 2, 3,...They are also called counting numbers as they are used in counting. It is assumed that the students are familiar with the operations of addition, subtraction, multiplication and division of natural numbers. We also assume that the students know that the sum and product of two natural numbers are natural numbers i. e., the set of natural numbers is closed with respect to addition and multiplication. But the set is not closed with respect to subtraction and division. In the language of algebra if a and b are natural numbers the solutions of the equations $a+b=x$ and $ab=x$, are natural numbers. But the solutions of the equations $a-b=x$ and $a \div b=x$ are not necessarily natural numbers.

Properties of natural numbers.

(1) 1 is the least natural number and there is no greatest natural number. The natural numbers are infinite in number and two consecutive natural numbers differ by 1.

(2) The natural numbers are well ordered.

i.e., (i) if a and b be two natural numbers then either $a>b$ or $a<b$.

(ii) if a, b, and c be three different natural numbers such that $a>b$ and $b>c$, then $a>c$.

Prime and composite natural numbers.

A natural number is said to be a prime if it has no factor other than 1 and itself; otherwise the natural number is composite. Two natural numbers are said to be mutually prime if they have no factor in common other than 1. 2, 7, 11 etc. are examples of Prime numbers. 15 and 77 are mutually Prime.

§ 1·2. Rational numbers. We have already mentioned in the last section that the solution of the equation $a \div b=x$ where a and b are natural mumbers is not necessarily a natural number. To have solutions of such equations we create new numbers, called fractional numbers. A number of the form $\frac{a}{b}$ where a and b are natural numbers is called *a* fractional number. The natural numbers are also fractional by definition. For $1=\frac{2}{2}$, $5=\frac{5}{1}=\frac{10}{2}$ etc. a and b are respectively called the numerator and denominator of the fractional number $\frac{a}{b}$. But the creation of fractional numbers prove insufficient for the solution of equations of the form $a-b=x$ where a and b are any two natural numbers. For overcoming this insufficiency we extend our number system and create the following numbers.

(i) The number zero.

The solution of the equation $a-a=x$ is denoted by '0' and is called the number zero. As $a-a=0$, so $a=a+0$.

(ii) If $a+b=0$, where a is *a* natural number then b is denoted by $-a$ and is called the negative of the natural number a. The totality of the natural numbers, the number zero and the negatives of the natural numbers constitute the set of integers. The natural numbers are now renamed as positive integers and their negatives are called the negative integers. The number '0' is an integer, but is neither negative nor positive. So, the sum and product of two integers is an integer. You have learnt the definition of multiplication of integers in the lower classes. Here we do not enter into the details of them.

(iii) The negative fractional numbers.

If f be a fractional number then the solution of the equation $f+x=0$ is called the negative of the fractional number f.

The totality of the fractional numbers and their negatives (including the positive and negative integers) and the number '0' (zero) constitute what is known as the set of rational numbers.

Rational numbers. Numbers of the from $\frac{p}{q}$ where p and q are integers but $q\neq 0$ are called rational numbers. The rational numbers are generally taken in their lowest form ; for, if $\frac{p}{q}$ be a rational number and p and q have a common factor, then we can cancel the common factor to start with. Generally q is taken to be positive.

If q be negative, then $\frac{p}{q}$ is expressed as $\left(-\frac{p}{-q}\right)$.

The four operations of addition, subtraction, multiplication and division of rational numbers are defined as follows :

If $\frac{p}{q}$ and $\frac{r}{s}$ $(q\neq 0, s\neq 0)$ be two rational numbers, then

(i) $\frac{p}{q}+\frac{r}{s}=\frac{ps+qr}{qs}$ (ii) $\frac{p}{q}-\frac{r}{s}=\frac{ps-qr}{qs}$ (iii) $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}$

(iv) $\frac{p}{q}\div\frac{r}{s}=\frac{ps}{qr}$ if $r\neq 0$ i.e., $\frac{r}{s}\neq 0$.

The sum, difference, product and quotients of two rational numbers are also rational. For, as $\frac{p}{q}$ and $\frac{r}{s}$ are rational numbers, so p, q, r, s are integers and each of q and s is not 0. So, both the numerators and denominators in the above results are integers and the denominators are not zero. In case of division we have assumed $r\neq 0$; for, if $r=0$, then the rational number $\frac{r}{s}$ is 0. But division by 0 is not possible.

§ 1·3. Properties of rational numbers.

(1) The rational numbers are closed with respect to addition, subtraction, multiplication and division (division by 0 being undefined) i.e. if $\frac{p}{q}$ and $\frac{r}{s}$ are two rational numbers then $\frac{p}{q}+\frac{r}{s}$, $\frac{p}{q}-\frac{r}{s}$, $\frac{p}{q}\cdot\frac{r}{s}$ and $\frac{p}{q}\div\frac{r}{s}$ $\left(\text{if } \frac{r}{s}\neq 0\right)$ are also rational numbers.

(2) Addition and multiplication of rational numbers are both commutative and associative ; i.e., for any three rational numbers a, b, c.

$a+b=b+a$ [commutative law of addition]

$ab=ba$ [commutative law of multiplication]

$(a+b)+c=a+(b+c)$ [associative law of addition]

$(ab)c=a(bc)$ [associative law of multiplication]

(3) The distributive laws

$a(b+c)=ab+ac$

and $a(b-c)=ab-ac$ hold for rational numbers.

(4) The rational numbers are well ordered. i.e. (i) if a and b be two unequal rational numbers then either $a>b$ or $a<b$.

(ii) for any three unequal rational numbers if $a > b$ and $b > c$, then $a > c$.

(5) Between two unequal rational numbers there exist an infinite number of rational numbers.

§ 1·4. Decimal expression of a rational number :

We shall show in this section that the decimal expression of a rational number is either terminating or non-terminating recurring.

Let $\frac{p}{q}$ be a rational number. We divide p by q. If p be divisible by q then $\frac{p}{q}$ is an integer. If p is not divisible by q, then there will be a remainder less than p. At this stage multiply the remainder by 10 i.e., put a '0' (zero) on the right of the remainder. We now divide this product number by q, the quotient will be the first digit after the decimal point in the decimal expression of $\frac{p}{q}$. In this case if there be no remainder the decimal expression terminates. If there be a remainder, we multiply the remainder by 10 and continue to repeat the process until the remainder is 0 or one of 1, 2, 3, ..., $(q-1)$. If we obtain a remainder '0' at any stage, the division is completed and we get a terminating decimal expression of the rational number $\frac{p}{q}$. On the other hand, if the division does not end i.e., does not give remainder 0, at any stage, then the successive remainders will be less than q i. e., one of 1, 2, 3, ...$(q-1)$. As these remainders are finite in number, so after a finite number (less than q) of divisions the remainder will be the same as one the preceding remainders. From this stage onwards, the successive remainders and also the successive quotients will repeat until this remainder comes back and the division will not be completed but the successive quotients from this stage will repeat in the same order. So, in this case the decimal expression of $\frac{p}{q}$ will be non-terminating but recurring. The following examples will clarify the above reasonings.

Ex. 1. Express $\frac{5}{16}$ as a decimal fraction.

$$\begin{array}{r l}16 \,)\; 50 \;(0\cdot3125 & \\ 48 & \\ \hline 20 & \\ 16 & \\ \hline 40 & \\ 32 & \\ \hline 80 & \\ 80 & \\ \hline \end{array}$$

Here $p(5)$ is less than $q(16)$. So, the integral part of the quotient is 0 and the first remainder may be taken as 5. Then bringing decimal point in the quotient the successive remainders are multiplied by 10, the successive quotients are 3, 1, 2, 5 and the division is completed at this stage. So $\frac{5}{16}=0\cdot3125$ and the decimal expression of $\frac{5}{16}$ is terminating.

Ex. 2. Determine the decimal expression of $\frac{20}{13}$.

$$\begin{array}{r l}13 \,)\; 20 \;(1\cdot538461 & \\ 13 & \\ \hline 70 & \\ 65 & \\ \hline 50 & \\ 39 & \\ \hline 110 & \\ 104 & \\ \hline 60 & \\ 52 & \\ \hline 80 & \\ 78 & \\ \hline 20 & \\ 13 & \\ \hline 7 & \end{array}$$

Here $p(20)$ is greater than $q(13)$ and the integral part of the quotient is 1. The first integral remainder is 7. We then multiply this remainder by 10 and continue divisions bringing in 0 at every stage, the successive quotients are 1, 5, 3, 8, 4, 6, 1 and the successive remainders are 7, 5, 11, 6, 8, 2 but the division never ends. When the remainder is 2, we get 20 after bringing in a 0 and the quotients and remainders go on repeating in the same order without terminating the division. So, in this case $\frac{20}{13}=1\cdot\dot{5}3846\dot{1}$ which is non-terminating but recurring.

§1·5. Geometrical representation of rational numbers by points of a straight line :

Let us take a straight line XX′ (extending to infinity on both sides.) Let us take a point O on this straight line and let this point O denote the number 0 (zero). We make the convention that points of the straight line will denote positive rational numbers if they are situated on the right of O and points of the straight line on the left of O will represent negative rational numbers. We take a number A on the right of O. Let it represent the number 1.

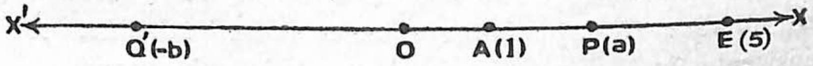


Fig. 1·1

Then OA will represent the unit length and our scale of representation is fixed. A positive number 'a' will be represented by a point P on the right of O on the straight line such that $OP = a \times OA$. The negative number $-b(b>0)$ will be represented by a point Q′ on the left of O such that $OQ' = b \times OA$.

Let us explain the method by the following examples.

Ex. 1. The number 5, will be represented by the point E on XX′ on the right of O such that $OE = 5.OA$.

Ex. 2. The number -3 will be represented by the point C′ on XX′ on the left of O such that $OC' = 3 \times OA$.

Ex. 3. To represent the number $\frac{5}{7}$, we divide OA into 7 equal parts and take 5 of them starting from O. Let P be the terminal point of the fifth part; Then $OP = \frac{5}{7}\ OA$ and P will represent the number $\frac{5}{7}$. To represent the number $-4\frac{3}{5}$, we take the points D′ and E′ on the left of O such that $OD' = 4.OA$ and $OE' = 5.OA$. D′ and E′ will

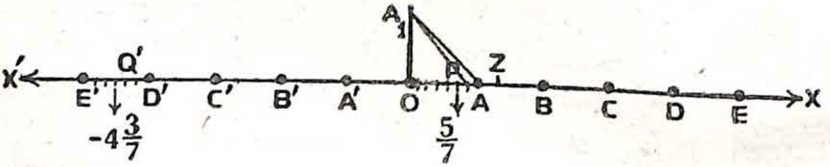


Fig. 1·2

represent the numbers -4 and -5. The line segment D′E′ is divided into 5 equal parts and we take the first 3 of them starting from D. Let Q′ be the terminal point of this third part. Then $OQ' = 4\frac{3}{5} \times OA$ and Q′ will represent the number $-4\frac{3}{5}$. The line XX′ is called the number line.

In the methods discussed above all rational numbers can be represented by points on the line XX′ and corresponding to every rational number we shall get one and only one point on the line XX′ (of course after fixing the point O to represent the number 0 and the point A to represent the number 1). Now the question that arises is that, is the converse of the above statement true ? Can we get a rational number corresponding to every point of the line XX′. That the answer is in the negative is shown below by showing at least one exception.

Let OA_1 be perpendicular at O on XX′ and $OA_1 = OA$. AA_1 is joined. Then $AA_1 = \sqrt{OA^2 + OA_1^2} = \sqrt{1^2 + 1^2}$ (as AA_1 is the hypotenuse of the right angled triangle AOA_1 and OA and OA_1 both represent unit length) $= \sqrt{2}$ units of length. With centre O and radius AA_1 we draw an arc of a circle to intersect XX' at the point Z. Then OZ is of length $\sqrt{2}$ units and the point Z will represent the number $\sqrt{2}$. We shall now show that $\sqrt{2}$ is not a rational number. For, if possible let $\sqrt{2}=\frac{p}{q}$ where p and q are integers and $q\neq 0$. We can assume that p and q have no factor in common, because if they have any common factor, we can cancel them to start with.

So $\sqrt{2}=\frac{p}{q}$ where p and q have no common factor.

or. $2=\frac{p^2}{q^2}$ (squaring both sides)

or, $p^2=2q^2$.

So p^2 is an even integer.

So, p is an even integer.

[For if p be not even, let $p=2m+1$, where m is an integer.

$\therefore \quad p^2=(2m+1)^2$

$=4m^2+4m+1$ which is odd as $4m^2$ and $4m$ are even.

But p^2 is even.

So p is an even number]

Let $p=2r$ where r is an integer.

$\therefore \quad (2r)^2=2q^2 \quad$ So $4r^2=2q^2$ or, $q^2=2r^2$

So q^2 in an even integer.

$\therefore \quad q$ is an even integer.

So both p and q are even integers and have a common factor 2. But this contradicts that p and q have no common factor. The contradiction proves that $\sqrt{2}$ cannot be written in the form $\frac{p}{q}$. So $\sqrt{2}$ is not rational.

Hence the point Z on the line XX' does not represent a rational number. Hence corresponding to every point on the line XX', there does not correspond a rational number. So, if the rational numbers are represented on the line XX' as discussed above, there will be gaps in the straight line. These gaps correspond to numbers which we call irrational numbers. So $\sqrt{2}$ is an irrational number.

Numbers which cannot be expressed as the ratio of two integers in the form $\frac{p}{q}$ $(q\neq 0)$ but corresponding to which there are points on the number line xx' are called irrational numbers. It is evident that the irrational number $\sqrt{2}$, can be represented by one and only point z on the line xx'. In the same way one can show that corresponding to every irrational number, there will be one and only one point in the number line xx'. Thus the points of the number line represent either rational or irrational numbers.

The totality of all rational and irrational numbers constitute what we call the real numbers. Hence corresponding to every real number there exists one and only one point on the line xx' and corresponding to every number on the straight line xx', there exists one and only one real number. Thus there is a one-one correspondence between the real numbers and the points on the line xx'. The number line xx' is also for this reason called the real number line. The real numbers and the points corresponding to them on the number line, due to the one-one correspondence between them are frequently referred interchangably.

The real numbers constitute the Arithmetic continuum and the points on the number line constitute what we call the Geometric continuum. The term continuum is used due to the compact or gapless character of real numbers and their representation on the number line.

§ 1·6. Decimal expression of the irrational numbers.

The irrational numbers, though they have definite values, can not be expressed as a fraction of the form $\frac{p}{q}$. That is why they are called *incommensurable.* The exact value of an irrational number connot be determined. But approximate value of every irrational number can be determined correct to any number of decimal places. Let us illustrate the above statement with an example and find the value of $\sqrt{3}$.

We know, $1<3<4$ i.e., $1^2<3<2^2$ $\therefore$ $1<\sqrt{3}<2$.

So, the value of $\sqrt{3}$ lies between 1 and 2.

Again $(1\cdot7)^2=2\cdot89$ and $(1\cdot8)^2=3\cdot24$ and so

$(1\cdot7)^2<3<(1\cdot8)^2$. i e, $1\cdot7<\sqrt{3}<1\cdot8$.

Again $(1\cdot73)^2=2\cdot9929$ and $(1\cdot74)^2=3\cdot0276$

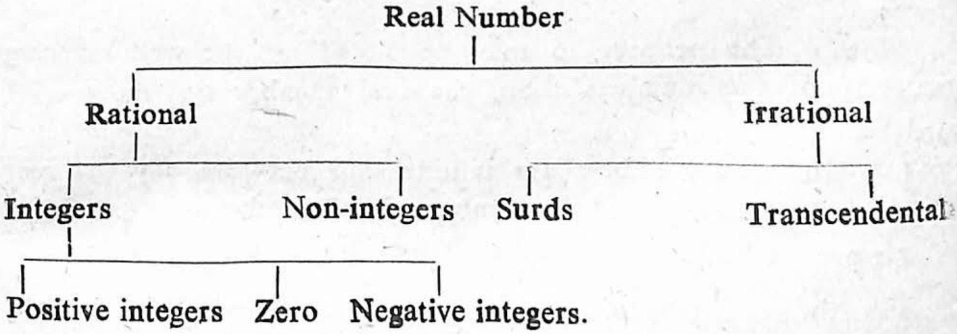
So, $(1\cdot73)^2<3<(1\cdot74)^2$ $\therefore$ $1\cdot73<\sqrt{3}<1\cdot74$.

Similarly it can be shown that $1\cdot732<\sqrt{3}<1\cdot733$.

In this manner the value of $\sqrt{3}$ can be determined to any number of decimal places This decimal representation is neither terminating nor recurring. It is not terminating as a terminating decimal expression is a rational number. It is not recurring, as here there is no division by the same divisior (when we extract the values by the method of division). Hence the decimal expression of an irrational number is non-terminating and non-recurring.

§ 1·7. Irrational numbers and Surds.

In algebra you have learnt that an irrational number is called a surd if it is a root of a polynomial with rational coefficients. But all irrational numbers are not surds. It can be proved that there are irrational numbers such as π (The constant ratio between the length of the circumference of a circle and the diameter of a circle) and e (the infinite series $1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\cdots$) which are not roots of polynomials with rational coefficients. These numbers are called transcendental. With these ideas we can now construct the Real Number—Tree as shown below.



§1·8. Properties of Real Numbers.

In algebra (in the first paper of this Book) we have studied the operations of addition, subtraction, multiplication and division of irrational numbers. In §1·2 and §1·3 we have defined addition, subtraction, multiplication and division of rational numbers and also discussed the properties of the rational numbers. Combining these operations of both the irrational and rational numbers we get ideas about the four operations of real numbers. In every case division by 0 has been restricted. Infact division by 0 is not defined in case of real numbers. In the next section, we shall endeavour, to make the students understand the inconvenience of defining division by 0. But, before that, let us state the following properties of real numbers.

(1) If a and b be two real numbers, then $a+b$, $a-b$, ab and $\frac{a}{b}$ $(b \neq 0)$ are real numbers ; i.e , the real number system is closed with respect to addition, subtraction, multiplication and division.

(2) For any two real numbers, a and b, $a+b=b+a$ and $ab=ba$ [Commutative laws of addition and multiplication].

Note : $a-b \neq b-a$ and $a \div b \neq b \div a$ for any two unequal real numbers.

(3) For any three real numbers a, b, c, $(a+b)+c=a+(b+c)$; $(ab)\ c=a(bc)$ associative laws of addition and multiplication.

(4) For any three real numbers a, b, c, $a\ b+c)=ab+ac$. [distributive law]

(5) (i) If a and b be two real numbers, then one and only one of $a>b$, $a=b$ or $a<b$ holds.

(ii) If a, b, c be three real numbers and $a>b$ and $b>c$, then $a>c$.

Note : The property 5 may be stated as 'the well ordering property of real numbers'. So, the real number system is well ordered.

(6) The real numbers are infinite and between any two real numbers there exist an infinite number of real numbers.

§ 1·9. Division by zero (0).

For the real numbers division by 0 is not defined. We explain below by an example the inconvenience caused if divisions by 0 were allowed.

We know $0.5=0$ and $0.6=0$.

Also we know when division by a real number c is allowed, and $a=b$, then $\frac{a}{c}=\frac{b}{c}$.

So if division by 0 is allowed,

$\frac{0.5}{0}=\frac{0.6}{0}$ or $5=6$. Cancelling the zeroes from the numerators and denominators. Which is a contradiction. So division by 0 cannot be allowed.

§ 1·10. Absolute value.

5 and -5 both have the numerical value 5. Numerical value or Absolute value of a number is also called the modulus of the number. Modulus of a real number x is denoted by the symbol $|x|$.

So, $|x|$ being the numerical value of x, $|x|=|-x|$. $|5|=|-5|$. $|x|$ may also be expressed as

$|x|=x$ when $x \geqslant 0$; and $=-x$ when $x \leqslant 0$; of course $|0|=0$.

So, $|3|=3$, here $x=3>0$; $|-3|=-(-3)=3$ as $-3<0$.

Geometrically, if the point P of the number line represents the number x, then $\text{OP}=|x|$ (O being the origin which represents the number 0). So, $|x|$ represents the distance between the point P, representing the number x, from the origin. If $x>0$, then P is on the right of O. But if x be negative, say $x=-u$ where $u>0$, then the point P is situated on the number line on the left of O. So, length $\text{OP}=u=|-u|=|x|$. So whether the point x (we have already said that numbers and points are used interchangably) is situated on the left or right of O, then $|x|$ is the distance of the point P (representing the number x) from the origin.

Again, $|a_1-a_2|$ is the distance between the points P and Q representing the numbers a_1 and a_2; clearly $\text{PQ}=a_1-a_2$ or a_2-a_1 according as $a_1<a_2$ or $a_1>a_2$

Now, $|a_1-a_2|$ is one of a_1-a_2 or a_2-a_1.

For example, $|5-3|=|2|=2$ which is the distance between the points 5 and 3.

$|7-(-3)|=|10|=10$, which is the distance between the points 7 and -3.

$|3-(-7)|=|10|=10$, which is also the distance between the points 3 and -7.

The following results about $|x|$ may be remembered.

1. $|x|=|-x|$
2. $|xy|=|x|\,|y|$

Example. $|2.3|=|6|=6$; Also $|2|=2$, $|3|=3$.

So, $|2|\,|3|=2\times 3=6$.

So $|2.3|=|2|\,|3|$.

3. $\left|\dfrac{x}{y}\right|=\dfrac{|x|}{|y|}$
4. $|x+y| \leqslant |x|+|y|$.

Proof. we know $|x| \geqslant 0$ whether $x \geqslant 0$ or $x<0$.
Also if $x \geqslant 0$, then $|x|=x$.
If $x<0$ say $x=-u(u>0)$, then $|x|=|-u|=u>x$
$(\because\ u>0, x<0)\ \therefore\ x<|x|$.
So $x \leqslant |x|$ whether $x \geqslant 0$ or $x<0$
Now, if $x+y \geqslant 0$, then $|x+y|=x+y \leqslant |x|+|y|$
$[\because\ x \leqslant |x|, y \leqslant |y|]$
if $x+y<0$, then $|x+y|=-(x+y)<|-x|+|-y|$

$$=|x|+|y|$$

5. $|x-y| \geqslant |x|-|y|$

Proof $|x|=|x-y+y| \leqslant |x-y|+|y|$ [by (5) above]
or $|x|-|y| \leqslant |x-y|$
i-e $|x-y| \geqslant |x|-|y|$.

6. $|x-a|<\delta$ means $a-\delta<x<a+\delta$.

Proof : $\because\ |x-a| \geqslant 0$ So, as $|x-a|<\delta$, so $\delta>0$.
Again, if $x>a$, then $x-a=|x-a|<\delta$.
So, $x<a+\delta$. Also $a-\delta<a<x$. $(\because\ \delta>0)$.
$\therefore\ a-\delta<x<a+\delta$.
If $x<a$, then $x<a+\delta$ as $\delta>0$.
Also in this case $a-x=|x-a|<\delta\ \therefore\ a-\delta<x$.
So in this case also $a-\delta<x<a+\delta$.
Hence in all cases $a-\delta<x<a+\delta$.

Examples 1

1. Show that between two real numbers there are an infinite number of real numbers.

Let a and b $(a<b)$ be two real numbers.

Then $\frac{a+b}{2}$ is a real-number between a and b so that $a<\frac{a+b}{2}<b$.

Now a and $\frac{a+b}{2}$ are two real numbers.

So, $a+\frac{\frac{a+b}{2}}{2}=\frac{3a+b}{4}$ is a real number and $a<\frac{3a+b}{4}<\frac{a+b}{2}<b$.

So, between a and b we get two real numbers. In this way from a and $\frac{3a+b}{4}$ we shall get another real number c and then from a and c we get another real number between a and b and so on. Our process will never end as $(b>a)$ and so we shall get an infinite number of real numbers between a and b.

Note : In this example one or both of a and b may be rational or irrational. So, the infinite number of real numbers found between a and b may be rational or irrational. We could find infinite number of real numbers between a and b in the manner shown in the example 2 below.

Ex. 2. Show that between any two unequal rational numbers there are an infinite number of rational numbers.

Let a and b be two given rational numbers and $b>a$ So $b-a>0$ and is a rational number.

$\therefore$ So $a<a+\frac{b-a}{n}<b$ where n is a positive integer>1.

As there are an infinite number of positive integers>1, so giving n those infinite number of values, we can get an infinite number of numbers between a and b, each of which is rational as both $b-a$ and n are rational.

Ex. 3. Suppose on the number line XX′, the points O and A represent the numbers 0 and 1 respectively.

Find the point P on the number line that will represent the number $\sqrt{3}$.

Let on the number line the point B (on the right of O) be such that $OB=2\ OA$. Then B represents the number $+2$ and length $OB=2$ units. OY is drawn perpendicular on XX′ at O. From OY, $OC=OA$ is cut off. With centre C and radius OB we drawn an arc of a circle which intersects XX′ at the point P. The point P will represent the number $\sqrt{3}$.

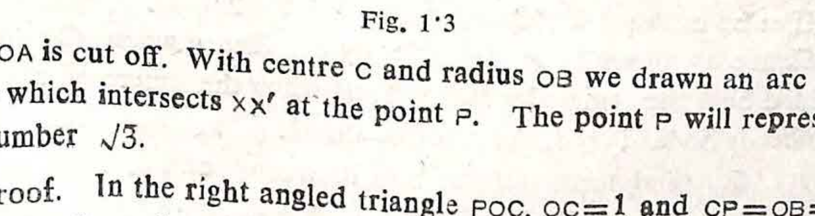


Fig. 1·3

Proof. In the right angled triangle POC, $OC=1$ and $CP=OB=2$.

$\therefore\ OP^2=CP^2-OC^2=2^2-1^2=4-1=3.$

$\therefore\ OP=\sqrt{3}$. So, the point P represents the number $\sqrt{3}$.

Ex. 4. Prove that $\sqrt{3}$ is not a rational number.

If possible let $\sqrt{3}$ be *a* rational number and $\sqrt{3}=\frac{p}{q}$ where p and q are integers and $q\neq 0$.

We take $\frac{p}{q}$ in the lowest form i.e., p and q have no common factor ; for had there been any, we could have cancelled that to start with.

Now squaring we get $3=\dfrac{p^2}{q^2}$

or $p^2=3q^2$. So p^2 is a multiple of 3.

We show below that p is also a multiple of 3.

For, if p is not a multiple of 3,

Then $p=3m+1$ or $3m+2$ where m is an integer.

If $p=3m+1$, then $p^2=9m^2+6m+1$.

Now $9m^2$ and $6m$ are both multiples of 3.

So $p^2=9m^2+6m+1$ is not a multiple of 3. So $p\neq 3m+1$

Similarly $p\neq 3m+2$.

So p is a multiple of 3, say $p=3m$. $\therefore$ $p^2=9m^2$

$\therefore$ $3q^2=9m^2$ or $q^2=3m^2$. So q^2 is *a* multiple of 3.

So q is a mulltiple of 3

$\therefore$ p and q have a common factor 3 which contradicts that p and q have no common factor. The contradiction proves that $\sqrt{3}$ cannot be represented in the form $\frac{p}{q}$ i. e. $\sqrt{3}$ is an irrational number.

Ex. 5. Prove that the square of an odd natural number when divided by 8 always gives the remainder 1. [A. I. H. S. 1972]

Let $2m+1$ be an odd natural number (m is a natural number).

$\therefore$ $(2m+1)^2=4m^2+4m+1=4m(m+1)+1$

Now if m be odd, then $m+1$ is even and let $m+1=2p$ (p is a natural number). $\therefore$ $4m\ (m+1)=4m.\ 2p=8mp$ and so $4m\ (m+1)$ is divisible by 8. So, in this case if $(2m+1)^2$ is divided by 8, then the remainder is 1.

If m be even say $2r$, then $4m\ (m+1)=8r\ (2r+1)$.

Hence $4m\ (4m+1)$ is divisible by 8. Hence when $(2m+1)^2$ is divided by 8 the remainder is 1. Thus when the square of a natural number is divided by 8, the remainder is 1.

Ex. 6 Find 4 rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$.

$\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$. $\therefore$ Each of $\frac{1}{2}+\frac{1}{2}.\frac{1}{6}$, $\frac{1}{2}+\frac{1}{3}.\frac{1}{6}$, $\frac{1}{2}+\frac{1}{4}.\frac{1}{6}$, $\frac{1}{2}+\frac{1}{5}.\frac{1}{6}$ is a rational number greater than $\frac{1}{2}$ and less than $\frac{2}{3}$. Hence $\frac{7}{12}$, $\frac{5}{9}$, $\frac{13}{24}$ and $\frac{8}{15}$ are four rational numbers between $\frac{1}{2}$ and $\frac{1}{3}$.

Ex. 7. Show that $a<x<b$ can be written as $\left| x-\frac{a+b}{2}\right| <\frac{b-a}{2}$

If $x>\frac{a+b}{2}$ then $x-\frac{a+b}{2}>0$

So $\left| x-\frac{a+b}{2}\right| =x-\frac{a+b}{2}<b-\frac{a+b}{2}$ $[\because \; x<b]$

or $\left| x-\frac{a+b}{2}\right| <\frac{b-a}{2}.$

If $x<\frac{a+b}{2}$, then $\left| x-\frac{a+b}{2}\right| =\frac{a+b}{2}-x<\frac{a+b}{2}-a$ $[\therefore \; a<x]$

i.e., $\left| x-\frac{a+b}{2}\right| <\frac{b-a}{2}.$

If $x=\frac{a+b}{2}$ then $x-\frac{a+b}{2}=0$ or $\left| x-\frac{a+b}{2}\right| =0<\frac{b-a}{2}$

[For, as $b>a$, so $\frac{b-a}{2}>0$]

So, in any case $\left| x-\frac{a+b}{2}\right| <\frac{b-a}{2}$

Ex. 8. If $| x-a | <\delta$, $| y-a | <\delta$, show that $| x-y | \leqslant 2\delta$.

$| x-y | = | (x-a)+(a-y) | \leqslant | x-a | + | a-y |$

Now $| a-y | = | y-a | <\delta$. Also $| x-a | <\delta$.

$\therefore$ $| x-a | + | a-y | <\delta+\delta$ or, $| x-y | <2\delta$.

Ex. 9. Express $| x+4 | \leqslant 1$ as inequalities without using modulus notation.

We know $| x-a | \leqslant\delta$ means $a-\delta\leqslant x\leqslant a+\delta$.

[see § 1·10 Property 6]

Here $a=-4$, $\delta=1$

$\therefore$ $| x+4 | \leqslant 1$ means $-4-1\leqslant x\leqslant -4+1$ or, $-5\leqslant x\leqslant -3$.

Ex. 10. Express the inequality $1\leqslant x\leqslant 5$ by modulus notation.

Here $\frac{1+5}{2}=3$. and $\frac{5-1}{2}=2$.

So, the given inequality can be expressed as $| x-3 | \leqslant 2$

[See Ex. 7 above]

Ex. 11. For which values of x, the following expressions are undefined ?

(i) $\frac{x^2-7x+12}{x^2-3x+2}$ (ii) $\frac{\tan x}{x}$ (iii) $\sqrt{x^2-7x+12}$

(i) The denominator $x^2-3x+2=(x-1)(x-2)$.

So, when $x=1$ or 2, the denominator becomes 0 and so the expression becomes undefined.

(ii) $\dfrac{\tan x}{x}$ becomes undefined when $x=0$ or $\tan x$ is undefined.

Now $\tan x$ is undefined when $x=(2n+1)\dfrac{\pi}{2}$ $[n=0, \pm 1, \pm 2, \cdots]$

So, $\dfrac{\tan x}{x}$ is undefined when $x=0$ or $(2n+1)\dfrac{\pi}{2}$ $[n=0, \pm 1, \pm 2, \cdots]$

(iii) $x_2-7x+12=(x-3)(x-4)$

When $3<x<4$, $x-3$ is positive and $x-4$ is negative ;

So $(x-3)(x-4)$ is negative.

In this case $\sqrt{(x^2-7x+12}=\sqrt{(x-3)(x-4)}$ is imaginary

So, when $3<x<4$, $\sqrt{x^2-7x+12}$ is undefined

For, other finite values of x, $x^2-7x+12$ is positive and so $\sqrt{x^2-7x+12}$ possesses finite real values and is defined.

Exercise 1

1. Which of the following statements are true and which are false ?

(i) Between two rational numbers, there are an infinite number of rational numbers.

(ii) The decimal expression of an irrational number is non-terminating and non-recuring.

(iii) As $x^2-4=(x+2)(x-2)$, so $\dfrac{x^2-4}{x-2}$ is defined for all values of x.

(iv) The sum of two irrational numbers is not always irrational.

(v) $|a-b| \geqslant |a|-|b|$

(vi) All irrational numbers are surds.

(vii) All irrational numbers are integers.

2. Fill up the gaps (with suitable terms given in the brackets).

(i) The number e is——. (irrational, a surd)

(ii) The decimal expression of a rational number is terminating or non-terminating——. (recurring, non-recurring)

(iii) The product of two——numbers is not necessarily a rational number. (rational, irrational).

(iv) Between two irrational numbers there are——number of irrational numbers (finite, infinite)

3. Show that between two irrational numbers there are an infinite number of irrational numbers.

4. Insert four rational numbers between $\sqrt{2}$ and $\sqrt{3}$.

5. Is there any rational number which is not real ?

[A. I. H. S. 1970]

6. Show that the product of any three consecutive natural numbers is divisible by 6.

7. The rational number which equals the number $2\cdot\dot{3}5\dot{7}$ with recurring decimal is (A) $\frac{2355}{1001}$ (B) $\frac{2339}{997}$ (C) $\frac{2355}{999}$ (D) None of these. Which is correct ?

[I. I. T. 1983]

8. Suppose on the number line xx′, the points O and A represent the numbers 0 and 1 respectively. Find the point P on the number line which will represent the number $\sqrt{5}$.

9. Show that $\sqrt{5}$ is not a rational number.

10. If $\sqrt{a}$ be a surd show that $\frac{x}{y}\sqrt{a}$ is irrational. (x, y are rational numbers).

11. Find the value of $\sqrt[3]{5}$ correct to two places of decimal.

12. "If the sum of two surds be rational, then they are conjugate of each other"—comment on the validity of the above statement.

13. Which of the following are rational numbers.

(i) $\frac{2}{3}$ (ii) $\cdot\dot{1}4285\dot{7}$ (iii) $2+\sqrt{3}$ (iv) $\dfrac{5}{5-5}$ (v) $\dfrac{1}{n-2}$ when n is an integer (vi) $(5+\sqrt{3})(5-\sqrt{3})$ (vii) $(5+\sqrt{3})(2-\sqrt{3})$.

14. (*a*) Express the following in the form $a \leqslant x \leqslant b$.

(i) $|x-3| \leqslant 2$ (ii) $|3-x| \leqslant 4$ (iii) $|x-a| \leqslant 1$ (iv) $|x| \leqslant 2$

(*b*) If $|2x-4| < 6$ then show that $|x-2| < 3$.

15. Express the following inequalities with the modulus notation.

(i) $3 \leqslant x \leqslant 5$ (ii) $-5 < x < 7$ (iii) $-11 \leqslant x \leqslant -1$ (iv) $a-\delta \leqslant x \leqslant \delta$.

16. If $|x-a| \leqslant \delta$, $|y-b| \leqslant \delta$, $|z-c| \leqslant \delta$, show that $|x+y+z-(a+b+c)| \leqslant 3\delta$.

17. For which values of x are the following expressions undefined.

(i) $\dfrac{x}{x}$ (ii) $\dfrac{\sin x}{x}$ (iii) $\dfrac{\cos 2x}{x}$ (iv) $\log_e(x-1)$ (v) $\sqrt{x^2-3x+2}$

(vi) $\dfrac{x^2+3x+2}{x^2+x-2}$ (vii) $\sec x$.

18. $|x-1|+|x-2|+|x-3 \geqslant 6$. Then (A) $0<x<4$ (B) $x<-2$ or $x \geqslant 4$ (C) $x \leqslant 0$ or $x \geqslant 4$ (D) None of these ; which is correct ?

[I. I. T. 1913]

CHAPTER TWO

Functions

§ 2·1. Variables and Constants : The concepts of functions demand as a prerequisite the concepts of variables. So let us start with variables.

Variable : A symbol or quantity which is capable of assuming more than one value in a mathematical discussion is called a variable.

Constant : A symbol which assumes one and only one value in a mathematical discussion is a constant.

Note : In mathematics we use other than variables and constants symbols like +, −, ( ), { } etc. They are called auxiliary symbols.

Let us discuss the above definitions of variables and constants with the help of an example. We know that area of a circle is πr^2 sq. cms. if r cm. be its radius. If A denotes the area of a circle, then the quantity A is related with the quantity r by the relation $A=\pi r^2$. Here the quantity r can assume any real value. So r is a variable. It must be noted that though r has a unit cm., in calculus, we are concerned with the numerical value of r. So if r is 7 cm., we shall be interested in the value 7. When r is in cenitmetres, the corresponding value of A is in square centimetres and we shall be interested only in the numerical value of A ; omitting the unit. (In a particular discussion we reduce quantities of the same kind in terms of the same unit. i.e,, if we take unit of length as centimeter then 2 meters will be reduced to 200 centimeters). You will agree that in the above relation $A=\pi r^2$, the numerical values of A and r will be real numbers. So, A and r are real variables. To be precise, a variable which assumes only real values is called a real variable. In fact, our subject matter is calculus of real variables and so all variables that will come in our discussion will assume only real values (i.e., real numbers as their values.) The same is the case with constants ; The quantity π in the relation $A=\pi r^2$, possesses one and only one value which is a real number. So, π is an example of a constant ; of course as for different values of r. A assumes different values so A is also a variable.

§ 2·2. Domain of a variable :

The totality of all values which a variable can assume or is made to assume is called its domain In x^2, the variable x can assume all real numbers as its value. So, in this case the domain of the variable x is the set of real numbers. But in $\sqrt{x}$, the variable x cannot assume any negative value (because, then $\sqrt{x}$ will become imaginary, but our variable is a real variable.) So in this case the domain of x is the set of all positive real numbers and zero. In many cases one can restrict the value of a variable between two real numbers or, even to some specified real numbers. For example, if we consider the monthly salary of workers in a factory, and Rs. x denote the monthly salary of a worker, then x varies from worker to worker and so x is a variable. If in the factory, under consideration, the minimum and maximum salary of a worker be Rs. 750 and Rs. 1600 per month, then the domain of the variable is restricted between 750 and 1600. Once again note that here, when we refer to the values of x, we omit the unit Rupees. In this case if the salary of a worker is paid in the nearest rupee, than x can assume only integral values between 750 and 1600 and not all values between 750 and 1600.

§ 2·3. Interval :

The totality of all real numbers between two real numbers a and b $(a<b)$ is called an Interval of real numbers. If both the numbers a and b are included in the interval, then we call it a closed interval and shall denote the closed interval as $a \leqslant x \leqslant b$. If both the end points a and b are excluded, then the interval is open and we denote it as $a<x<b$. If a is included and b is excluded, then the interval is half open or half closed ; it is closed on the left and open on the right and the interval is denoted as $a \leqslant x<b$. Similarly if a is excluded and b is included, then also, we get a half open interval. This interval is open on the left and closed on the right and it is denoted as $a<x \leqslant b$.

Note : Many authors use the symbols (a, b), $[a, b]$, $(a, b]$, $[a, b)$ for closed intervals, open intervals, intervals closed on the left and open on the right and intervals open on the left and closed on the right respectively. Some authors again use the symbols [ ] and ( )

for closed and open intervals ; they use [ ) and ( ] symbols for half-open intervals.

§ 2·4. Continuous and discrete Variables :

A variable is said to be continuous in a given interval, if it can assume all values within the interval. A variable is discrete or discontinuous in an interval if it cannot assume all values within the interval.

A variable assumes only rational values in the interval $0 \leqslant x \leqslant 1$; between 0 and 1 there are an infinite number of irrational values and the variable, in this case does not assume these values. So, in this case the variable is discrete or not continuous.

A variable which assumes only integral values in the interval $10 \leqslant x \leqslant 100$ is another example of a discrete variable. If x be a variable, then $\frac{1}{x}$ is also a variable.

In the interval $1 \leqslant x \leqslant 2$ or any other interval which does not include the value '0' (zero), $\frac{1}{x}$ can assume all real values in the intervals concerned. So, in these intervals $\frac{1}{x}$ is a continuous variable. But in the interval $-1 \leqslant x \leqslant 1$ or any interval including '0', $\frac{1}{x}$ can not assume the value '0' for $\frac{1}{0}$ is undefiined. Hence the variable $\frac{1}{x}$ is discrete or discontinuous in every interval including '0'.

§ 2·5. Functions : If two variables x and y be so connected that within a given interval, for every real value of x, we can get at least one value of y, then y is said to be a function of x.

If $y = x^2$, then for all real values of x, we get one and only one value of x^2. Here the set of real numbers is called the domain of definition of y. The set of all real numbers is denoted as the interval $-\infty < x < \infty$. Note that $-\infty$ and $+\infty$ are not real numbers. The set of all values which y assumes in its domain of definition is called the range of y. In § 2·8 we shall discuss more about the domain of definition of a function.

A variable y is said to be a single valued function of another variable x, if for every value of x in the domain of definition of y, there exists one and only one value of y. Strictly speaking by a function is meant a single valued function. But we shall stick to the former definition.

$y = x^2$ is a single valued function of x.

$y=\sin\theta$ is a single valued function of θ in $-\infty<\theta<\infty$.

For, corresponding to every value of θ, we get one and only one value of $y=\sin\theta$.

But $y=$square root of x is a two valued function of x.

For, corresponding to every value of x, we get two and only two square roots of x (when $x=0$, the two square roots coincide).

$y=\text{Sin}^{-1}x$ is a multiple valued function of x, as for every value of x we get many values of y (here the values of y are infinite in number).

But $y=\sin^{-1}x$ (the principal value of $\sin^{-1}x$), is a single valued function of x.

So, y is an n valued function (n is a positive integer) of x if for every value of x, we get n values of y.

Note. 1. In an inverse circular function, when we consider the multiple valued function, we use a capital in the first letter.

So., $\text{Sin}^{-1}x$, $\text{Tan}^{-1}x$ denote the multiple valued inverse circular functions.

But when we consider principal values, the functions begin with a small letter. Thus $\sin^{-1}x$, $\tan^{-1}x$ denote principal values of the functions.

2. In the above examples we have used the letter y for a function. But we can use other letters also. For example when θ denotes heat and 't' temperature of a body in calories and selsiers, θ is a function of t. In $s=ut+\frac{1}{2}ft^2$, the distance variable s is a function of the time variable t.

§ 2·6. Notation for functions.

A function of x is denoted as $f(x)$. The letter f is the first letter of the term function. Note that $f(x)$ is not the product of f and x; it is just a symbol. So, $f(x+y)$ is generally not equal to $f(x)+f(y)$.

For different functions, you must appreciate, one should use different symbols using different letters. If $f(\theta)$ denotes the function $\sin\theta$, i.e., if $f(\theta)=\sin\theta$ then we must use a different symbol (i. e., a different letter) to use the function $\cos\theta$. Let $F(\theta)=\cos\theta$. Generally, letters f, F, ϕ, g, h, θ etc. are used to denote functions.

As a constant possesses a finite value (the constant itself) for every value of a variable x, constant is taken as a function of x. So, constants are frequently described as constant functions.

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§ 2·7. Dependent and Independent variables

If y be a function of x, say $y=f(x)$, then the values of y depend upon the values of x. So, y is called the dependent variable and x is called the independent variable. Thus if two variables be connected by a functional relation, then the variable whose value depends on the other is the dependent variable and the other is the independent variable. In the relation $v=u+ft$, where v is the velocity of a particle at time t after start with initial velocity u and uniform acceleration f (here u and f are constants), the value of v depends on the value of t and so v is a function of t. Here v is the dependent variable and t is the independent variable.

§ 2·8. Domain of definition of a function.

The totality of values of the independent variable, for which the dependent variable (a function of the independent variable) has a definite value is called the domain of definition of the function. The totality of all values of the dependent variable corresponding to the values of the independent variable in the domain of definition of a function, is called the range of the function. Let us discuss these two concepts with examples.

Ex. 1. The domain of definition of the function $y=x-1$ is the interval $-\infty<x<\infty$; as for every value of x, y has one (and only one) value. Here the range of y is also $-\infty<y<\infty$.

Ex. 2. In $y=\sin\theta$, the domain of definition of y is $-\infty<\theta<\infty$. Here for every value of θ we get one and only one value of $y=\sin\theta$. But since the value of the sine of an angle lies between -1 and $+1$ (inclusive of -1 and $+1$) so the range of $y=\sin\theta$ is the interval $-\infty<\theta<\infty$.

Ex. 3. In $0\leqslant x\leqslant 2\pi$, $\tan x$ is not defined when $x=\frac{\pi}{2}$ or $\frac{3\pi}{2}$. So, the domain of $\tan x$ in $0\leqslant x\leqslant 2\pi$ is $0\leqslant x<\frac{\pi}{2}, \frac{\pi}{2}<x<\frac{3\pi}{2}$ and $\frac{3\pi}{2}<x\leqslant 2\pi$. This can also be stated as $0\leqslant x\leqslant 2\pi$, other than $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Note: In example 3, we have restricted our discussion within the interval $0\leqslant x\leqslant 2\pi$. This type of restriction is permissible when desired or when required. If we do not restrict the discussion, then the domain of definition of tan x is $-\infty<x<+\infty$ other than the values $(2n+1)\frac{\pi}{2}$ [where $n=0, \pm 1, \pm 2, \pm 3, \cdots$] of x.

Ex. 4. In case of $y=\sqrt{x^2-3x+2}$, the demain of definition of y is $-\infty<x\leqslant 1$; $2\leqslant x<+\infty$.

For, $y=\sqrt{(x^2-3x+2)}=\sqrt{(x-1)(x-2)}$

If x lies in $1<x<2$, then $x-1$ is positive and $x-2$ is negative and so $(x-1)(x-2)$ is negative. So, when $1<x<2$, y is imaginary. So, x cannot assume any value in $1<x<2$ to make y real.

§ 2·9. Single valued, Multipled valued and constant functions.

A function is a said to be a single valued function if for every value of the independent variable, in the domain of definition of the function, the dependent variable (i.e., the function) possesses one and only one value.

Ex. 1. If $y=x+1$, y is a function of x. Here x is the independent variable and y is the dependent variable. In the domain of definition $(-\infty<x<\infty)$ of the function, for every value of x, y has one and only one value. So, y is here a single valued function of x.

Ex. 2. If A denotes the area of a circle and r denotes its radius, then $A=\pi r^2$. Here for every value of r, A possesses one and only one value. So here the area A of a circle is a single valued function of the radius r.

Ex. 3. $y=\sqrt[3]{x}$ is a single valued function of x. For, corresponding to every real value of x, we get one and only one real value of y.

Note : We know that for every value of x, one can get three cube roots of x viz, the real cube root of x, say θ and two imaginary cube roots $\theta\omega$, $\theta\omega^2$. But these later cube roots being imaginary, do not come into our consideration.

If corresponding to every value of the independent variable, the dependent variable $y=f(x)$, possesses more than one value, then y is said to be a multiple valued function of x.

Ex. 4. If y be the square root of x, then y is a multiple (two) valued function of x. For, corresponding to every positive value of $x \neq 0$, we get two square roots (one positive and the other negative) of x.

Note : $y=\sqrt{x}$, as $\sqrt{x}$ denotes the positive square root of x, is a single valued function of x.

2. The domain of definition of the square root of x or of $\sqrt{x}$ are both non-negative real values of x.

Ex. 5. Inverse circular functions like $\text{Sin}^{-1}x$, $\text{Tan}^{-1}x$ are multiple valued functions of x. When $x=\frac{1}{\sqrt{2}}$, the value of $\text{Sin}^{-1}x$ are $n\pi+(-1)^n\frac{\pi}{4}$ $[n=0, \pm 1, \pm 2, \cdots]$ when $x=\frac{1}{\sqrt{3}}$, values of $\text{Tan}^{-1}x$ are $n\pi+\frac{\pi}{6}$ $[n=0, \pm 1, \pm 2, \cdots]$.

Note 1. You have already learnt in Trigonometry that capital letters S, C, T etc. are used in the multiple valued functions $\text{Sin}^{-1}x$, $\text{Cos}^{-1}x$, $\text{Tan}^{-1}x$ etc.

2. The Principal values $\sin^{-1}x$, $\cos^{-1}x$. $\tan^{-1}x$ etc. are single valued functions of x.

§ 2·10. The symbol $f(a)$:

The value of a function $f(x)$ when $x=a$, is denoted as $f(a)$.

So if $f(x)=\sin x$, $f\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$. If $\phi(x)=x^2$, $\phi(2)=2^2=4$.

If $F(x)=\frac{1}{x-2}$, when $x=2$, then the value of $F(x)$ is denoted as $F(2)$; but $F(2)=\frac{1}{2-2}$ which is undefined. So, in this case we see that $F(2)$ does not exist. In fact, if $f(a)$ is not finite, we say that $f(a)$ does not exist or $f(a)$ is undefined. If $f(a)$ is imaginary, then also $f(a)$ does not exist or is undefined. So $\sqrt{x}$ does not exist when $x=-1$.

§ 2·11. (a) Even and odd functions :

If $f(-x)=f(x)$, for every x, for which $f(x)$ is defined, then $f(x)$ is said to be an even function of x.

$f(x)=x^2$ is an even function of x.

For $f(-x)=(-x)^2=x^2=f(x)$,

$\phi(x)=\cos x$ is an even function of x.

For, $\phi(-x)=\cos(-x)=\cos x=\phi(x)$.

A function $f(x)$ is said to be an odd function of x, if $f(-x) = -f(x)$ for every x in the domain of definition of the function. x^3, $\sin x$ are two standard examples of odd functions.

Note : It is not necessary that every function should be either even or odd.

The function $f(x) = x^2 + x^3$ is neither even nor odd.

For, $f(-x) = (-x)^2 + (-x)^3 = x^2 - x^3$

which is neither $f(x)$ nor $f(-x)$.

Properties of odd and even functions :

(i) The sum (difference) of two even functions is an even function and the sum (difference) of two odd functions is an odd function.

(ii) The product (or quotient) of two even or two odd functions is an even function.

(iii) The product (or quotient) of an even function and an odd function is an odd function.

These properties are proved as examples in Examples 2.

(*b*) Explicit and Inplicit functions :

When a function $f(x)$ of a single variable x, is expressed directly in terms of x, then the function is said to be an explicit function of x. $y = x^3$, $y = \cos 2x$ etc. are explicit functions of x.

If two variables x and y be such that y is not explicitly expressed in terms of x but the two variables are connected by an equation, then y is said to be an implicit function of x. In the relation $x^2 + y^2 = a^2$, y is not expressed directly in terms of x, but x and y are connected by the equation. Here y is an implicit function of x or x is an implicit function of y. From the implicit relation $x^2 + y^2 = a^2$, the two explicit functions $y = \pm\sqrt{a^2 - x^2}$ or $x = \pm\sqrt{a^2 - y^2}$ can be obtained. $x^3 + y^3 = 3axy$, $y = \sin(x+y)$, $y^{\log x} = x$ are examples in each of which y is an implicit function of x.

(*c*) Parametric Representation of a function or Parametric functions :

Sometimes, though two variables, x and y may not be functionally connected, implicitly or explicitly they are expressed in terms of a third variable. This third variable is called a parameter.

In the two relations, $x=a\cos\phi$, $y=b\sin\phi$, the two variables x and y are both expressed in terms of a third variable or parameter ϕ. Eliminating ϕ, between the two relations, we get $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. So, here y is an implicit function of x. $x=a\cos\phi$, $y=b\sin\phi$ is the parametric representation of the implicit function y of x given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. $x=at^2$, $y=2at$ is the parametric representation of the variables in the equation $y^2=4ax$ of a parabola.

(*d*) Periodic functions :

If $f(x)$ be a function of the variable x such that $f(x+a)=f(x)$ for every x where a is a constant, then the function $f(x)$ is said to be a periodic function of x with period a. As, $\sin(x+2\pi)=\sin x$, so $\sin x$ is a periodic function of x with period 2π. As $\tan(\theta+\pi)=\tan\theta$, so $\tan\theta$ is a periodic function of θ with period π.

(*e*) Inverse functions :

Let $y=f(x)$ be a function of x ; when x is expressed in terms of y, then x is said to be the inverse function of $f(x)$. If from the relation $y=f(x)$, one can get the relation $x=\phi(y)$, then f and ϕ are inverse functions of each other. From the function $y=2x+1$, we get the inverse function $x=\frac{1}{2}(y-1)$. If $y=\sin x$, then $x=\text{Sin}^{-1}y$. Note that here the inverse function is multiple-valued, though the original function is single-valued. In the function $y=x^2$, y is defined for all real values of x, but the inverse function $x=\pm\sqrt{y}$ is not defined for negative values of x.

§ 2·12. The Fundamental Functions :

The following functions viz, Algebraic functions, Trigonometric functions, Inverse circular functions, Exponential functions and Logarithmic functions are called fundamental functions. From these functions, by addition, subtraction, multiplication, division, evolution of one or more of these functions we can get many other funcitons. We shall discuss these functions one by one.

(1) Algebraic functions :

A function of the type x^n where n is a constant is called a power function. Sum, difference, multiplication and divisions of constants of different power functions give rise to algebraic functions. Algebraic functions are of the following types :

(i) Polynomial functions : An expression of the form $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a$, when n is an integral constant, the coefficients $a_0, a_1, a_2, \cdots\cdots\cdots, a_n$ are real constants and x is a variable is called a polynomial or a polynomial function.

(ii) Functions of the form $\dfrac{a_0x^n+a_1x^{n-1}+\cdots\cdots+a_n}{b_0x^m+b_1x^{m-1}+\cdots\cdots+b_m}$ where the numerators and denominators are polynomials are called rational algebraic functions.

(iii) Functions of the forms $\sqrt{x}$, $\sqrt{x^2+6x+2}$, $\sqrt[3]{x+2}$, $\dfrac{x^2}{\sqrt{x^3+7}}$, $\dfrac{(x+7)^{\frac{5}{3}}}{(x^2+1)^{\frac{6}{5}}}$ are called irrational functions.

(2) Trigonometric functions. $\sin x$, $\cos 2x$, $\tan\sqrt{x}$, $\sec^2 x$ etc. are Trigonometric functions.

(3) Inverse circular functions. $\sin^{-1}x$, $\tan^{-1}2x$, $(\operatorname{cosec}^{-1}3x)$ etc. are examples of Inverse circular functions.

(4) Exponential functions. e^x, 2^{2x}, 10^{5x}, $100^{\sqrt{x}}$ etc. are examples of exponential functions.

(5) Logarithmic functions. $\log_e{}^x$, $\log_{15}^{2x}$, $\log_e{}^{(1+x)}$ etc. are examples of logarithmic functions.

Note. 1. In calculus the base of the logarithm of a number is generally taken as e. So, the base e, is frequently kept understood. $\log x$ should be read as $\log_e x$. But other bases must be explicity indicated.

2. Of the above examples, let us consider the functions $\sqrt[3]{x+2}$, $\tan\sqrt{x}$, $\sec^2 x$, $\sqrt[3]{\sec^{-1}x}$, 2^{2x}.

In, $\sqrt[3]{x+2}$, $x+2$ itself is at the first instance, a function of x. $\sqrt[3]{x+2}$ is a function of this function of x. So, here we get an example of what is called a function of a function.

$\tan\sqrt{x}$ is a function of the function $\sqrt{x}$ of x ; $\sec^2 x=(\sec x)^2$ is a function of the function $\sec x$ of x. $\sqrt[3]{\sec^{-1}x}$ is a function of the function $\sec^{-1}x$ of x. $2^{2x}=(2^x)^2$ is a function of the function 2^x of x. Again $\sin(e^{x^2})$ is an example of a function of a function of a function of x.

Here x^2 is a function of x ; e^{x^2}, is a function of the function x^2 $\sin(e^{x^2})$ is a function of the function e^{x^2}, a function of a function of x.

3. On many occasions functions are defined by more than one mathematical relations.

For example,
$$\begin{aligned} f(x)&=2x+1 \text{ when } x>1\\ &=4 \quad\;\; \text{ when } x=1\\ &=2x-1 \text{ when } x<1. \end{aligned}$$

§ 2·13. **Monotonic functions.** A function is said to be monotonic increasing in an interval $a\leqslant x\leqslant b$ included in its domain of definition, if in the interval values of the function increases with x. If for any two points x_1 and x_2 in the interval $a\leqslant x\leqslant b$, within the domain of definition of the function $f(x)$, $f(x_1)\geqslant f(x_2)$ when ever $x_1>x_2$, then the function is monotonic increasing in the interval.

A function $f(x)$ is monotonic decreasing in the interval $a\leqslant x\leqslant b$, included in its domain of definition if $f(x)$ decreases as x increases in theinterval. If x_1 and x_2 be any two points in $a\leqslant x\leqslant b$ and $f(x_1)\leqslant f(x_2)$ whenever $x_1>x_2$, then $f(x)$ is monotonic decreasing in the interval. Students may verify the following :—

(1) The function $\sin x$ is monotonic increasing in the interval $0\leqslant x\leqslant\frac{\pi}{2}$ but is monotonic decreasing in the interval $\frac{\pi}{2}\leqslant x\leqslant\pi$.

(2) The function $\cos x$ is monotonic decreasing in the interval $0\leqslant x\leqslant\pi$ and monotonic increasing in the interval $\pi\leqslant x\leqslant 2\pi$.

(3) The function x^2 is monotonic increasing in $0\leqslant x<\infty$ and is monotonic dercreasing in $-\infty<x\leqslant 0$.

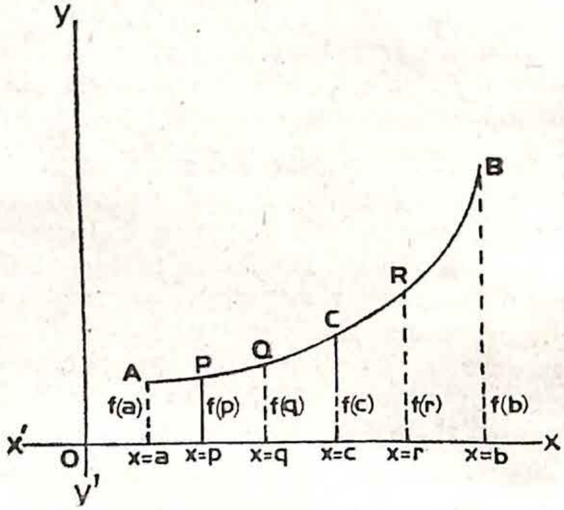
§ 2·14. **Bounds of a functjon.**

If in an interval $a\leqslant x\leqslant b$, a function $f(x)$ is defined and if there exist two numbers M and m such that for all values of x in the interval, $m \leqslant f(x) \leqslant M$ then the function $f(x)$ is said to be bounded in the interval $a \leqslant x \leqslant b$. In this case M is called an upper bound of $f(x)$ and $f(x)$ is a lower bound of $f(x)$. If an upper bound M be such that $M - f(x)$ is less than every positive number, however, small it may be, for at least one x in the interval then M is said to be a strictly upper Bound of $f(x)$. If there exists a point d in $a \leqslant x \leqslant b$, so that $f(d) = M$ (the strictly upper bound), then the function is said to attain its upper bound. If a lower bound m be such that $f(x) - m$ is less that every positive number, however small it may be, for at least one value of x in the interval then m is said to be a strictly lower Bound of the function $f(x)$. If there exists a point c in $a \leqslant x \leqslant b$, so that $f(c) = m$ (the strictly lower Bound), then the function is said to attain its lower Bound.

§ 2·15. Graphs of functions and continuity of a function.

The graph of a function is its pictorial or geometrical representation ; the graph of a function helps us to study the properties of the function. To draw the graph of a function we choose two straight lines at right angles to each other, according to our convenience, as the axes of co-ordinates. The point of intersection of these axes is the origin. Next we choose suitable scales of length along these axes. Note that the two scales may be the same or different according to convenience.

Next let c be any number within the interval in which we are to draw the graph of the function and we determine the value $f(c)$. We now plot the point $C\{c, f(c)\}$. Similarly if p, q, r,$\cdots$are numbers within the interval then the points P, Q, R,$\cdots$are also plotted with co-ordinates $\{p, f(p)\}$, $\{q\ f(q)\}$, $\{r, f(r)\}\cdots$care must be taken to see that $f(p), f(q), f(r)\cdots$are all finite. If corresponding to a value c of x within the interval, $f(x)$ does not possess any finite value, then the point $\{c, f(c)\}$ will not be a point of the graph. Sufficient number of points (depending on the nature of the function) starting from $x = a$ and ending at $x = b$ are taken in the interval $a \leqslant x \leqslant b$. Joining the points $A\{a, f(a)\}, \cdots P, Q, R, \cdots B\{b, f(b)\}$ free-hand, the graph of the function will be obtained. If the interval be $-\infty < x < \infty$, then there will be no end points of the graph. If $f(c)$ is not finite, we have already said that the point $C\{c, f(c)\}$ will not be a point of the graph ; the graph will have a break or gap at the point C corresponding to $x = c$. The graph of a function is said to be continuous if there be no break or gap in the graph i. e., from one end of the graph to the other end we can move our pencil or chalk along the graph without lifting it. If in moving our pencil or chalk along the graph of a function we have to lift our pencil or chalk at a point C corresponding to $x = c$, then the graph has a break at that point and the graph is discontinuous at the point C. C is the point of discontinuity. If the graph of a function be continuous, then the function is also continuous. In Chapter Four we shall study the continuity of a functions analytically. We shall there prove the converse proposition that the graph of a continuous function is also continuous. If at a point C$\{c, f(c)\}$, the graph of a function has a break, then the function is discontinuous at $x=c$ and $x=c$ is a point of discontinuity of the function.



Continuous graph.

Fig. 2·1

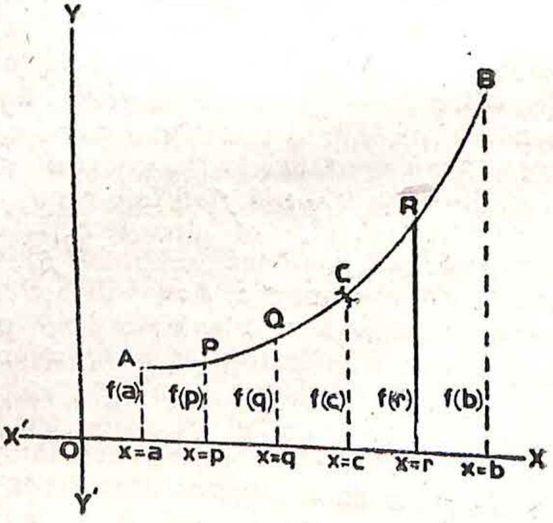


Fig. 2·2

Discontinuous graph having break at C.

Note : In a graph if there be a break or gap at a point C, then we cannot show the break by keeping a gap however small ; for, even in a very small gap there exist an infinite number of points. But we are to show the break or discontinuity only at a point. To avoid this difficulty we conventionally show the point of discontinuity by drawing two small arcs of circles opening in opposite directions and touching each other at the point. This has been illustrated in fig. 2·2.

We conclude this section by showing the graphs of some elementary functions. The domains of definitions of the functions and points of discontinuities are also indicated.

(i) Graph of the function

$$f(x)=x.$$

Let $y=f(x)$.

The graph is a straight line AB through the origin making an angle of 45° with the positive direction of the x-axis. It has no discontinuity and so the function is continuous. The domain of definition is $-\infty<x<\infty$.

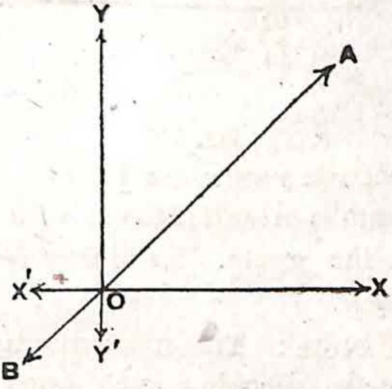


Fig. 2·3

Note : 1. The graph may be drawn by plotting points from the following table

x	0	1	-2
$y=f(x)$	0	1	-2

2. From geometrical consideration the gradient of the straight line (We know from analytical geometry that the equation $y=x$ represents a straight line) $y=x$ is 1 i.e., the straight line makes an angle of 45° with the positive direction of the x-axis. It also passes through the origin.

3. x is the simplest form of a polynomial function. It will be shown later (in Chapter Four) that every polynomial function is continuous.

(ii) The function $f(x)=\dfrac{x^2}{x}$

When $x=0$, you can understand, the function is not defined as division by zero is not defined. So the function $f(x)$ may be expressed as follows

$$f(x)=x \text{ when } x\neq 0 \text{ is undefined when } x=0$$

Let $y=f(x)$.

Then $y=x$ when $x\neq 0$ and y is undefined when $x=0$.

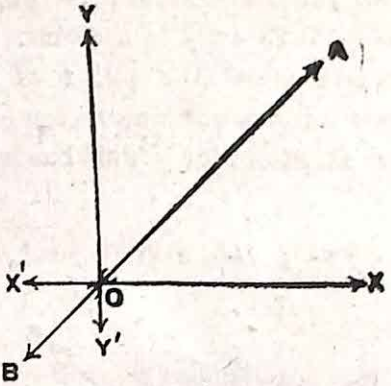


Fig. 2·4

So, the graph of the function will be the same as the graph of $y=x$; only when $x=0$, the graph will have no point and so will have a break at the origin [for, when $x=0$, $y=0$ for the function $y=f(x)=x$]. The graph is shown in fig. 2·4. So the graph is dicontinuous at $x=0$. Hence the function is discontinuous at the origin. Its domain of definition is $-\infty<x<\infty$ excluding $x=0$.

Note : The discontinulty at $x=0$ is shown by two arcs of circles touching each other at the origin. The upper one opens upwards in the form of a cup and the lower one opens downwards in the form of a cap. Use of cups and caps to show a single point of discontinuity is a matter of convention. This convention is used to show that there is only one point of discontinuity; otherwise if we left a gap at the point of discontinuity, however small the gap may be, the gap would contain an infinite number of points which would indicate infinite number of points of discontinuity.

2 The distance between the two branches is smaller than every positive number however small and so may be said to be zero.

(iii) $f(x)=x-1$ when $x\geqslant 0$
$=x+1$ when $x<0$

Here the function is defined in two forms one for positive values of x and 0 and the other for negative values of x.

Let $y=f(x)$. Then we get

$y=x-1$ when $x\geqslant 0$ and $y=x+1$ when $x<0$ and the graph will have two branches. It is evident that both the branches being graphs of linear equations $x-y-1=0$ and $x-y+1=0$ will be straight lines, the graph is shown in fig. 2·5.

The branch $y=x-1$ is a straight line starting from the point $(0, -1)$ of the y-axis and goes upward. The branch $y=x+1$ is a straight line parallel to the first branch starting from the point $(0, 1)$ of the y-axis. In this case the point $(0, 1)$ is not included in the graph; but $(0, -1)$ is a point of the first branch. Clearly the graph is discontinuous and the discontinuity is at $x=0$.

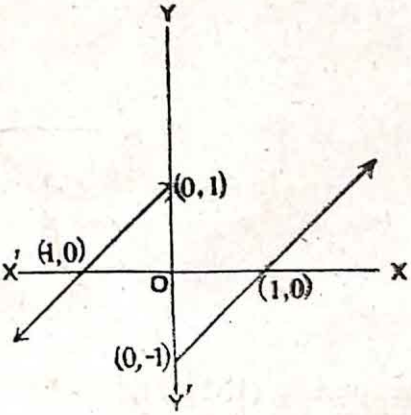


Fig. 2·5

Here it may be noted that the distance between the two branches is finite. In the next section, before we proceed to the Examples Set, we conclude our discussion by showing graphs of elementary functions and briefly discussing their properties.

§ 2·16. Graphs of Elementary Functions.

(A) Algebraic functions.

(i) $y=f(x)=x$. The graph has been shown and discussed in fig. 2·3 of § 2·15.

(ii) $y=f(x)=x^2$.

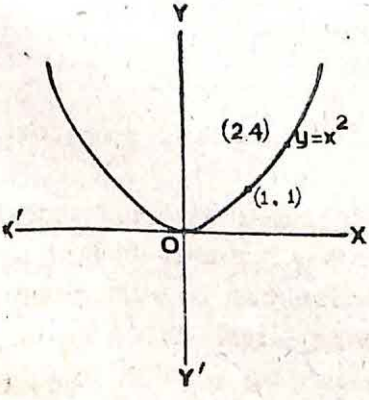


Fig. 2·6

The graph is shown in fig. 2·6. The graph is a continuous graph having two branches on the two sides of the y-axis. The graph has no portion below the x-axis. So the function is non-negative for all values of x. As x increases in magnitude, the value of $y=f(x)$ also increases on both sides of the y-axis and so opens upwards to infinity. Hence the curve is an open curve symmetrical about the y-axis.

(iii) $y=f(x)=x^3$.

The graph is a continuous one passing through the origin and

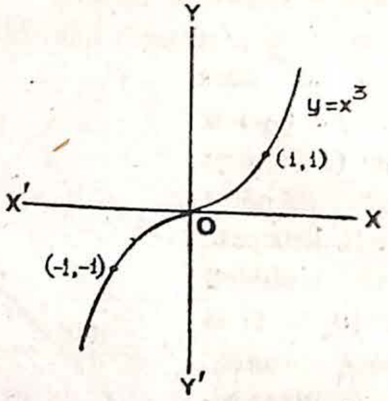


Fig. 2·7

the points (1, 1) and (−1, −1) and extends upto infinity on both sides of the x-axis.

(iv) $y=f(x)=x^4$.

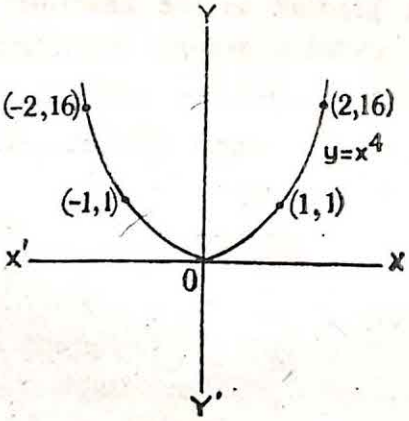


Fig. 2·8

The graph is shown in fig. 2·8. It has the same properties as $f(x)=x^2$.

Note : We find from the above graphs that when the index of x is a positive odd integer the graph has branches on both sides of the x-axis and the values of the function can be numerically as large as possible whether positive or negative. When the index is an even positive integer, the graph has no portion below the x-axis and is symmetrical about the y-axis. The function can attain positive values as large as one desires. In both cases the graphs are continuous. In fact graphs of polynomial functions are continuous. Hence we may conclude that polynomial functions are continuous for all values of x. When n is odd, the graph of $y=x^n$ in every case passes through the points (0, 0), (1, 1) (−1, −1). When n is even the graph of $y=x^n$ passes in every case through the points (0, 0), (1, 1) and (−1, 1).

(v) $y=f(x)=\frac{1}{x}$.

The graph is shown in fig. 2·9. The function is undefined at $x=0$. So, the graph as well as the function is discontinuous at $x=0$. The graph has two branches one in the first quadrant and the other in the third quadrant. The axes of co-ordinates are asymptotes of the graph.

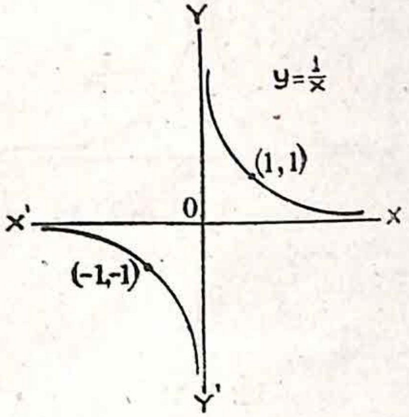


Fig. 2·9

Note. 1. The graph is a rectangular hyperbola.

2. The graph is an example of the graph of a rational algebraic function.

In the following figures we show the graphs of the functions $y=f(x)=\frac{1}{x^n}$ where $n=2, 3, 4$. Every graph is discontinuous at the

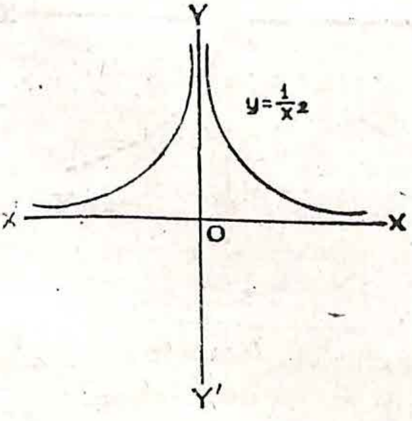


Fig. 2·10

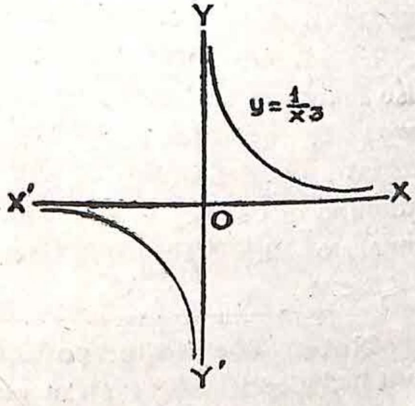


Fig. 2·11

origin and passes through the points (1, 1) and (−1, −1) when n is odd and passes through the points (1, 1) and (−1, 1) when n is even.

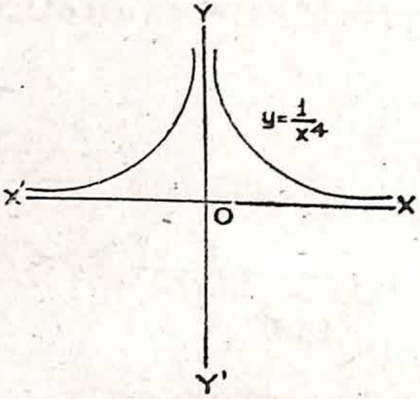


Fig. 2·12

When n is even both the branches are above the x-axis in the first and second quadrant and so the values of the function are also positive. When n is odd the two branches are in the first and third quadrants and the function may have, positive and negative values as large as one desires. The function never attains the value 0.

(vi) $y=f(x)=x^{\frac{1}{2}}$

$y=x^{\frac{1}{2}}$ or, $y^2=x$. But $x^{\frac{1}{2}}=\sqrt{x}$ is the positive square root of x so y is here always positive. So, the graph has no portion below the x-axis. Again $\sqrt{x}$ is not defined for negative values of x. Hence the graph has no portion on the left of the y-axis. Hence the graph is situated only in the first quadrant. As x increases from 0, the value of the function also increases and may be as large as we like. So the domain of definition of the function is $0\leq x<\infty$ and the range of the function is also $0\leq x<\infty$.

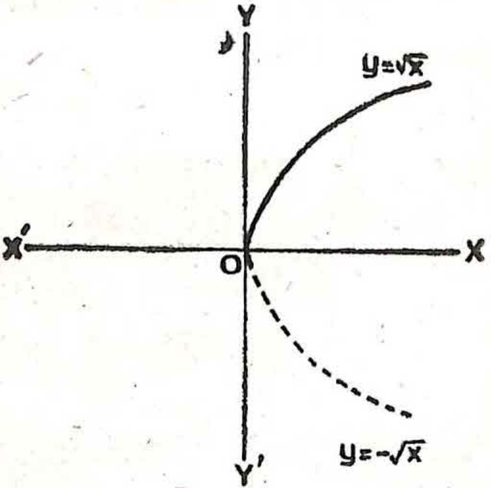


Fig. 2·13

Note. The dotted portion of the graph as shown in fig. 2·13 is not included in the graph of $y=\sqrt{x}$. It is the graph of $y=-\sqrt{x}$. Its domain of definition is $0\leq x\leq\infty$ but range is $-\infty<x\leq 0$.

(vii) Graphs of $y=x^{\frac{1}{3}}$, $y=x^{\frac{2}{3}}$ and $y=x^{\frac{3}{2}}$.

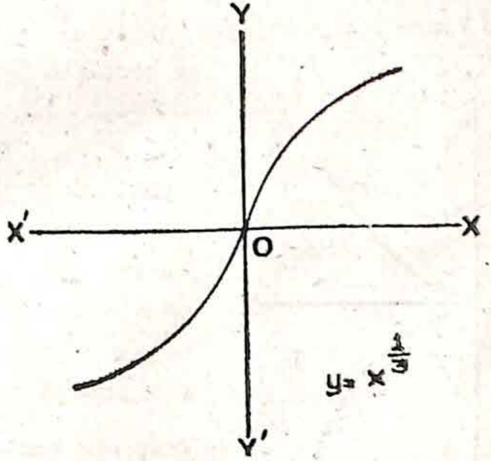


Fig. 2·14

The graph of $x^{\frac{1}{3}}$ is a continuous graph through the origin. Its domain of definition is $-\infty < x < \infty$ and range is also $-\infty < y < \infty$.

The graph of $x^{\frac{2}{3}}$ is a continuous graph through the origin. It has two branches on both sides of the y-axis. The graph is

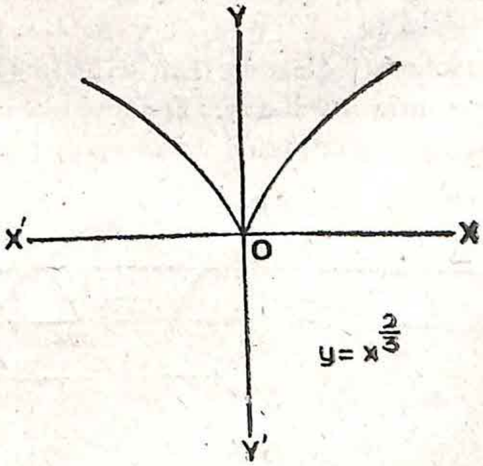


Fig. 2·15

symmetrical about the origin. It has no part below the x-axis and so y is always positive.

The domain of definition of the graph is $-\infty < x < \infty$ and its range is $0 \leqslant y < \infty$.

The function $y = x^{\frac{3}{2}}$ is defined for values of $x \geqslant 0$. If $x < 0$, then y becomes imaginary. So its domain of definition is $0 \leqslant x < \infty$. The range of the function is also $0 \leqslant y < \infty$. The graph is situated only in the first quadrant. It is a continuous graph through the origin. The dotted portion in the fourth quadrant is the graph of $y = -x^{\frac{3}{2}}$. The two functions $y = x^{\frac{3}{2}}$ and $y = -x^{\frac{3}{2}}$ together constitute the implicit function $y^2 = x^3$. It is called the semi-cubical parabola.

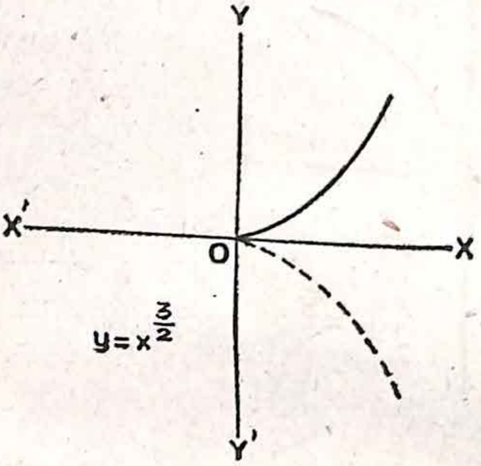
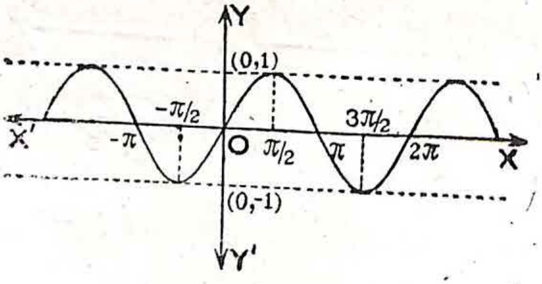


Fig. 2·16

(B) Graphs of Trigonometric Functions.

You are already acquainted with the graphs of Trigonometric functions. We show in the following figures graphs of the six trigonometric functions indicating their domain of definition, range and points of discontinuity if any. It should be mentioned that the functions are periodic with period 2π as $\sin(x + 2\pi) = \sin x$ etc.



Graph of $y = \sin x$

Fig. 2·17

The graph is a continuous graph through the origin. The domain of definition of the function $\sin x$ is $-\infty < x < \infty$ and range is $-1 \leqslant y \leqslant 1$. So the maximum and minimum values that sin x can attain are **1** and -1 respectively.

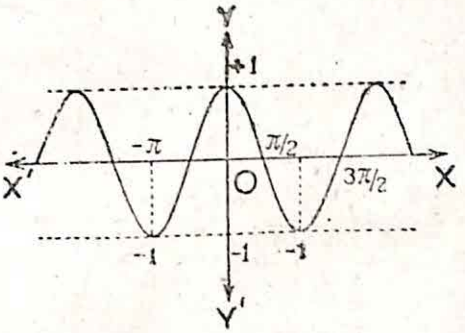


Fig. 2·18

Graph of $y = \cos x$

The graph is a continuous graph through the origin. The domain of definition and range of the function $y = \cos x$ are $-\infty < x < \infty$ and $-1 \leqslant y \leqslant 1$ respectively. So the maximum and minimum values that a co-sine function can attain are -1 and $+1$ respectively. See fig. 2·18.

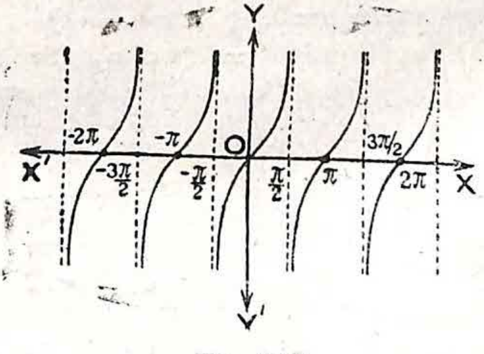


Fig. 2·19

Graph of $y = \tan x$

The graph consists of an infinite number of branches of the same shape and size but are distinct from one another at the points $x = (2n+1)\frac{\pi}{2}$ [$n = 0, \pm 1, \pm 2, \ldots\ldots$] which are its points of discontinuity. Every branch is itself a continuous graph. The domain of definition of the function tan x is $-\infty < x < \infty$ other than the points $x=(2n+1)\frac{\pi}{2}$ $[n=0, \pm 1, \pm 2, \cdots\cdots]$. The range of the function is $-\infty<y<\infty$. The graph passes through the origin. See fig. 2·19.

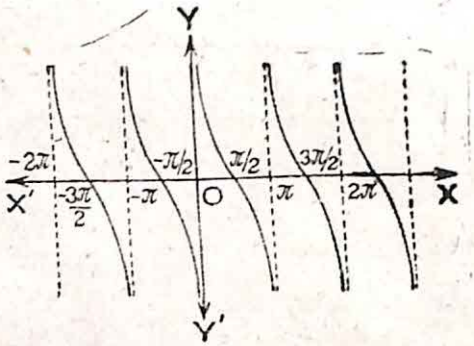


Fig. 2·20

Graph of $y=\cot x$

The graph consists of several branches distinct from one another at points $x=n\pi$ $[n=0, \pm 1, \pm 2, \cdots\cdots]$. So these are the points of discontinuity of the function and so the graph does not pass through the origin. The branches themselves are continuous. The domain of definition of the function cot x is $-\infty<x<\infty$ other than the points $x=n\pi$ $[n=0, \pm 1, \pm 2, \cdots\cdots]$

The range of the function is $-\infty<y<\infty$. See fig. 2·20.

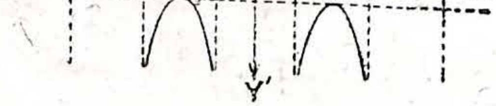


Fig 2·21

Graph of $y=\sec x$

This graph also consists of several distinct branches having breaks or discontinuities at points $x=(2n+1)\frac{\pi}{2}$ $[n=0, \pm 1, \pm 2\cdots]$ So, these points are also points of discontinuities of the function. The domain of definition of the function is $-\infty<x<\infty$ other than the points $x=(2n+1)\frac{\pi}{2}[n=0, \pm 1, \pm 2, \cdots\cdots]$. The range of the function is $y \geqslant 1$ or $y \leqslant -1$. So the graph has no branch between $y=-1$ and $y=1$. See fig. 2·21.

Graph of $y=\operatorname{cosec} x$

This graph is discontinuous at $x=n\pi[n=0, \pm 1, \pm 2, \cdots\cdots]$

This graph also consists of several distinct branches. The domain of definition of the function is $-\infty<x<\infty$ other than $x=n\pi[n=0, \pm 1, \pm 2, \cdots\cdots]$. The range of the function is $y \geqslant 1$ or $y \leqslant -1$. So, this graph has also no portion between $y=-1$ and $y=1$. See fig. 2·22.

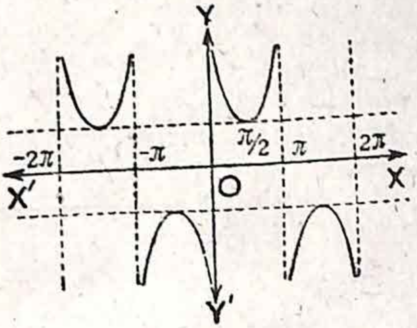
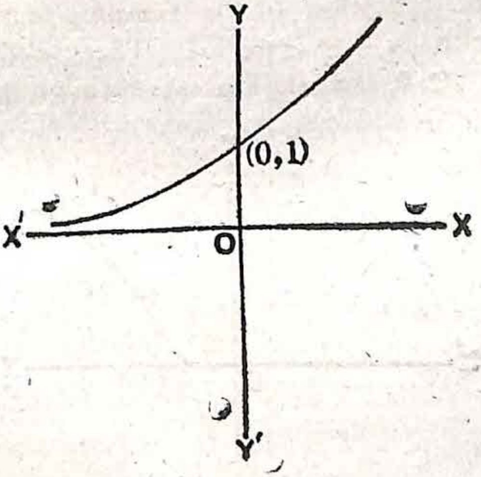


Fig. 2·22

(C) Exponential function : a^x, e^x.

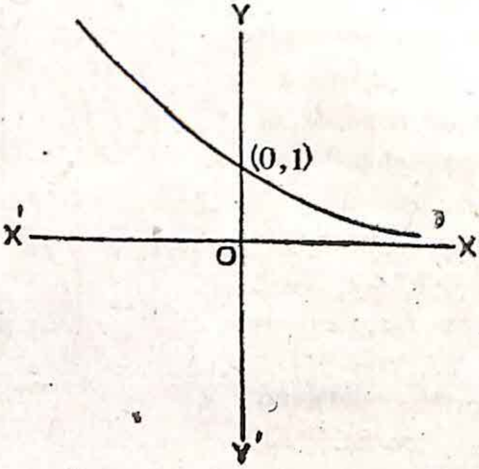
The graph of $y=a^x$ is a continuous graph through the point (0, 1) and is defined for positive values of a. Whatever be the



Graph of $y=a^x (a>1)$

Fig. 2·23

value of a, the different graphs pass through the point (0, 1). The function e^x is a particular case of a^x. Every graph is situated above the x-axis and so y is always positive. The domain of definition of a^x is $-\infty < x < \infty$ and its range is $0 < y < \infty$. We show the graphs of the functions a^x in the two cases when $a > 1$ and $a < 1$ respectively. Note that $e > 1$ and so you can draw the graph by taking approximate value of e. When $a = 1$, $y = x$ whose graph has already been discussed.

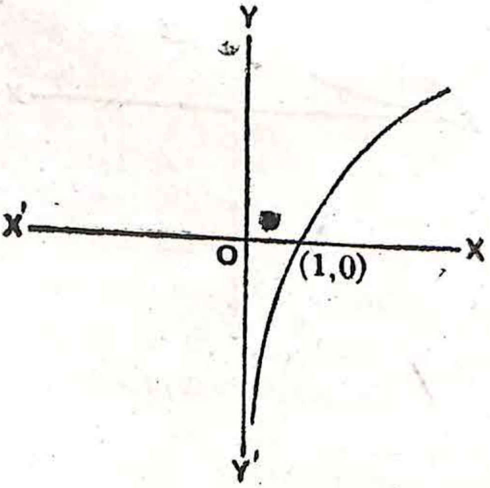


Graph of $y = a^x$ $(0 < a < 1)$

Fig. 2·24

(D) · Graph of $y = \log_a^x$. $a > 0$

The domain of definition of the function is $0 < x < \infty$ and the range of the function is $-\infty < y < \infty$. The graph is shown below. For all values of $a > 0$, the graph passes through the point (1, 0).



Graph of $y = \log_a^x$

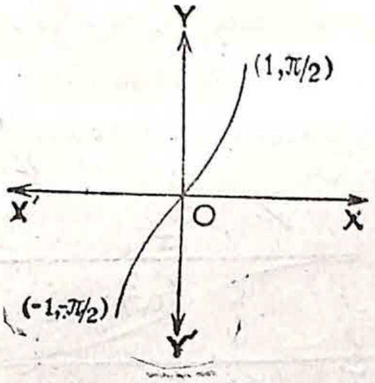
Fig. 2·25

(E) Graphs of inverse circular functions

You know that the inverse circular functions are multiple valued functions.

The range of every inverse circular function is $-\infty < x < \infty$. But their domains of definition are different. We show in the following figures the graphs of different inverse circular functions for values of y within particular intervals.

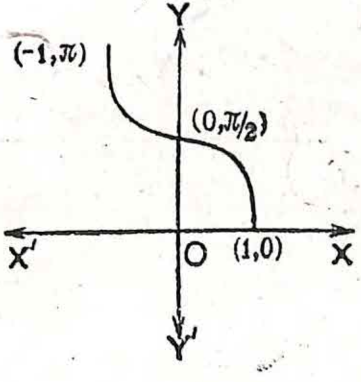
(i) $y=\sin^{-1}x$; domain of definition of the function is $-1 \leqslant x \leqslant 1$.



Graph of $y=\sin^{-1}x$ in $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$

Fig, 2·26

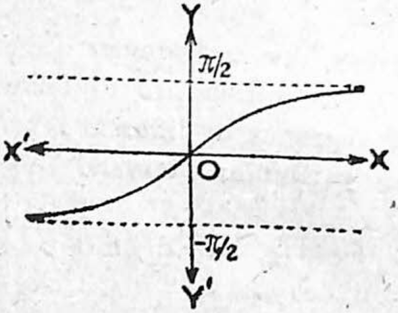
(ii) $y=\cos^{-1}x$; domain of the function is $-1 \leqslant x \leqslant 1$.



Gaaph of $y=\cos^{-1}x$ in $0 \leqslant y \leqslant \pi$.

Fig. 2·27

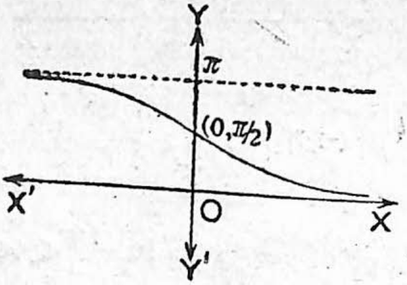
(iii) $y=\tan^{-1}x$; domain of definition of the function $-\infty<x<\infty$.



Graph of $y=\tan^{-1}x$ in $-\frac{\pi}{2}<y<\frac{\pi}{2}$

Fig. 2·28

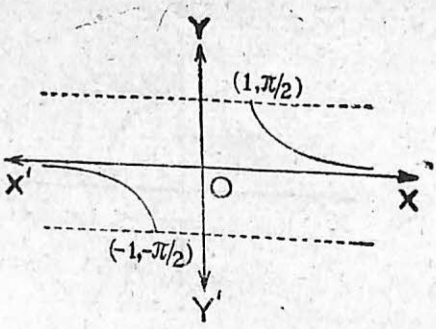
(iv) $y=\cot^{-1}x$; Range of the function $-\infty<y<\infty$.



Graph of $y=\cot^{-1}x$ in $0<y<\pi$.

Fig. 2·29

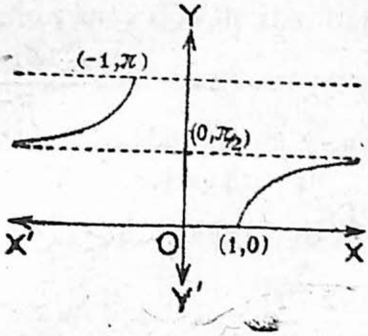
(v) $y=\sec^{-1}x$; Range of the function $-\infty<y\leqslant-1$ and $1\leqslant y<\infty$.



Graph of $y=\sec^{-1}x$ in $0\leqslant y\leqslant\pi$

Fig. 2·30

(vi) $y=\operatorname{cosec}^{-1}x$; Range of the function $-\infty<y\leqslant-1$ and $1\leqslant y<\infty$.



Graph of $y=\operatorname{cosec}^{-1}x$ in $-\frac{\pi}{2}<y<\frac{\pi}{2}$

Fig. 2·31

Examples 2

Ex. 1. Find the domain of definition of the following functions :

(i) $f(x)=\frac{x}{x}$ (ii) $f(x)=\sqrt{6-x}$ [Joint Entrance, 1984]

(iii) $f(x)=\frac{x}{x^2-5x+6}$ (iv) $f(x)=\frac{x^2-4}{x-2}$ when $x\neq2$; $f(x)=3$ when $x=2$. (v) $\sqrt{\frac{(x+1)(x-3)}{(x-2)}}$ [I. I. T., 1980]

Solution (i) The domain of definition of the function is all real values of x other than x. For, when $x=0$, then $f(x)$ becomes undefined as division by 0 is undefined. The function is defined for all other values of x.

(ii) When $x>6$, $6-x$ is negative and $\sqrt{6-x}$ becomes imaginary. When $x\leqslant6$, $6-x$ is $\geqslant0$ and so $\sqrt{6-x}$ is real. So the domain of definition of $f(x)$ is all real values of $x\leqslant6$.

(iii) The function is defined for all values of x other than those which make the denominator 0. Now the denominator $x^2-5x+6=(x-2)(x-3)$ becomes 0 when $x=2$ or $x=3$. So, the domain of definition of the function is all real values of x other than 2 or 3.

(iv) When $x \neq 2$, $x-2 \neq 0$ and so $f(x)=\dfrac{x^2-4}{x-2}$ have finite values. Again when $x=2$, $f(x)$ is given to be 3. Hence the domain of definition of the function is all real values of x i.e., $-\infty < x < \infty$.

(v) When $x=2$, then $x-2=0$ and $\dfrac{(x+1)(x-3)}{x-2}$ is undefined; so when $x=2$, then $f(x)$ is undefined. Again when $-1<x<3$, then $(x+1)(x-3)$ is negative. When $-1<x<2$, then $x-2$ is negative; so when $-1<x<2$ then $\dfrac{(x+1)(x-3)}{x-2}$ is positive and $\sqrt{\dfrac{(x+1)(x-3)}{x-2}}$ is real and finite. When $2<x<3$, then $x-2$ is positive and $\dfrac{(x+1)(x-3)}{x-2}$ is negative; so $\sqrt{\dfrac{(x+1)(x-3)}{x-2}}$ is imaginary. When $x>3$, then each of $x+1$, $x-3$, $x-2$ is positive and so $\dfrac{(x+1)(x-3)}{x-2}$ is positive and $\sqrt{\dfrac{(x+1)(x-3)}{x-2}}$ is real. When $x<-1$, then all of $(x+1)$, $(x-3)$ and $x-2$ are negative. So, $\dfrac{(x+1)(x-3)}{x-2}$ is negative and $\sqrt{\dfrac{(x+1)(x-3)}{x-2}}$ is imaginary. Hence when $x<-1$ or $2<x<3$ then $\sqrt{\dfrac{(x+1)(x-3)}{x-2}}$ is imaginary. For all other values of x i.e., in the intervals $-1 \leqslant x < 2$ and $x \geqslant 3$ (i.e., $3 \leqslant x < \infty$) the function is defined and so these two intervals constitute the domain of definition of $\sqrt{\dfrac{(x+1)(x-3)}{x-2}}$.

Ex. 2. (*a*) If $y=f(x)=\dfrac{x+1}{x+2}$, find $f(0)$ and $f(-1)$. Also show that $f(y)=\dfrac{2x+3}{3x+5}$.

[Tripura, 1978]

(*b*) If $f(x)=b\left(\dfrac{x-a}{b-a}\right)+a\left(\dfrac{x-b}{a-b}\right)$, show that $f(a+b)=f(a)+f(b)$.

(*c*) If $y=f(x)=\dfrac{ax+b}{bx-a}$, show that $f(y)=x$.

Solution : (*a*) $f(0)=\dfrac{0+1}{0+2}=\dfrac{1}{2}$;

$$f(-1)=\frac{-1+1}{-1+2}=\frac{0}{1}=0.$$

$$f(y)=\frac{y+1}{y+2}=\frac{\frac{x+1}{x+2}+1}{\frac{x+1}{x+2}+2}\quad\left[\because\ y=f(x)=\frac{x+1}{x+2}\right]$$

$$=\frac{\frac{2x+3}{x+2}}{\frac{3x+5}{x+2}}=\frac{2x+3}{3x+5}$$

(*b*) $f(a)=b\left(\frac{a-a}{b-a}\right)+a\left(\frac{a-b}{a-b}\right)=b.0+a.1=a.$

$f(b)=b\left(\frac{b-a}{b-a}\right)+a\left(\frac{b-b}{a-b}\right)=b.1+a.0=b.$

$$f(a+b)=b\left(\frac{a+b-a}{b-a}\right)+a\left(\frac{a+b-b}{a-b}\right)$$

$$=b\left(\frac{b}{b-a}\right)+a\left(\frac{a}{a-b}\right)=\frac{b^2}{b-a}+\frac{a^2}{a-b}=\frac{a^2-b^2}{a-b}$$

$$=a+b=f(a)+f(b).$$

(*c*) $$f(y)=\frac{ay+b}{by-a}=\frac{a\left(\frac{ax+b}{bx-a}\right)+b}{b\left(\frac{ax+b}{bx-a}\right)-a}\quad\left[\because\ y=f(x)=\frac{ax+b}{bx-a}\right]$$

$$=\frac{\frac{a^2x+ab+b^2x-ab}{bx-a}}{\frac{abx+b^2-abx+a^2}{bx-a}}=\frac{(a^2+b^2)x}{(a^2+b^2)}=x$$

Ex. 3. (*a*) If $f(x)=e^{px+q}$; (p, q) are constants, then show that $f(a).f(b).f(c)=f(a+b+c)e^{2q}$ [H. S. 1982]

(*b*) (i) If $f(x+3)=2x^2-3x-1$, find $f(x+1)$

(ii) If $f(x)=\tan^{-1}x$, prove that $f(x)$, $f(y)$ and $f(x+y)$ are related ; find the relation. [H. S. 1987]

Solution : (*a*) $f(x)=e^{px+q}$

$\therefore\ f(a)=e^{pa+q}$; $f(b)=e^{pb+q}$, $f(c)=e^{pc+q}$

$$\therefore\ f(a).f(b).f(c)=e^{pa+q}.e^{pb+q}.e^{pc+q}$$

$$=e^{p(a+b+c)+3q}=e^{p(a+b+c)+q}.e^{2q}$$

$$=f(a+b+c).e^{2q}.$$

(*b*) (i) Let $f(x)=ax^2+bx+c$. $\therefore\ f(x+3)=a(x+3)^2+b(x+3)+c$
$=ax^2+(6a+b)x+9a+3b+c.$

Again $f(x+3)=2x^2-3x-1$

$\therefore\ a=2$; $(6a+b)=-3$ and $9a+3b+c=-1$

Now, $6a+b=-3$ or, $12+b=-3\quad \therefore\ b=-15.$

Again $9a+3b+c=-1$ or, $18-45+c=-1\quad \therefore\ c=26$

$\therefore\ f(x)=2x^2-15x+26\quad \therefore\ f(x+1)=2(x+1)^2-15(x+1)+26$
$=2x^2+4x+2-15x-15+26=2x^2-11x+13.$

(ii) As $f(x)=\tan^{-1}x\quad \therefore\ x=\tan\{f(x)\}$

$f(y)=\tan^{-1}y\quad \therefore\ y=\tan\{f(y)\}$

$f(x+y)=\tan^{-1}(x+y)\quad \therefore\ x+y=\tan\{f(x+y)\}$

or, $\tan\{f(x)\}+\tan\{f(y)\}=\tan\{f(x+y)\}\cdots$(i)

Hence $f(x)$ and $f(y)$ are related by the relation (i).

Ex. 4. If $f(x)=x-3$ and $g(x)=4-x$, find those values of x for which $|f(x)+g(x)|<|f(x)|+|g(x)|$ holds. [H. S. 1986]

$f(x)+g(x)=x-3+4-x=1\quad \therefore\ |f(x)+g(x)|=|1|=1.$

First let $x<3$, then $x-3<0\quad \therefore\ |f(x)|=|x-3|=-(x-3)$
$=3-x$. Also $4-x>0\quad \therefore\ |g(x)|=|4-x|=4-x.$

$\therefore\ |f(x)|+|g(x)|=3-x+4-x=7-2x>1$ (as $x<3$)

or, $|f(x)|+|g(x)|>|f(x)+g(x)|$ i.e., $|f(x)+g(x)|$
$<|f(x)|+|g(x)|$

Next let $x>4$, then $x-3>0\quad \therefore\ |f(x)|=|x-3|=x-3$
$4-x<0\quad \therefore\ |g(x)|=|4-x|$
$=-(4-x)=x-4$

$\therefore\ |f(x)|+|g(x)|=x-3+x-4=2x-7>1$(as $x>4$).

$\therefore\ 1<|f(x)|+|g(x)|$

or, $|f(x)+g(x)|<|f(x)|+|g(x)|$

If $3\leqslant x\leqslant 4$, $f(x)=x-3\geqslant 0$ and $g(x)=4-x\geqslant 0$

$\therefore\ |f(x)|=|x-3|=x-3$ and $|g(x)|=|4-x|=4-x.$

$\therefore\ |f(x)|+|g(x)|=x-3+4-x=1$

$\therefore\ |f(x)|+|g(x)|=|f(x)+g(x)|.$

Hence $|f(x)+g(x)|<|f(x)|+|g(x)|$ holds when $x<3$ or $x>4$.

Ex. 5. Find the inverse function of $f(x)=\dfrac{ax+b}{cx-a}$.

[Tripura, 1985]

Let $y=f(x)=\dfrac{ax+b}{cx-a}$. or, $cxy-ay=ax+b$.

or, $x(cy-a)=ay+b$ $\therefore$ $x=\dfrac{ay+b}{cy-a}$.

$\therefore$ The inverse function of $f(x)$ is $\dfrac{ay+b}{cy-a}=f(y)=f\{f(x)\}$.

Ex. 6. The entire graph of $y=x^2+kx-x+9$ is entirely above the x-axis if and only if (*a*) $k<7$ (*b*) $-5<k<7$ (*c*) $k>-5$ (*d*) None of these. Which is true ?

[I. I. T., 1979]

Solution : $y=x^2+kx-x+9=x^2+(k-1)x+9$

$$=x^2+(k-1)x+\left(\frac{k-1}{2}\right)^2+9-\left(\frac{k-1}{2}\right)^2$$

$$=\left(x+\frac{k-1}{2}\right)^2+9-\left(\frac{k-1}{2}\right)^2$$

Now $\left(x+\dfrac{k-1}{2}\right)^2\geqslant 0$ for all real values of x.

So, y will be positive for all x if and only if $9-\left(\dfrac{k-1}{2}\right)^2>0$

$\left[\because\right.$ minimum value of $\left.\left(x+\dfrac{k-1}{2}\right)^2=0\right]$ or, $9>\left(\dfrac{k-1}{4}\right)^2$

or, $36>(k-1)^2$ or, $6>\pm(k-1)$ i.e., $-5<k<7$.

Hence (*b*) is true.

Ex. 7 Find the domain and range of the function $f(x)=\dfrac{x}{1+x^2}$. Is the function one to one ?

[I. I. T., 1978]

Solution : The given function is an algebraic rational function and will be defined for those values of x which will not make the denominator $(x^2+1)=0$. Now for all real values of x, $x^2\geqslant 0$ so, $x^2+1>0$. Hence the function is defined for all real values of x. So the domain of the function is the interval $-\infty<x<\infty$.

Again $1+x^2-x=x^2-x+\frac{1}{4}+\frac{3}{4}=(x-\frac{1}{2})^2+\frac{3}{4}>0$

$\therefore$ $1+x^2>x$ i.e., $x<1+x^2$ or $\dfrac{x}{1+x^2}<1$ when $x\geqslant 0$, both x and $1+x^2$ are positive and $\dfrac{x}{1+x^2}>0$.

So when $x>0$, $0<\frac{x}{1+x^2}<1$. When $x<0$, then $1+x^2>0$,

So. $\frac{x}{1+x^2}<0$.

Let in this case $x=-y$ $(y>0)$

$\therefore\ 1+x^2-y=1+y^2-y=\left(y-\frac{1}{2}\right)^2+\frac{3}{4}>0.$

So, $1+x^2>y$ or $y<1+x^2$ i.e., $\frac{y}{1+x^2}<1$.

So, $\frac{x}{1+x^2}=-\frac{y}{1+x^2}>-1$

So, in this case $-1<\frac{x}{1+x^2}<0$.

So, the range of the function is $-1<f(x)<1$.

The function is not one to one. For let $f(x)=\frac{1}{3}$.

$\therefore\ \frac{1}{3}=\frac{x}{1+x^2}$ or, $1+x^2=3x$. or, $1+x^2-3x=0$

or, $x=\frac{3\pm\sqrt{9-4}}{2}=\frac{3\pm\sqrt{5}}{2}$.

So, corresponding to the value $\frac{1}{3}$ of $f(x)$, x has two values and the function is not one—one.

Ex. 8. If $f(x)=ax^2+bx+c$, find a, b so that $f(x+1)=f(x)+x+1$ may hold identically. [H. S., 1988]

Solution : $f(x)=ax^2+bx+c$

$\therefore\ f(x+1)=a(x+1)^2+b(x+1)+c=ax^2+2ax+a+bx+b+c$
$=ax^2+(2a+b)x+a+b+c$.

If $f(x+1)=f(x)+x+1$ be an identity,

Then $ax^2+(2a+b)x+a+b+c=ax^2+bx+c+x+1$
$=ax^2+(b+1)x+c+1$ is true for all values of x.

$\therefore\ 2a+b=b+1$ and $a+b+c=c+1$ i.e., $2a=1$ and $a+b=1$

Solving we get $a=b=\frac{1}{2}$.

Hence the required values are $a=b=\frac{1}{2}$

Ex. 9. If $f(x)=x-|x|$, then determine whether $f(4)$ and $f(-4)$ are equal or not ? [Tripura, 1979]

Solution : $f(4)=4-|4|=4-4=0$

$f(-4)=-4-|-4|=-4-4=-8$

So $f(4)\neq f(-4)$.

Ex. 10. If $f(x)=\frac{1-x}{1+x}$, then find the value of $f\{f(\frac{1}{x})\}$ $[x\neq 0]$

[Joint Entrance, 1984]

Solution : $f(x)==\frac{1-x}{1+x}$ $\therefore$ $f(\frac{1}{x})=\frac{1-\frac{1}{x}}{1+\frac{1}{x}}=\frac{x-1}{x+1}$

$$\therefore \quad f\{f(\tfrac{1}{x})\}=\frac{1-f(\frac{1}{x})}{1+f(\frac{1}{x})}=\frac{1-\frac{x-1}{x+1}}{1+\frac{x-1}{x+1}}=\frac{\frac{2}{x+1}}{\frac{2x}{x+1}}=\frac{1}{x}.$$

Ex. 11. A function $f(x)$ is such that $f(x)+f(y)=f(x+y)$. Show that (i) $f(0)=0$ (ii) $f(-x)=-f(x)$ (iii) if x is an integer and $f(1)=c$, then $f(x)=cx$.

Solution : $f(x)+f(y)=f(x+y)$

Putting $x=0$ we get $f(0)+f(y)=f(0+y)=f(y)$.

$\therefore$ $f(0)=0 \cdots\cdots$(i)

Putting $y=-x$ we get

$f(x)+f(-x)=f(x-x)=f(0)=0$ [proved in (i)]

$\therefore$ $f(-x)=-f(x) \cdots\cdots$(ii)

Next let n be a positive integer.

Then $f(nx)=f(x+x+\cdots\cdots$to n terms)

$=f(x)+f(x)+\cdots$to n terms.

[By repeated application of $f(x+y)=f(x)+f(y)$]

$=nf(x)$.

Putting $x=1$ we get $f(n)=n\,f(1)=nc$.

Putting $n=x$ we get $f(x)=cx\cdots$(iii)

Now let n be a negative integer and $n=-m(m>0)$

$\therefore$ $f(-mx)=-f(mx)$ [by (ii)]

$=-mf(x)$ [as m is a positive integer.]

Putting $x=1, f(-m)=-mf(1)=-mc$ or $f(n)=nc$.

Putting $n=x$, we get $f(x)=cx\cdots$(iv)

Again when $x=0$, $f(x)=f(0)=0$.

Also when $x=0$, $cx=c.0=0$

So $f(x)=cx$, is satisfied when $x=0\cdots$(v)

So by (iii), (iv) and (v) we get,

$f(x)=cx$ when x is an integer and $f(1)=c$

Ex. 12. If s is the set of all real x such that $\dfrac{2x-1}{2x^3+3x^2+x}$ is positive, then s contains.

(A) $(-\infty, \frac{3}{2})$; (B) $(-\frac{3}{2}, -\frac{1}{4})$; (C) $(-\frac{1}{4}, \frac{1}{2})$; (D) $(\frac{1}{2}, \frac{1}{3})$; (E) None of these.

Which are the correct answers ? [I. I. T., 1986]

[In this question we are to determine in which of the open intervals (A), (B), (C), (D) or none of them the given function is positive.]

Solution : Let $f(x)=\dfrac{2x-1}{2x^3+3x^2+x}=\dfrac{2x-1}{x(2x^2+3x+1)}$

$$=\frac{2x-1}{x(2x+1)(x+1)}.$$

When $2x-1=0$, i. e., $x=\frac{1}{2}$, then the denominator does not vanish ; but the function becomes 0. So when $x=\frac{1}{2}$, $f(x)$ is not positive. The denominator becomes 0, when $x=0$, $x=-\frac{1}{2}$ or $x=-1$. So when $x=0$, $-\frac{1}{2}$ or -1 the function is undefined.

Now (1) when $x<-1$, all the factors of the denominator and also the numerator are negative. So, $f(x)$ is positive. $\therefore$ $x<-1$ and so we can say that in the interval $(-\infty, -1)$ $f(x)$ is positive. This interval contains the interval $(-\infty, -\frac{3}{2})$. Hence (A) is correct.

(2) The interval $(-\frac{3}{2}, -\frac{1}{4})$ contains points for which $x<-1$ and also the point $x=-\frac{1}{2}$. When x lies between -1 and $-\frac{1}{2}$, the function is negative. For, in this case $2x-1$, x and $2x+1$ are negative and $x+1$ is positive. Also at $x=-\frac{1}{2}$ the function is undefined. So the answer (B) is not correct.

(3) The interval $(-\frac{1}{4}, \frac{1}{2})$ contains $x=0$, for which $f(x)$ is undefined. Hence (C) is not correct.

(4) When $\frac{1}{2}<x<3$, all the factors are positive and none of them is 0. So $f(x)$ is defined and positive in the interval $(\frac{1}{2}, 3)$.

Hence the answer (D) is correct. So (E) is not correct.

Thus the correct answers are (A) and (D).

Ex. 13. (*a*) If $f(x)=\dfrac{x}{1-x}$, show that $\dfrac{f(x+h)-f(x)}{h}$

$$=\frac{1}{(1-x)(1-x-h)}.$$

[C. U. 1958]

(*b*) If $f(x)=\sin x$, show that $\dfrac{f(x+h)-f(x)}{h}=\dfrac{\cos(x+\frac{h}{2})\sin\frac{h}{2}}{\frac{h}{2}}$.

Solution : (a) $f(x)=\frac{x}{1-x}$; $f(x+h)=\frac{x+h}{1-x-h}$

$$\therefore \quad \frac{f(x+h)-f(x)}{h}=\frac{\frac{x+h}{1-x-h}-\frac{x}{1-x}}{h}=\frac{\frac{x-x^2+h-hx-x+x^2+hx}{(1-x-h)(1-x)}}{h}$$

$$=\frac{h}{h(1-x-h)(1-x)}=\frac{1}{(1-x-h)(1-x)}.$$

(b) $f(x)=\sin x$; $\therefore$ $f(x+h)=\sin (x+h)$

$$\therefore \quad \frac{f(x+h)-f(x)}{h}=\frac{\sin (x+h)-\sin x}{h}$$

$$=\frac{2 \cos(x+\frac{h}{2}) \sin \frac{h}{2}}{h}=\cos (x+\tfrac{h}{2}) \frac{\sin \frac{h}{2}}{\frac{h}{2}}.$$

Ex. 14. Prove that

(i) The product of two even functions or two odd functions is an even function

(ii) The product of an even function and an odd function is an odd function.

(iii) The sum of an even function and an odd function is neither even nor odd.

Solution : (i) First let $f(x)$ and $g(x)$ be two even functions.

$\therefore$ $f(-x)=f(x)$; $g(-x)=g(x)$. Let $F(x)=f(x)g(x)$.

$\therefore$ $F(-x)=f(-x)g(-x)=f(x)g(x)=F(x)$.

So $F(x)$, the product of the two even functions $f(x)$ and $g(x)$, is even.

Next let $f(x)$ and $g(x)$ be two odd functions $\therefore$ $f(-x)=-f(x)$; $g(-x)=-g(x)$.

Now let $F(x)=f(x)g(x)$

$\therefore$ $F(-x)=f(-x)g(-x)=\{-f(x)\}\{-g(x)\}=f(x)g(x)=F(x)$.

So $F(x)$, the product of the two odd functions $f(x)$ and $g(x)$ is an even function.

(ii) Let $f(x)$ be an even function and $g(x)$ be an odd function. $\therefore$ $f(-x)=f(x)$ and $g(-x)=-g(x)$.

Let $F(x)=f(x)\,g(x)$.

$\therefore$ $F(-x)=f(-x)g(-x)=f(x)\{-g(x)\}=-f(x)\;g(x)=-F(x)$.

So, $F(x)$ the product of the even function $f(x)$ and the odd function $g(x)$ is an odd function.

(iii) Let $f(x)$ be an even function and $g(x)$ be an odd function. Then $f(-x)=f(x)$ and $g(-x)=-g(x)$.

Now, let $F(x)=f(x)+g(x)$.

$\therefore\ F(-x)=f(-x)+g(-x)=f(x)-g(x)$.

Which is neither $F(x)$ nor $-F(x)$.

Hence $F(x)$ is neither even nor odd.

i.e., the sum of an even function and an odd function is neither even nor odd.

Ex. 15. Express the following implicit functions as explicit functions.

(i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (ii) $x^3y^3=a^3+b^3$.

Solution : (i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ or, $\dfrac{y^2}{b^2}=1-\dfrac{x^2}{a^2}=\dfrac{a^2-x^2}{a^2}$

$\therefore\ y^2=\dfrac{b^2}{a^2}(a^2-x^2)$ or, $y=\pm\dfrac{b}{a}\sqrt{a^2-x^2}$

(ii) $x^3y^3=a^3+b^3$ or, $y^3=\dfrac{a^3+b^3}{x^3}$ or, $y=\dfrac{\sqrt[3]{a^3+b^3}}{x}$.

Ex. 16. Show that the function $f(x)=\dfrac{x}{x+1}(x>0)$ is monotonic increasing.

Solution : Let $x_1>0$ and $x_2>0$ and $x_1>x_2$.

Now $f(x_1)=\dfrac{x_1}{x_1+1}$, $f(x_2)=\dfrac{x_2}{x_2+1}$

$$\therefore\ f(x_1)-f(x_2)=\frac{x_1}{x_1+1}-\frac{x_2}{x_2+1}$$

$$=\frac{x_1x_2+x_1-x_1x_2-x_2}{(x_1+1)(x_2+1)}=\frac{x_1-x_2}{(x_1+1)(x_2+1)}.$$

As $x_1>x_2$ $\therefore$ $x_1-x_2>0$.

Also as $x_1>0$, $x_2>0$, so $x_1+1>0$, $x_2+1>0$.

$\therefore\ (x_1+1)(x_2+1)>0$.

$\therefore\ f(x_1)-f(x_2)>0$. or, $f(x_1)>f(x_2)$.

Thus if $x_1>x_2$, then $f(x_1)>f(x_2)$.

Hence the function is monotonic increasing.

Ex. 17. If $f(x)=5^x$, show that

(i) $f(x+1)=5f(x)$. (ii) $f(a)f(b)=f(a+b)$.

Solution : (i) $f(x+1)=5^{x+1}=5^x.5^1=5.5^x=5f(x)$

(ii) $f(a)=5^a$; $f(b)=5^b$.
$f(a+b)=5^{a+b}=5^a.5^b=f(a)f(b)$

Ex. 18. (i) When are two functions said to be identical.

(ii) Are the functions x and $\frac{x^2}{x}$ identical? Support your answer with reasons. [*c.f.* H.S. 1979]

Solution : (i) Two functions are said to be identical if for all values of the independent variable, the corresponding values of the function be also the same.

(ii) The functions x and $\frac{x^2}{x}$ are not identical. For, the functions have the same values for $x \neq 0$; when $x=0$, the function x has the value 0 but the function $\frac{x^2}{x}$ is undefined when $x=0$.

Ex. 19. Show that the function $\frac{x^2-3x+4}{x^2+3x+4}$ is bounded and find their bounds.

Let $y=\frac{x^2-3x+4}{x^2+3x+4}$

$\therefore$ $y(x^2+3x+4)=x^2-3x+4$.
or, $x^2(y-1)+3x(y+1)+4(y-1)-0 \cdots$(i)

For real values of x, the quadratic equation (i) will have its discriminant$\geqslant 0$.

i.e., $9(y+1)^2-16(y-1)^2 \geqslant 0$.
or, $-(7y^2-50y+7) \geqslant 0$
or, $7(y-\frac{1}{7})(y-7) \leqslant 0$.

Now if $y<\frac{1}{7}$, then both $y-\frac{1}{7}$ and $y-7$ are negative and so $7(y-\frac{1}{7})(y-7)>0$. $\therefore$ $y \not< \frac{1}{7}$.

If $y>7$, then both $y-\frac{1}{7}$ and $y-7$ are positive, and so

$$7(y-\tfrac{1}{7})(y-7)>0 \quad \therefore \quad y \not> 7.$$

If $\frac{1}{7} \leqslant y \leqslant 7$, then $y-\frac{1}{7}$ is positive or zero and $y-7$ is negative or zero and so $7(y-\frac{1}{7})(y-7) \leqslant 0$.

So y must lie between $\frac{1}{7}$ and 7.

i.e., the given function must lie between $\frac{1}{7}$ and 7. So the function is bounded and $\frac{1}{7}$ and 7 are respectively its lower and upper bounds.

Ex. 20. Express the length of the minor axis of an ellipse of given major axis as a function of the eccentricity of the ellipse ; state also the domain and range of the function.

Let $2a$ be the length of the given major axis and y and e donote length of the minor axis and eccentricity.

$$\therefore \quad \frac{y^2}{4}=a^2(1-e^2) \quad \text{or,} \quad y=2a\sqrt{1-e^2}.$$

Here $e<1$; which is the domain of the function. Also when $e<1$, then $y<2a$. Also $0<y$. Hence the range of y is $0<y<2a$.

Ex. 21. Let f and g be increasing and decreasing functions respectively from $(0, \infty)$ to $(0, \infty)$. Let $h(x)=f(g(x))$. If $h(0)=0$ then $h(x)-h(1)$ is

(A) always Zero (B) always negative (C) always positive (D) strictly increasing (E) None of these.

Which is the correct answer ?

Solution : Let $0<x_1<\infty$, $0<x_2<\infty$ and $x_1<x_2$.

$\therefore$ $g(x_1)>g(x_2)$ [as g is a decreasing function]

$\therefore$ $f(g(x_1))>f(g(x_2))$ [as f is an increaing function]

or, $h(x_1)>h(x_2)$.

So $h(x)$ is a decreasing function.

So if $0\leqslant x<\infty$, then $h(0)\geqslant h(x)$...(i)

Again for all x in $0\leqslant x<\infty$, $0\leqslant g(x)<\infty$, $0\leqslant f(x)<\infty$

So, for all x in $0\leqslant x<\infty$, $0\leqslant f(g(x))<\infty$ i.e., $0\leqslant h(x)<\infty$...(ii)

Combining (i) and (ii) we get $h(x)=0$ for all x in $0\leqslant x<\infty$.

$\therefore$ $h(1)=0$ as $0<1<\infty$.

$\therefore$ $h(x)-h(1)=0-0=0$ for all x in $0\leqslant x<\infty$ and (A) is true. So (B), (C), (D) and (E) are incorrect. So (A) only is true.

Ex. 22. (*a*) Draw the graph of the function $\dfrac{x^2-4}{x-2}$

From the graph state whether the function is continuous at $x=2$.

[Joint Entrance, 1984]

(*b*) A function $f(x)$ is defined as follows.

$$f(x)=\frac{x^2-4}{x-2},\ x\neq 2,\ f(2)=0.$$

Draw the graph of the function and state whether the function is continuous at $x=2$.

[Tripura, 1982]

Solution : (*a*) The function is undefined at $x=2$, as $x-2=0$ when $x=2$. At points $x\neq 2$, $\frac{x^2-4}{x-2}=x+2$

Let $y=f(x)$. So $y=x+2$ when $x\neq 2$ and is undefined when $x=2$.

The graph of $y=x+2$ may be drawn by plotting points corresponding to the following table with respect to a set of rectangular axes OX and OY.

x	0	1	-1	2
y	2	3	1	No value

The ordinate drawn from the point (2, 0) corresponding to $x=2$, would intersect the straight line at the point P (2, 4). But at $x=2$, the function is undefined. So the point P (2, 4) is not included in the graph. The graph is drawn in fig 2·32(*a*). As there is a break at the point P corresponding to $x=2$, so the graph and so the function is discontinuous at $x=2$.

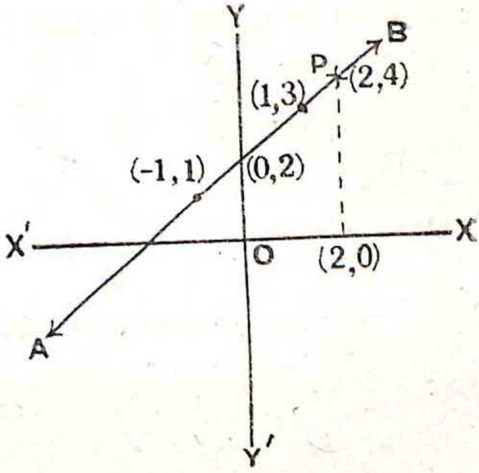


Fig. 2·32(a)

(*b*) In this case when $x=2$, the function has the value 0. So corresponding to $x=2$, we shall get the isolated point (2, 0) as a part of the graph. In this case also the point P (2, 4) of the straight line is not included in the graph and so the graph has a break corresponding to $x=2$. Hence the graph and so the function is discontinuous at $x=2$. The graph is shown in fig. 2·32 (*b*).

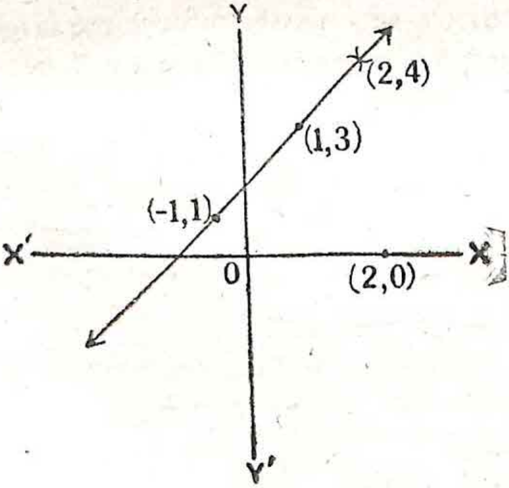


Fig. 2·32(b)

Ex. 23. Draw the graph of the function $| x |$. Also discuss the continuity of the function.

[*c.f.* H. S. 1978 ; H. S. 1981, Tripura, 1981 ; 1983]

Solution : Let $y = | x |$. The function $| x |$ can be defined as follows : $y = x$ when $x \geqslant 0$; $= -x$ when $x < 0$.

The equation $y = x$ $(x \geqslant 0)$ represents the straight line drawn through the origin which bisects the angle between the axes of co-ordinates in the first quadrant. As $x \geqslant 0$, so this straight line has no portion beyond the first quadrant. The graph of the equation $y = -x$ $(x < 0)$ is the straight line drawn in the second quadrant bisecting the angle between the co-ordinate axes. As $x < 0$, so this straight line does not pass through the origin but its distance from the origin is smaller than any positive number however small i.e., it practically passes through the origin. So the graph of $| x |$ has two branches ; though the second branch $(y = -x)$ does not pass through the origin, as the first branch passes through the origin, so the graph is a continuous graph and hence the function is continuous. The graph is drawn in fig. 2·33.

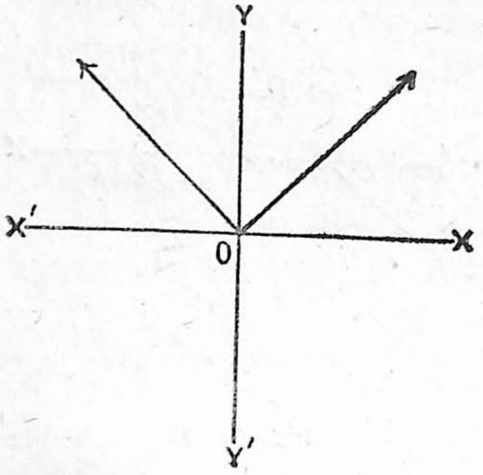


Fig 2·33

The graph could be drawn by plotting points corresponding to the following table with respect to a set of rectangular axes OX and OY.

when $x \geqslant 0$

x	0	1	2	···
y	0	1	2	···

when $x < 0$

x	-1	-2	$-·0001$	···
y	1	2	·0001	···

Note : The graph has no portion below the x-axis. So $| x |$ is positive for all values of x.

Ex. 24. Draw the graph of the function $\dfrac{| x |}{x}$ without using graph paper.

[Joint Entrance, 1983 ; *c.f.* H. S. 1984]

Let $y=\frac{|x|}{x}$. The function is undefined at the origin i.e., when $x=0$. So we prepare the following two tables for $x>0$ and $x<0$.

When $x>0$

x	1	2	·0001
$\lvert x\rvert$	1	2	·0001
$y=\frac{\lvert x\rvert}{x}$	1	1	1

When $x<0$

x	−1	−2	−3	−·0001	......
$\lvert x\rvert$	1	2	3	·0001	......
$y=\frac{\lvert x\rvert}{x}$	−1	−1	−1	−1	......

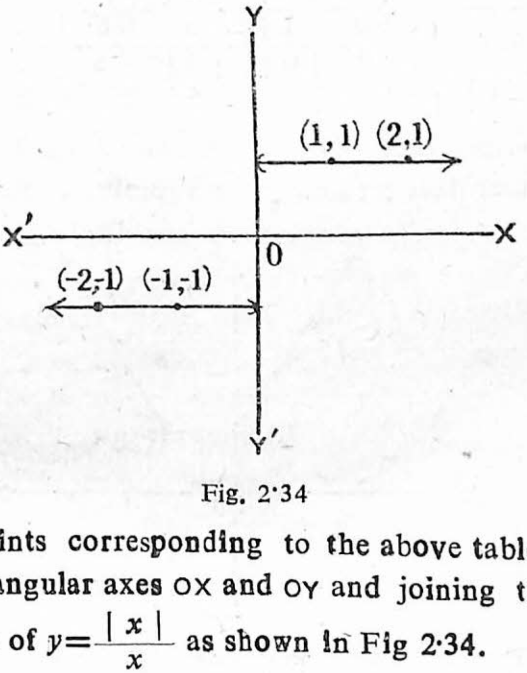


Fig. 2·34

Plotting points corresponding to the above tables with respect to a set of rectangular axes OX and OY and joining the points, we draw the graph of $y=\frac{|x|}{x}$ as shown in Fig 2·34.

The graph consists of two branches parallel to the x-axis. 1. A branch above the x-axis and on the right of the y-axis and starting from the y-axis without intersecting it. This branch extends upto infinity on the right of the y-axis. 2. A branch below the x-axis and on the left of the y-axis and starting from the y-axis without intersecting it. This branch also extends upto infinity. It is evident from the figure that the two branches are disconnected and are at a distance 2 from each other. So, the graph is discontinuous and the discontinuity is at the origin.

Note : As the distance between the two branches is 2 which is finite, so if we redefine the function as $y=\frac{|x|}{x}$ when $x \neq 0$ and $y=a$ when $x=0$, then also the function will remain discontinuous. Only the isolated point $(a, 0)$ will be included in the graph.

Ex. 25. Without using graph paper draw the graph of $y=|x|+|x-1|$ and from the graph determine the points of discontinuity.

[Joint Entrance 1980]

To draw the graph of $y=|x|+|x-1|$ we prepare the following table.

x	0	0·4	1	2	3	−0·6	−1	−2	−3	…
$\|x\|$	0	0·4	1	2	3	0·6	1	2	3	…
$\|x-1\|$	1	0·6	0	1	2	1·6	2	3	4	…
$y=\|x\|+\|x-1\|$	1	1	1	3	5	2·2	3	5	7	…

Plotting points according to the above table with respect to a set of rectangular axes OX and OY and joining those points we get

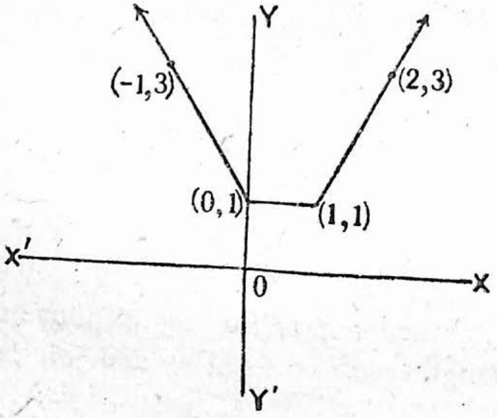


Fig. 2·35

the graph as shown in Fig. 2·35. The graph extends upwards upto infinity on both sides of the y-axis. The graph has no break and so is a continuous graph. Hence the function has no point of discontinuity.

Ex. 26. Draw the graph of the function $[x]$, where $[x]$ denotes the greatest integer less than or equal to x.

$[x]$ is the greatest interger le than or equalss to x.

So $[0{\cdot}01]=0$; $[0]=0$; $[1]=1$; $[1{\cdot}5]=1$; $[2]=2$; $[2{\cdot}3]=2$; $[-0{\cdot}1]=-1$; $[-1]=-1$; $[-1{\cdot}5\,]=-2$; $[-2]=-2$.

Let $y=[x]$.

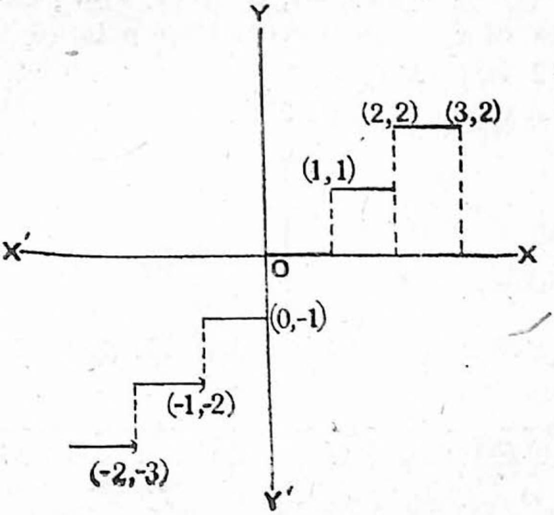


Fig. 2·36

So, when $0\leqslant x\leqslant 1$, $y=0$

$1\leqslant x<2$, $y=1$

$2\leqslant x<3$, $y=2$

$-1\leqslant x<0$, $y=-1$

$-2\leqslant x<0$, $y=-2$ etc.

The graph is drawn as shown in figure 2·36.

Note 1. : The graph consists of infinite number of branches each of which is a line segment of one unit lengh parallel to the x-axis. The distance between consecutive branches is 1 unit. From the graph it is seen that the graph and hence the function is discontinuous at every integral value if x.

Note 2. The graph resembles the steps of a staircase. So, the function has been named as a Step function.

Ex. 27. Draw the graph of the function defined as follows :

$f(x)=1$ when x is an integer

$=0$ when x is not an integer.

Let $y=f(x)$.

So, according to the definition of the function,

$$y=1 \text{ when } x=n\ [n=0, \pm 1, \pm 2, \cdots]$$
$$=0 \text{ when } x \neq n\ [n=0, \pm 1, \pm 2, \cdots]$$

So, the graph will be the x-axis other than the points (0, 0), (1, 0), (2, 0), (3, 0),…and (−1, 0), (−2, 0)… ; corresponding to integral values of x, we shall get isolated point (0, 1), (1, 1), (2, 2), …(−1, 1), (−2, 1), (−3, 1)…

The graph is shown in Fig 2·37.

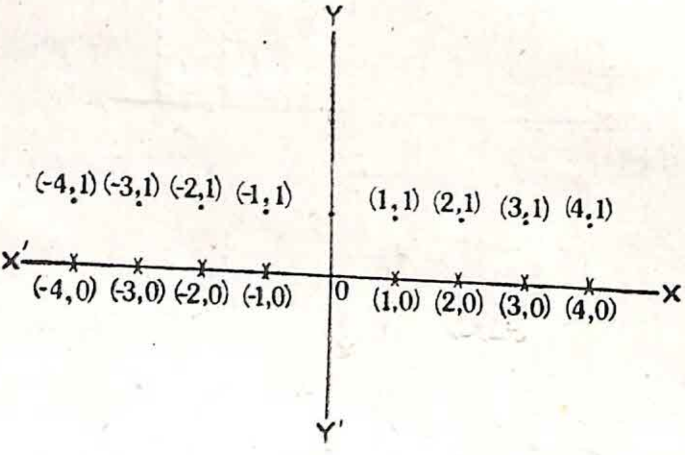


Fig 2·37

The cup and cap touching each other at each of the points $(n, 0)$ $[n=0, \pm 1, \pm 2, \cdots]$ show that the graph is discontinuous at each of these points.

Ex. 28. Without using graph paper draw the graph of the function.

$$y=f(x)=0 \text{ when } 0 \leqslant x \leqslant 1$$
$$=x-1 \text{ when } x \geqslant 1.$$

State whether the function is continuous at the point $x=1$

[Joint Entrance, 1987]

The function is defined for all values of x in $0 \leqslant x < \infty$. We prepare the following table.

x	0	·5	·6	1	2	3
y	0	0	0	0	1	2

Plotting the above points we find that the graph consists

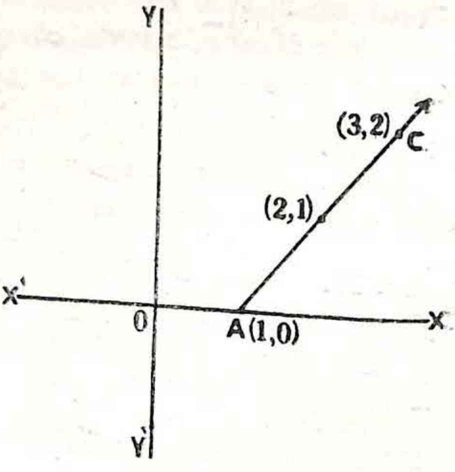


Fig 2·38

of two branches ; (i) the portion OA of the x-axis where O is the origin and A is the point (1, 0) and (ii) the ray $\overrightarrow{AB}$ drawn from A and extending upwards to ∞.

The graph is a continuous one and so the function is continuous at the origin.

Ex. 29. The definition of a function $f(x)$ is given below.

$f(x)=x$ when $x<1$...(i)

$=1+x$ when $x>1$...(ii)

$=\frac{3}{2}$ when $x=1$...(iii)

Draw the graph of $f(x)$ and examine whether $f(x)$ is continuous at $x=\frac{1}{2}$ and $x=1$. [H. S. 1980]

Solution : Alternative method (without plotting points)

Let $y=f(x)$.

From equation—(i) we get the straight line AO bisecting the $\angle$XOY and extending downwards upto infinity starting from the point A (1, 1). As $x<1$, in this case, so the point A (1, 1), is not included in the graph.

From equation (ii) we get a straight line through B parallel to AO starting from the point B and extending upwards upto infinity. (As gradients of the straight lines represented by equations (i) and (ii) are both 1, so the lines are parallel. As $x>1$, the point B is not included in the graph.

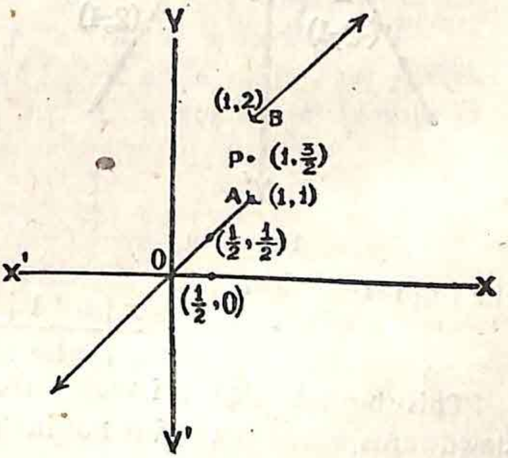


Fig 2·39

(iii) When $x=1$, $y=\frac{3}{2}$. So, the isolated point $(1, \frac{3}{2})$ is also included in the graph.

The graph has no break at $x=\frac{1}{2}$. So, the function $f(x)$ is continuous at $x=\frac{1}{2}$. The graph has break at the point $x=1$. So the function $f(x)$ is discontinuous at $x=1$.

Ex. 30. A function $f(x)$ is defined as follows.

$f(x)=3+2x$, when $x\leqslant 0$

$=3-2x$, when $x>0$.

Draw the graph of $f(x)$ and examine whether $f(x)$ is continuous at $x=0$. [H. S. 1982]

Solution : Let $y=f(x)$. As the function is defined by two different equations $y=3+2x$ when $x\leqslant 0$ and $y=3-2x$ when $x>0$, so the graph consists of two branches.

When $x\leqslant 0$: The graph is the straight line $\overrightarrow{AB}$ passing through the points (0, 3), (−1, 1), (−2 −1) as found from the following table.

x	0	−1	−2
y	3	1	−1

This branch originates from A (0, 3) and extends downwards upto infinity. The point A is included in the graph.

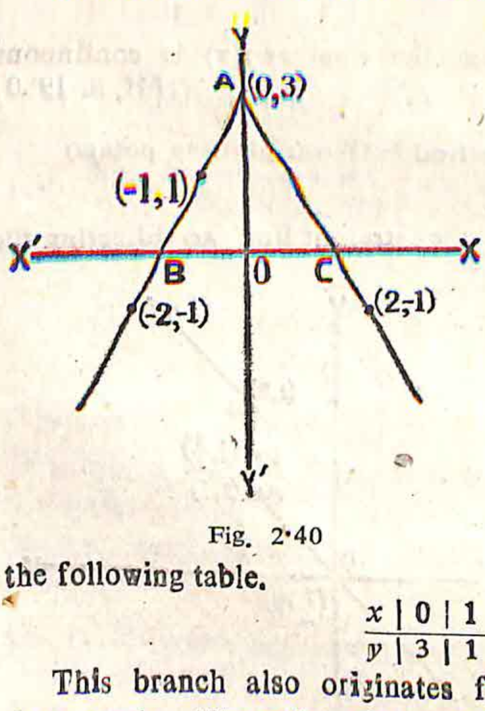


Fig. 2·40

When $x>0$: The graph is the straight line $\overrightarrow{AC}$ passing through the points (0, 3), (1, 1), (2, −1) as found from the following table.

x	0	1	2
y	3	1	−1

This branch also originates from the point A(3, 0) extending downwards. The point A is not included in this branch.

Though only the point A of the ray $\overrightarrow{AC}$ is not included in this branch, as the point A is included in the first branch, the continuity is maintained. We find the graph is a continuous one without any break. So, the function $f(x)$ is continuous every where and so at $x=0$.

Ex. 31. Let $\phi(x)=\frac{1}{2}-x,\ 0<x<\frac{1}{2}$

$=-\frac{1}{2}+x,\ \frac{1}{2}<x<1.$

and $\phi(0)=0,\ \phi(\frac{1}{2})=\frac{1}{2},\ \phi(1)=1.$

(i) Draw the graph of $y=\phi(x)$ without using graph paper.

From your graph verify whether the following statements are correct:

(ii) The function $\phi(x)$ assumes every value between 0 and 1 as x increases from 0 to 1.

(iii) $\phi(x)$ is continuous at $x=\frac{1}{2}$. [Joint Entrance 1982]

Solution: Let $y=\phi(x)$. For the interval $0<x<\frac{1}{2}$, we prepare the following table.

x	·1	·2	·3
y	·4	·3	·2

The points corresponding to the table are plotted and joining them we get the line segment AB joining the points A(0, ·5), B(·5, 0). But both these points are not included in this branch as $x=0$, and $x=\frac{1}{2}$ are not points of the interval.

For the interval $\frac{1}{2}<x<1$ we plot points corresponding to the following table.

x	·6	·7	·8
y	·1	·2	·3

Joining these points we get the line segment BC where C is the point (1, ·5). Both the points B and C are not included in this branch as $x=\frac{1}{2}$ and $x=1$ are not points of the interval $\frac{1}{2}<x<1$.

For the end points; when $x=0$; $y=0$ when $x=\frac{1}{2}$, $y=\frac{1}{2}$; when $x=1$, $y=1$. These are isolated points of the graph as none of them are included in the main two branches.

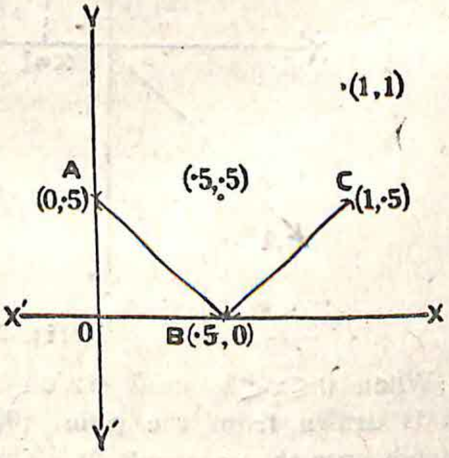


Fig. 2·41

From the branch AB of the graph it is seen that as x assumes values from 0 to $\frac{1}{2}$ (without assuming the values 0 and $\frac{1}{2}$), $\phi(x)$ assumes every value between $\frac{1}{2}$ and 0 (the values $\frac{1}{2}$ and 0 are not assumed in this case). Again from the branch BC we find that as x assumes every value in $\frac{1}{2}<x<0$, $y=\phi(x)$ assumes every value in $0<y<\frac{1}{2}$. It is also found that $\phi(x)$ assumes the values 0, $\frac{1}{2}$ and 1 at the isolated points (0, 0), ($\frac{1}{2}$, $\frac{1}{2}$) and (1, 1). Thus $\phi(x)$ does not assume every value between 0 and 1 and the statement (ii) is not true.

Again the graph is broken at $x=\frac{1}{2}$. Hence the graph is discontinuous at $x=\frac{1}{2}$. So, the function $\phi(x)$ is also discontinuous at $x=\frac{1}{2}$. Thus the statement (iii) also is not true.

Ex. 32. Draw the graph of the function defined below :

$$f(x)=x \text{ if } x<0$$
$$=2, \text{ if } 0\leqslant x<2$$
$$=4-x \text{ if } x\geqslant 2$$

From your graph examine if the function is continuous at $x=0$ and $x=2$.

Solution : Let $y=f(x)$.

So, when $x<0$, $y=x$ and the graph of $y=x$ in ($x<0$) is the st. line OA bisecting the angle X′OY′ (O is the origin). This branch emanates from O downwards. As $x<0$. So, O is not included in this branch of the graph.

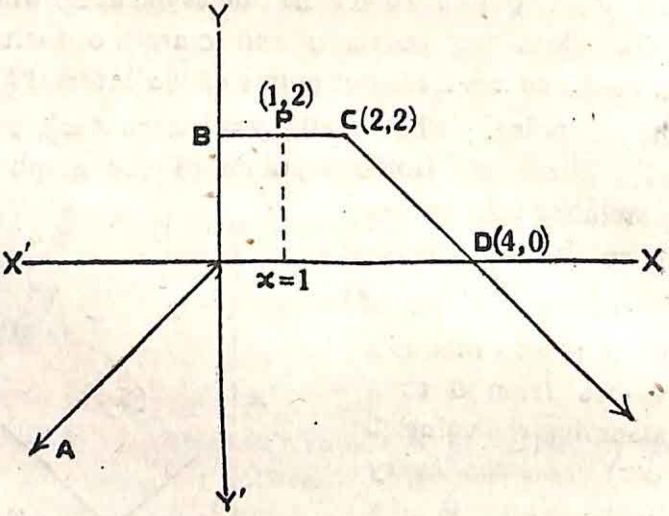


Fig. 2·42

When $0\leqslant x<2$, $y=2$ which is a straight line parallel to the x-axis drawn from the point (0, 2) on the y-axis. This branch extends upto the point C(2, 2). Though the point B(0, 2) is included in the graph, the point C(2, 2) is not included in this branch.

When $x\geqslant 2$, $y=4-x$. The branch corresponding to the above relation is the portion of the straight line $\overrightarrow{CD}$ joining the points C(2, 2) and the point D(4, 0). The point C(2, 2) is included in this branch. This branch starting from C extends downwards upto infinity.

The graph is drawn in fig. 2·42.

From the graph it is seen that the graph is broken at $x=0$ but not broken at $x=1$. [From one side of the point P(1, 2) we can go along the graph to the other side without lifting the pen or pencil].

Exercise 2

1. Find the domain of definition of the following functions.

(i) $f(x)=\frac{x^2}{x-1}$ (ii) $f(x)=\frac{x^2-9}{x-3}$ (iii) $f(x)=\sqrt{4-x^2}$

(iv) $f(x)=\sqrt{5-x}$ (v) $f(x)=\sqrt{\frac{x^2-x-2}{x-3}}$.

(vi) $f(x)=\frac{x}{x^2-3x+2}$ (vii) $f(x)=\frac{x^2-9}{x-3}$ when $x\neq 3$, $f(x)=0$ When $x=3$.

2. If $f(x)=4x^4-(a-1)x^3+ax^2-6x+1$ and $f(\frac{1}{2})=0$, show that $a=1$.

3. If $f(x)=x+\frac{1}{x}$, show that $f(\frac{1}{x})=f(x)$.

4. If $f(x)=\frac{x}{1+x}$, show that $f(\frac{a}{b})\div f(\frac{b}{a})=\frac{a}{b}$.

5. If $f(x)=\frac{3x+2}{5x-3}$, show that $f^2(x)=x$ [$f^2(x)=f\{f(x)\}$].

6. If $\phi(x)=\frac{x-1}{x+1}$, show that $\frac{\phi(x)-\phi(y)}{1+\phi(x)\,\phi(y)}=\frac{x-y}{1+xy}$.

7. (*a*) If $f(x)=2^x$, show that $f(x+1)=2f(x)$

(*b*) If $f(x)=a^x$, show that $f(x-1)=\frac{f(x)}{a}$.

8. If $f(x)=e^{ax+b}$, show that $f(r).f(s)=f(r+s)e^b$.

9. If $f(x)=\frac{(x-q)(x-r)}{(p-q)(p-r)}+\frac{(x-r)(x-p)}{(q-r)(q-p)}+\frac{(x-p)(x-q)}{(r-p)(r-q)}$, show that $f(0)=1$ and $f(p)=f(q)=f(r)=f(0)$.

10. If $f(x+1)=2x^2-11x+3$ then find $f(x+3)$.

11. If $f(x)=\frac{x-a}{x}+\frac{x-b}{x}$, find the value of $f(a+b)$.

12. Determine the inverse function of the function $f(x)=\frac{x+1}{2x-1}$.

13. Show that the function $f(x)=\dfrac{x^2-5x+6}{x^2-8x+12}$ is undefined at $x=2$. Find the values of $f(-5)$ and $f(6)$.

14. If $f(x)=\log\dfrac{1+x}{1-x}$, show that $f\left(\dfrac{2x}{1+x^2}\right)=2f(x)$.

15. If $f(x)=\dfrac{e^x+1}{e^x-1}$, show that $f(-x)=-f(x)$.

16. Are the two functions $f(x)=\dfrac{x^2-a^2}{x+a}$ and $\phi(x)=x-a$ identical ? Support your answer with reasons.

17. If $f(x)=\dfrac{1}{1-x}$ show that $f(1-x)=\dfrac{1}{x}$ and $f\left(\dfrac{1}{1+x}\right)=\dfrac{1+x}{x}$.

18. If $f(x)=\cos(\log x)$, find the value of
$f(x)\,f(y)-\frac{1}{2}[\,f(\frac{x}{y})+f(xy)\,]$ [c.f. I. I. T. 1983]

19. If $f(x)=x^2$, what is the value of $f(2{\cdot}01)$?
Show that $\dfrac{f(2{\cdot}01)-f(2)}{{\cdot}01}=4{\cdot}01$

20. Show that $\dfrac{f(x+h)-f(x)}{h}$ is equal to

(i) $e^x.\dfrac{e^h-1}{h}$ when $f(x)=e^x$.

(ii) $-2\dfrac{\sin(2x+h)\sin h}{h}$ when $f(x)=\cos 2x$.

(iii) $\dfrac{1}{\cos(x+h)\cos x}\cdot\dfrac{\sin h}{h}$ when $f(x)=\tan x$.

(iv) $-\dfrac{1}{x(x+h)}$ when $f(x)=\dfrac{1}{x}$.

(v) $\dfrac{\log(1+\frac{h}{x})}{h}$ when $f(x)=\log x$.

21. $f(x)=\dfrac{\sin x}{x}$ and $g(x)=\dfrac{\cos x}{x}$. Show that $f(x)$ is an even function and $g(x)$ is an odd function.

22. Show that $f(x)=x^2+\cos x+4$ is an even function.

23. If $f(x)=\tan x$, show that $f\left(\dfrac{\pi}{2}-x\right)=\dfrac{1}{f(x)}$.

24. For any function $f(x)$ prove that
(i) $f(x)+f(-x)$ is an even function and
(ii) $f(x)-f(-x)$ is an odd function.

25. Which of the following statements are true and which are false ?

(i) The quotient of an even function and an odd function is an odd function.

(ii) Both the sum and product of two odd functions are two even functions.

(iii) Every function is either even or odd.

(iv) If $f(x)$ and $g(x)$ be two even functions then $f(x)-g(x)$ is an even function.

(v) $\sin x-\cos x$ is an even function.

26. (*a*) Show that the function $x-[x]$, (where $[x]$ denotes the greatest integer less than or equal to x) is periodic. [*c.f.* I.I.T. 1983]

(*b*) Are the functions $f(x)=\sin\frac{1}{x}$, $x\neq 0$, $f(0)=0$ and $g(x)=x\cos x$ periodic ? [*c.f.* I.I.T. 1983]

27. Show that $\dfrac{e^x+1}{e^x-1}$ is an odd function.

28. Show that $\dfrac{x^2}{1+x^2}$ is an even function and.

$\dfrac{x}{1+x^2}$ is an odd function.

29. If $f(x)=\sin x$, $g(x)=\cos x$ and $h(x)=\tan x$, show that

(i) $\{f(x)\}^2+\{g(x)\}^2=1$ (ii) $\dfrac{1}{\{g(x)\}^2}=1+\{h(x)\}^2$

(iii) $f(2x)=\dfrac{2h(x)}{1+\{h(x)\}^2}$ (iv) $g(3x)=4\{g(x)\}^3-3g\{(x)\}$

(v) $h(x+y)=\dfrac{h(x)+h(y)}{1-h(x)h(y)}$ (vi) $g(2x)=\{g(x)\}^2-\{f(x)\}^2$

30. (i) Show that the function $\dfrac{x-1}{2x+3}$ is monotonic increasing in the interval $0\leqslant x<\infty$.

(ii) Show that the function cosec x is monotonic decreasing in the interval $0<x\leqslant\frac{\pi}{2}$.

31. Can the constant 3 be called a function of x ? Give reasons in support of your answer. If 3 be a function of x, then

(i) The function is even, odd or none of these. which is the correct answer ?

(ii) Is the function monotonic.

32. Express y as an explicit function of x.
(i) $x^2y^2=5$ (ii) $xy+x+y+1=0$ (iii) $x^2+y^2=a^2$.

33. Make the following functions free from fractions and radicals and then express y as an implicit function of x.

(i) $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$. (ii) $y=\frac{x-5}{\sqrt{x}}$ (iii) $y=\frac{3x+1}{2x-7}$.

34. (*a*) Determine the inverse function of the function $y=x^2+3$. Is the function single valued ? If not, what restriction should be imposed on this inverse function so that, it will be single valued ? In this case determine the range of the inverse function.

35. If $y=\frac{1-\cos x}{1+\cos x}$, express x as a function $f(y)$ of y.

36. If $f(x)=\frac{x^2-2x+4}{x^2+2x+4}$, show that $f(x)$ is a bounded function.

37. Draw the graphs of the following functions. Comment from the graphs on the continuity of the functions. In case of discontinuity indicate the points of discontinuity.

(i) $f(x)=\frac{5x}{x}$ (ii) $f(x)=\frac{x^2-1}{x-1}$ (iii) $f(x)=-x$.

38. Draw the graph of the function $f(x)=|x|+1$ and show that the function is continuous every where.

39. Draw the graph of the function $|x-1|+|x+1|$.

40. Draw the graph of the function $\sqrt{4-x^2}$.

41. (*a*) Draw the graph of the function $f(x)=\frac{|x|}{x}$. From the graph show that the function is discontinuous at $x=0$.

[Tripura 1978]

(*b*) The definition of a function $f(x)$ is given below. $f(x)=2-\frac{|x|}{x}$, $-2\leqslant x\leqslant 2$.

Draw the graph of the function $f(x)$. Is the function continuous at $x=0$?

[Tripura 1980]

42. Draw the graph of the function $f(x)=x^2+1$ and from the graph show that $f(x)$ is continuous at $x=0$

[Tripura 1979]

43. Draw the graph of the function $f(x)$:—

$f(x)=x$ when $0\leqslant x<1$

$=2$ when $x=1$

$=2-x$ when $1\leqslant x<2$.

Examine whether the function is continuous at $x=1$.

[Tripura 1985]

44. Can the following conditions imposed on the variable quantity y, define y as a function of x.

$y=1$ when x is rational.

$y=3$ when x is irrational.

Draw the graph of the above relation between x and y.

45. If A and x denote the area and length of a side of an equilateral triangle, express A as a function of x.

46. Postal rates for parcels are as follows :

Rs. 5 for the first 500 gms or less, and for every additional 200 gms or part thereof additional Rs. 3. If x and y denote weight of parcel and postal charges respectively, then express y as a function of x upto 5 kg.

47. A person prepared a box open at the top by cutting off from the four corners of a square board each of whose side is of length a. If x and y denote the length of a side of each square cut off from the board and the volume of the box, express y as a function of x.

48. The distance between two places A and B is 520 kilometers. A man drived to B at the uniform speed of 75 km/hr, waited at B for 5 hours and then returned to A at the uniform rate of 70 km/hour. If x denotes distance from A in km. and t denotes time in hours measured from the instant of start from A, then express x as a function of t.

49. Let $f(x)=|x-1|$. Then

(A) $f(x^2)=\{f(x)\}^2$ (B) $f(x+y)=f(x)+f(y)$

(C) $f(|x|)=|f(x)|$ (D) None of these.

Which is correct ?

[I. I. T. 1983]

50. If $f(x)=\frac{1}{x^2}$, show that $f(x)-f(x+1)=\frac{2x+1}{x^2(x+1)^2}$.

Hence find the sum of the first n terms of the series

$$\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\cdots.$$

CHAPTER THREE

LIMIT

§ 3·1. Determination of the limit of a function i. e., the limit operation occupies the central place in calculus. Differentiation and integration, the two main operations of calculus are nothing but special types of limit operation. When a variable, in course of changing its value gradually assumes a value very close to (as close as one likes) a given constant, then we say that the variable approaches the constant. When a variable approaches a constant, then the values of any variable depending on the first variable, also changes. The object of the limit operation is the determination of the value of the dependent variable (The second variable) as the first variable approaches the constant. So, we must have a mathematical discussion on the behaviour of a variable x, when it approaches a constant 'a'.

§ 3·2. Meaning of a variable approaching a constant

A variable x approaches a constant a means that x assumes values so close to a (but not a) so that the difference of x and a may be smaller than any positive number however small. Here the value of x may be less than or greater than (but not equal to) a. So, x approaches a means that $x-a$ or $a-x$ must be less than any positive number, small at pleasure. For emphasis, we repeat once again that x approaches a means x does **not** assume the value a. As the value of x may be less than or greater than a, so the difference of the values of x and a is $|x-a|$. As x does not assume the value a, so $0<|x-a|$. Again as x is smallar than any positive number, small at pleasure, so $0<|x-a|<$any positive number, small at pleasure. When x approaches the value a from values less than a, we say that x approaches a from the left. Again, when x approaches the value a from values greater than a, then it is said that x approaches a from the right.

Symbol : That 'x approaches a' is expressed by the symbol '$x \to a$'. When x approaches a from the left or from the right, then we use the symbols "$x \to a-$" and "$x \to a+$" respectively. Let us now discuss with a particular value of a, say 5. $x \to 5$ or x approaches the value 5 means x assumes values very close to 5 but does not attain the value 5. $x \to 5$ means both $x \to 5-$ and $x \to 5+$. When $x \to 5-$, the values of x may be 4·9, 4·99, 4·998, 4·999, 4·9999,...or even more close to 5; but x in this case will never be equal to 5 or greater than 5. Again when $x \to 5+$, x will assume values close to but greater than 5 such as, 5·5, 5·1, 5·01, 5·001, 5·0001 etc. or even values more close to 5. But in this case x does not assume the value 5 or values less than 5.

We shall now endeavour to make the students understand the above discussion with the help of the number line. Let the straight line in fig. 3·1 be the number line so that its different points represent the different values of x. The origin O represents the number '0' and points on the right or left of O represent respectively positive and negative real numbers. Let the point P represent the number 5 and the points A and B represent the numbers 4 and 6 respectively. When $x \to 5$, x approaches the value 5 from values less than or greater than 5. When $x \to 5-$, x approaches the value 5, from the left i.e., through values less than 5. Here as $4 < 5$, let us assume that x initially takes the value 4 and then gradually changing its values, approaches the value 5. As a result x approaches the point P (representing the value 5) through the points A_1, A_2, A_3, A_4, A_5,......representing the numbers 4·5, 4·9, 4·99, 4·999, 4·99999 etc. but never reaches the point P or cross it. Again when $x \to 5+$, let us assume that x changes its values from the initial value 5 at B and gradually crosses the points, B_1, B_2, B_3, B_4, B_5 etc. attaining successively the values 5·5, 5·1, 5·01, ·5001, 5·0001 etc. or even values more close to 5. But in this case x does not attain the value 5 or values less than 5 i.e. does not reach the point P or does not cross P.

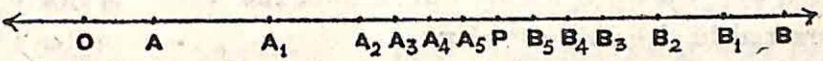


Fig. 3·1

Let us understand the above discussion more clearly. We have assumed above, that when the variable x approaches the value 5, from the left or right, then the initial values of x are 4 and 6 respectively. But $5-4=6-5=1$ is a definite quantity and cannot be taken as a small quantity (in calculus we use the term

Definitions.

1. $\underset{x \to a-}{\mathrm{Lt}} f(x) = l_1$.

If $f(x)$ be a function of the variable x and l_1 and a be two constants such that by taking values of x sufficiently close to a (always remaining less than a), the difference $|f(x) - l_1|$ of $f(x)$ and l_1 can be made less than any positive number, however small then we say, $\underset{x \to a-}{\mathrm{Lt}} f(x) = l_1$ and this is read as, 'when x approaches a from the left then the limit or limiting value of $f(x)$ is l_1.

2. $\underset{x \to a+}{\mathrm{Lt}} f(x) = l_2$.

If $f(x)$ be a function of the variable x and l_2 and a be two constants such that by taking values of x sufficiently close to a (but greater than a) one can make the difference $|f(x) - l_2|$ of $f(x)$ and l_2 smaller than any positive number, however small, then we say that $\underset{x \to a+}{\mathrm{Lt}} f(x) = l_2$ and this is read as 'when x approaches a from the right, then the limit or limiting value of $f(x)$ is l_2.

3. $\underset{x \to a}{\mathrm{Lt}} f(x) = l$.

Let $f(x)$ be a function of the variable x and l and a be two constants. If one can find an interval (as small as it may be) with centre, the point a, so that for every value of x ($x \neq a$) in the interval, $|f(x) - l|$ is less than any preassigned small positive number, small at pleasure, then we say, as x approaches a, the limit or limiting value of $f(x)$ is l and write $\underset{x \to a}{\mathrm{Lt}} f(x) = l$.

Note 1. The length of the small interval, referred to in the definition of $\underset{x \to a}{\mathrm{Lt}} f(x) = l$, depends on the preassigned small positive number.

In Example 1, we have seen that $\underset{x \to 3}{\mathrm{Lt}} (4x+1) = 13$. Here let the preassigned small positive number be $0{\cdot}001$. So, $|4x+1-13| < 0{\cdot}001$ or $|4x-12| < 0{\cdot}001$.

or, $|x-3| < \dfrac{0{\cdot}001}{4} = 0{\cdot}00025$. So, when the preassigned positive number is $0{\cdot}001$, then for values of x lying in the interval,

$3-\cdot 00025<x<3+\cdot 00025$ or, $2\cdot 99975<x<3\cdot 00025$, $|\,4x+1-13\,|<\cdot 001$.

2. In order that $\underset{x\to a+}{\text{Lt}}\ f(x)$ exist, each of

$\underset{x\to a-}{\text{Lt}}\ f(x)$ and $\underset{x\to a+}{\text{Lt}}\ f(x)$ must exist and they must be equal in value. This equal value is the value of $\underset{x\to a}{\text{Lt}}\ f(x)$.

So, $\underset{x\to a-}{\text{Lt}}\ f(x)=\underset{x\to a+}{\text{Lt}}\ f(x)=\underset{x\to a}{\text{Lt}}\ f(x)=l.$

3. For the existence of $\underset{x\to a}{\text{Lt}}\ f(x)$, it is not necessary, that $f(a)$ must have a finite value. We once again repeat that as x approachea a, x does not attain the value a; so the value of, $f(a)$ does not come under consideration. But this does not mean that x cannot attain the value a, in this case we are not interested in the value $f(a)$.

4. For $\underset{x\to a}{\text{Lt}}\ f(x)$, different symbols like $\underset{x\to a}{\text{Lim}}\ f(x)$ or $\underset{x\to a}{\text{L}}\ f(x)$ etc. are also used.

Let us now consider a few more examples.

Ex. 2. $\underset{x\to 2}{\text{Lt}}\ (x^2+1)$

Here $f(x)=x^2+1$ and $a=2$. First we tabulate values of $f(x)$ for values of x very close to 2 but less than 2.

x	1	1·5	1·9	1·99	1·999	1·999999
$y=x^2+1$	2	3·25	4·61	4·9601	4·996001	4·999996000001

From the above table, we find that as x assumes values very close to 2 (but less then 2), the value of $f(x)=x^2+1$ becomes closer and closer to 5. In this way taking values of x sufficiently close to 2 but less than 2, one can make $|\,5-(x^2+1)\,|$ less than any positive number, small at pleasure.

Hence $\underset{x\to 2-}{\text{Lt}}\ (x^2+1)=5$

Now let us prepare the table as x approaches the value 2 from values greater than 2 (without attaining the value 2)

x	3	2·5	2·2	2·1	2·01	2·001	2·00001
$y=x^2+1$	10	7·25	5·84	5·41	5·0401	5·004001	5·0000400001

In this case also we find from the above table that as x becomes closer and closer to 2, its value becomes closer and closer to 5. In this way making x sufficiently close to 2 we can make the difference $|x^2+1-5|$ smaller than any preassigned positive number however small. So $\underset{x\to 2+}{\text{Lt}}(x^2+1)=5$

$\therefore \quad \underset{x\to 2-}{\text{Lt}}(x^2+1)=\underset{x\to 2+}{\text{Lt}}(x^2+1)$ both exist.

and $\underset{x\to 2-}{\text{Lt}}(x^2+1)=\underset{x\to 2+}{\text{Lt}}(x^2+1)=5. \quad \therefore \underset{x\to 2}{\text{Lt}} f(x)=5.$

Note. In this case if $\cdot 00001$ be the preassigned small positive nubmer, then let us determine the interval with 2 at centre, so that for every x in the interval $|f(x)-5|<\cdot 00001$ or, $|x^2+1-5|<\cdot 00001$

or, $|x^2-4|<\cdot 00001$ or, $|x+2|\ |x-2|<\cdot 00001$

Here $x\to 2$, so x is very close to 2. So $|x+2|>2$.

$\therefore \quad 2|x-2|<|x+2|\ |x-2|<\cdot 00001$

$\therefore \quad |x-2|<\cdot 000005$ and this is the required interval.

Ex. 3. $\underset{x\to 1}{\text{Lt}}\dfrac{x^2-1}{x-1}$.

In this case $f(x)=\dfrac{x^2-1}{x-1}$ and as $x\to 1$, so x does not assume the value 1. So in our discussion $x\neq 1$ or, $x-1\neq 0$ and so $\dfrac{x^2-1}{x-1}$ will have finite values.

We prepare the following two tables, showing values of $f(x)$ corresponding to values of x very close to 1. The first table corresponds to values of $x<1$ and the other corresponds to values of x greater than 1.

Table 1

x	·9	·99	·999	·999992	·9999930029
$f(x)=\dfrac{x^2-1}{x-1}$	1·9	1·99	1·999	1·999992	1·9999930029

Table 2

x	1·5	1·1	1·01	1·003	1·00001	1·0000001
$f(x)=\dfrac{x^2-1}{x-1}$	2·5	2·1	2·01	2·003	2·00001	2·0000001

From the table 1, we find that as x approaches the value 1 from values less than 1 i.e., approaches the value 1 from the left, then $f(x)=\frac{x^2-1}{x-1}$ assumes values closer and closer to 2 and taking x sufficiently close to 1 (but less than 1) one can make the difference $|f(x)-2|$ less than any positive number however small. So $\underset{x\to 1-}{\text{Lt}} f(x)=2$.

Again from the table 2, we find that as x approaches the value 1 from values greater than 1 i.e., approaches 1 from the right, then $f(x)=\frac{x^2-1}{x-1}$ assumes values closer and closer to 2 and taking x sufficiently close to 1 (but always remaing greater than 1), we can make the difference $|f(x)-1|$ less than any positive number small at pleasure. So $\underset{x\to 1+}{\text{Lt}} f(x)=2$.

$$\therefore \underset{x\to 1-}{\text{Lt}} f(x)=\underset{x\to 1+}{\text{Lt}} f(x)=2.$$

Hence $\underset{x\to 1}{\text{Lt}} f(x)=2$ i.e., $\underset{x\to 1}{\text{Lt}} \frac{x^2-1}{x-1}=2$.

Note 1. Though we are not interested in the value 1 of x as $x\to 1$, yet note that here $f(1)$ does not exist but $\underset{x\to 1}{\text{Lt}} \frac{x^2-1}{x-1}$ exists This will make our future discussions easier.

Note 2. $x\to a$ means, x assumes values greater than or less than a but very close to a. In example 1, as values close to 3, we have taken the values 2·9, 2·99, 2·999999 etc. or the values 3·1, 3·01, 3·001 etc. Again in example 2 we have taken as values of x close to 2, the values 2·1, 2·01, 2·001,…or 1·9, 1·99, 1·999 etc. But in every case the values may be taken other wise ; to illustrate this in example 3 we have considered, the values ·999992, ·999993029 in tabulating the values of $\frac{x^2-1}{x-1}$.

Ex. 4. The variable y is a function of the variable x defined as follows :

$$\begin{aligned} y&=x^2 \text{ when } x<-1\\ &=x^2+2 \text{ when } -1\leqslant x<0\\ &=x+1 \text{ when } x\geqslant 0. \end{aligned}$$

Find $\underset{x\to 0}{\text{Lt}}\ y$.

Let us first determine the left hand limit i.e., $\underset{x\to 0-}{\text{Lt}}\ y$. As $x\to 0-$, we must consider values of x close to 0. Here when $x<0$, $f(x)$ is given in two forms viz., one when $x<-1$ and the other when $-1\leqslant x<0$. Evidently the values of $x<-1$ cannot be said to be so close to 0 as the values of x in $-1\leqslant x<0$.

As we are considering values of x very close to 0 (as $x\to 0-$), so consideration of the values in $-1\leqslant x<0$ will be sufficient.

So, when $x\to 0-$, $y=x^2+1$.

For determination of $\underset{x\to 0-}{\text{Lt}}\ f(x)=\underset{x\to 0-}{\text{Lt}}\ (x^2+2)$ we prepare, as in the previous examples, the following table.

x	-1	$-\cdot 9$	$-\cdot 1$	$-\cdot 01$	$-\cdot 001$	$-\cdot 000000001$
$y=x^2+2$	3	2·81	2·01	2·0001	2·000001	2·000000000000000001

It easily follows from the above table (as in the previous examples) that $\underset{x\to 0-}{\text{Lt}}\ y=2$.

Let us consider $\underset{x\to 0+}{\text{Lt}}\ y$. Here as $y\to 0+$ i.e., $y\geqslant 0$, so we take $y=x+1$ and prepare the following table with values of x very close to 0 but greater than 0.

x	0 1	0·01	0·0001	0·000000001	0·00000000001
$y=x+1$	1·1	1·01	1·0001	1·000000001	1·00000000001

In this case we find that as x assumes values closer and closer to 0, y assumes values closer and closer to 1 and as in the previous examples we can say $\underset{x\to 0+}{\text{Lt}}\ y=1$.

So, in this case $\underset{x\to 0-}{\text{Lt}}\ y\neq\underset{x\to 0+}{\text{Lt}}\ y$.

Hence $\underset{x\to 0}{\text{Lt}}\ y$ does not exist.

Note 1. Here $\underset{x\to 0-}{\text{Lt}}\ y$ and $\underset{x\to 0+}{\text{Lt}}\ y$ both exist but they are not equal, so $\underset{x\to 0}{\text{Lt}}\ y$ does not exist.

Ex. 5. A function $f(x)$ is defined as follows :

$f(x)=x$ when $x>90°$

$=\tan x$ when $x<90°$

Determine $\underset{x\to 90}{\text{Lt}}\ f(x)$. [Here the unit of x is in degrees but we omit the unit.]

We first consider the values of $f(x)$, when x is less than 90 (in degrees) but very close to 90 and prepare the following table.

x	89°	89°10′	89°40′	89°50
$f(x)=\tan x$	57·290	68·75	171·89	343·077

From the above table we find that as x assumes values 89°, 89°10′, 89°40′, 89°50′, the values of $\tan x$ changes too fast. It has been seen in Higher Trigonometry that when $89°50'<x<90°$, the changes in the values of x are so fast, that the corresponding values of $\tan x$ cannot be tabulated but the values are always positive. By taking values of x sufficiently close to 90 (but less than 90), we can make the value of $\tan x$ greater than any positive number large at pleasure. In other words we say $\underset{x\to 90-}{\text{Lt}}\ \tan x$ is infinity, i.e. $\underset{x\to 90-}{\text{Lt}}\ f(x)$ does not possess any finite value. i.e., $\underset{x\to 90-}{\text{Lt}}\ f(x)$ does not exist. Again, existence of $\underset{x\to 90-}{\text{Lt}}\ f(x)$ is a necessary condition for the existence of $\underset{x\to 90}{\text{Lt}}\ f(x)$. So, $\underset{x\to 90}{\text{Lt}}\ f(x)$ does not exist.

Note 1. In trigonometry, generally measures of angles are expressed in radians. Here we have used degrees as the students are used to trigonometric tables prepared in degrees. Moreover the values of a variable are taken without units. Here we have kept the units for convenience.

§ 3·4. Graphical discussion on limits.

Ex. 1. Draw the graph of the function $f(x)=3x$ and evaluate $\underset{x\to 2}{\text{Lt}}\ f(x)=3x$

Let $y=f(x)$ $\therefore$ $y=3x$.

Fig 3·2 is the graph of $y=3x$ or, the function $f(x)$ according to a preassigned scale. The graph is a straight line and point A on the x-axis is the point (2, 0). The ordinate AP drawn at A intersects the graph at the point P. Evidently length AP$=3\times2=6$ units and the co-ordinates of the point P are $(2, 6)$. The points B_1, B_2, B_3……on the x-axis are very close to the point A on its left and the points C_1, C_2, C_3……on the x-axis are very clos to A on its right. The ordinates drawn at these points intersect the graph at the points Q_1, Q_2, Q_3……R_1, R_2, R_3,……respectively. As $x\to2-$ *i.e.*, x approaches the point A from the left through the points B_1, B_2, B_3,……, y gradually approaches the ordinate AP through B_1Q_1, B_2Q_2, B_3Q_3,…*i.e.*, gradually y approaches the value 6. So, $\underset{x\to2}{\text{Lt}}\ f(x)=6$. Again as $x\to2+$ *i.e.*, x approaches the point A from the right through the points C_1, C_2, C_3,……, y gradually approaches the ordinate AP through C_1R_1, C_2R_2, C_3R_3,……, *i.e.* y gradually approaches the value 6. So $\underset{x\to2+}{\text{Lt}}\ f(x)=6$.

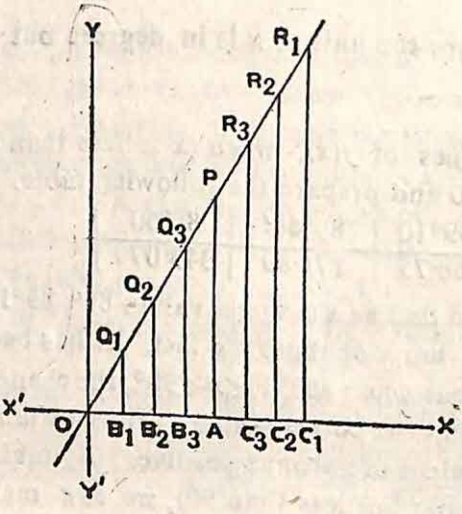


Fig. 3·2

$$\therefore \underset{x\to2-}{\text{Lt}}\ f(x)=\underset{x\to2+}{\text{Lt}}\ f(x)=6 \quad \text{or,} \quad \underset{x\to2}{\text{Lt}}\ f(x)=6$$

Ex. 2. $\underset{x\to1}{\text{Lt}}\ \dfrac{x^2-1}{x-1}$.

Let $y=\dfrac{x^2-1}{x-1}$. when $x=1$, then y is undefined ; for all other values of x and so for values of x very close to 1 (as close as it may be) y is defined.

So, $y=x+1$ when $x\neq1$
is undefined when $x=1$

Fig 3·3 is the graph of $y=\dfrac{x^2-1}{x-1}$.

The graph is a straight line having a break only at the point P $(1, 2)$. Let the point A on the x-axis be the point $(1, 0)$. As the graph has a break at the point P (1, 2), so the ordinate drawn at A does not intersect the graph. Evidently lenght of AP is 2 units. The points $B_1, B_2, B_3, \ldots\ldots, C_1, C_2, C_3, \ldots\ldots$ are points on the x-axis very close to A on the left and right of A respectively. The ordinates

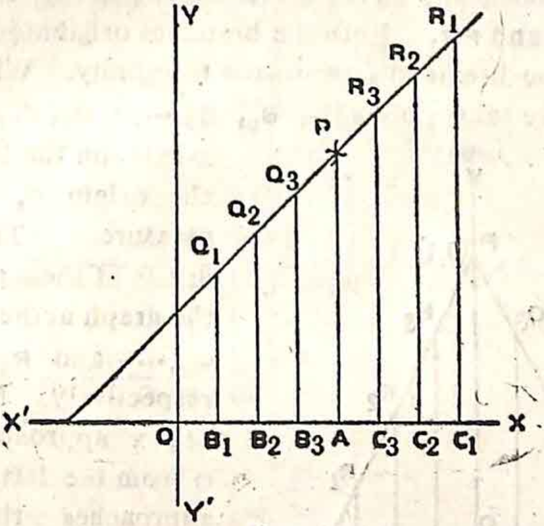


Fig. 3·3

$B_1Q_1, B_2Q_2, B_3Q_3, \ldots\ldots$ and C_1R_1, C_2R_2, C_3R_3, drawn at these points intersect the graph at the points $Q_1, Q_2, Q_3, \ldots\ldots$ and $R_1, R_2, R_3, \ldots\ldots$ respectively. As x approaches the value 1 from the left, it approaches the point A through the points $B_1, B_2, B_3, \ldots\ldots$ So, y approaches the ordinate AP, through $B_1Q_1, B_2Q_2, B_3Q_3, \ldots\ldots$ from the left of AP, *i.e.*, y approaches the value 2 from left.

$$\therefore \underset{x\to 1-}{\text{Lt}}\ y = \underset{x\to 1-}{\text{Lt}}\ \frac{x^2-1}{x-1} = 2$$

As x approaches the value 1 from the right, it approaches the point A through the points $C_1, C_2, C_3, \ldots\ldots$. So, y approaches the ordinate AP through $C_1R_1, C_2R_2, C_3R_3, \ldots\ldots$ from the right of AP; *i.e.*, y approaches the value 2 from the right.

$$\text{So, } \underset{x\to 1+}{\text{Lt}}\ y = 2 \quad \text{or} \quad \underset{x\to 1+}{\text{Lt}}\ \frac{x^2-1}{x-1} = 2.$$

$$\therefore \underset{x\to 1-}{\text{Lt}}\ \frac{x^2-1}{x-1} = \underset{x\to 1+}{\text{Lt}}\ \frac{x^2-1}{x-1} = 2. \quad \therefore \underset{x\to 1}{\text{Lt}}\ \frac{x^2-1}{x-1} = 2,$$

Ex. 3. $y=2x+1$ when $x\leqslant 0$.

$\qquad =1-2x$ when $x\geqslant 0$.

Determine $\underset{x\to 0}{\mathrm{Lt}}\ (2x+1)$.

Fig 3·4 is the graph of the given function. It consists of two branches PQ_1 and PR_1. Both the branches originate from the point $P\ (0, 1)$ and both extend downwards to infinity. When $x=0$, then $y=1$. Now we take points $B_1, B_2, B_3, \ldots\ldots, C_1, C_2, C_3, \ldots\ldots$ on the x-axis on the left and right of the origin O, close to O at pleasure. The ordinates drawn at these points intersect the graph at the points $Q_1, Q_2, Q_3, \ldots\ldots$ and $R_1, R_2, R_3, \ldots\ldots$ respectively. Now as $x\to 0-$ i.e., x approaches the value O from the left, it gradually approaches the point O, through the points $B_1, B_2, B_3, \ldots\ldots$ and comes as close as O, as one likes without being coincident with O. With x approaching O i.e., the value 0, y approaches the ordinate OP through the ordinates $B_1Q_1, B_2Q_2, B_3Q_3, \ldots\ldots$ and comes very close to OP without being coincident with OP i.e. y approaches the value 1 from the left.

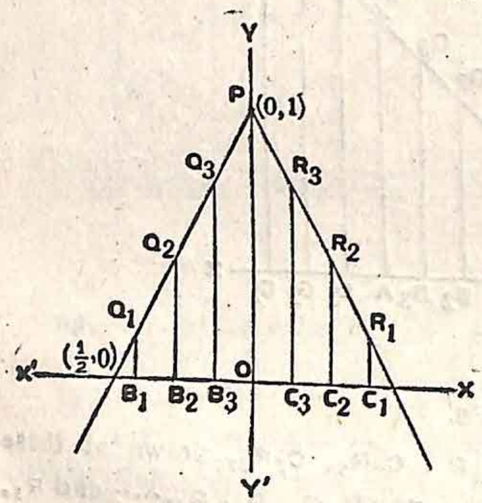


Fig. 3·4

$$\therefore \underset{x\to 0-}{\mathrm{Lt}}\ y=1.$$

Again, as $x\to 0+$ i.e., x approaches the value 0 from the right, it approaches the point O i.e., the value 0 through the points, $C_1, C_2, C_3\ldots\ldots$. So, correspondingly, y approaches the ordinate OP i.e., the value 1 through the ordinates $C_1R_1, C_2R_2, C_3R_3, \ldots\ldots$. Hence as $x\to 0+$, y approaches the value 1 and $\underset{x\to 0+}{\mathrm{Lt}}\ y=1$.

$$\therefore \underset{x\to 0-}{\mathrm{Lt}}\ y=\underset{x\to 0+}{\mathrm{Lt}}\ y=1. \quad \therefore \underset{x\to 0}{\mathrm{Lt}}\ y=1.$$

Ex. 4. Evaluate : $\underset{x \to 0}{\text{Lt}} \frac{|x|}{x}$

Let $y = \frac{|x|}{x}$. Figure 3·5 is the graph of $y = \frac{|x|}{x}$ the graph consists of two branches, one above the x-axis and on the right of the y-axis and the other below the x-axis and on the left of it.

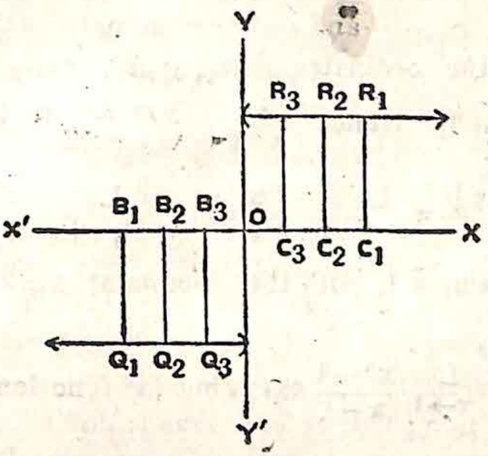


Fig. 3·5

Both the branches originate from the y-axis without intersecting it. The upper branch originates from the point (0, 1) and the lower branch from the point (0, -1). The function $\frac{|x|}{x}$ is undefined at $x=0$ and so the points (0, 1) and (0, -1) are not points on the graph.

We take points B_1, B_2, B_3,……on the left of O, very close to O on the x-axis. They are situated on the negative side of the x-axis and so the value of x at each of these points is negative. We draw ordinates B_1Q_1, B_2Q_2, B_3Q_3,……at these points to intersect the graph at the points Q_1, Q_2, Q_3,……As when $x<0$, $\frac{|x|}{x}$ is -1, so the length of each of these ordinates is -1 unit *i e.* the value of y at each of these points is -1. When x-approaches the origin O through these points *i.e.*, from the left, y approaches the y-axis *i e.*, the oridinate at O, through B_1Q_1, B_2Q_2, B_3Q_3,…… each of which has the value -1.

$$\text{So } \underset{x \to 0-}{\text{Lt}} y = \underset{x \to 0-}{\text{Lt}} \frac{|x|}{x} = -1$$

Again, we take points $C_1, C_2, C_3, \ldots\ldots$ on the right of O and very close to it on the x-axis and draw ordinates $C_1R_1, C_2R_2, C_3R_3, \ldots\ldots$ at these points to interesect the graph at the points $R_1, R_2, R_3, \ldots\ldots$ respectively. Evidently the points $R_1, R_2, R_3 \ldots\ldots$ are all situated above the x-axis and $C_1R_1, C_2R_2, C_3R_3 \ldots\ldots$ each have the value 1. So as x approaches 0 from the right, it approaches the origin O through $C_1, C_2, C_3, \ldots\ldots$ and correspondingly y approaches the y-axis through the ordinates $C_1R_1, C_2R_2, C_3R_3, \ldots\ldots$ *i.e.*, always having the value $+1$. Hence $\underset{x \to 0+}{\text{Lt}} \; y = \underset{x \to 0+}{\text{Lt}} \dfrac{|x|}{x} = 1.$

$\therefore \underset{x \to 0-}{\text{Lt}} \dfrac{|x|}{x} \neq \underset{x \to 0+}{\text{Lt}} \dfrac{|x|}{x}$. $\therefore \underset{x \to 0}{\text{Lt}} \dfrac{|x|}{x}$ does not exist.

Note. In example 1, both the function at $x=2$ and the limit when $x \to 2$, exist.

In example 2, $\underset{x \to 1}{\text{Lt}} \dfrac{x^2-1}{x-1}$ exists but the function does not exist at $x=1$.

In example 1 $f(x)$ is defined for all values of x and has the same form on both sides of $x=2$. But in ex. 2. The function has different forms on the two sides of $x=1$. In ex, 3 the function has different forms on the two sides of the origin but is defined for all values of x. Here also the limit exists. In ex. 4 the function has different forms on the two sides of the origin and here neither $\underset{x \to 0}{\text{Lt}} \dfrac{|x|}{x}$ nor the function at $x=0$ exist. In ex. 16 (iii) of Examples 3, though the function exists at $x=2$, yet $\underset{x \to 2}{\text{Lt}} f(x)$ does not exist.

§ 3·5. Difference between $\underset{x \to a}{\text{Lt}} f(x)$ and $f(a)$

By $f(a)$ we mean the value of $f(x)$ when $x=a$. When $x=a$, then the value of $f(x)$ may be finite or not. In the first case $f(x)$ exists and in the second case it does not.

In case of $\underset{x \to a}{\text{Lt}} f(x)$ we consider values of $f(x)$ when x is close to a (as close as one desires), but not at $x=a$. So, $\underset{x \to a}{\text{Lt}} f(x)$

does not depend on the existence or the value of $f(x)$. When we discuss $\underset{x\to a}{\text{Lt}}\, f(x)$ we may face the following situations.

(1) $\underset{x\to a}{\text{Lt}}\, f(x)$ exists and (i) $\underset{x\to a}{\text{Lt}}\, f(x)=f(a)$

Or, (ii) $\underset{x\to a}{\text{Lt}}\, f(x)\neq f(a)$, though $f(a)$ exists.

Or, (iii) $f(a)$ may not have any finite value.

(2) $\underset{x\to a}{\text{Lt}}\, f(x)$ does not exist and

(i) $f(a)$ exists or (ii) $f(a)$ does not exist.

§ 3·6. Some Important Limits.

The following results relating to limits are very important. Though their proves are not included in the syllabus, their applications are prescribed. Of these limits one thing is noteworthy that the constant to which each variable approaches, the corresponding function is undefined at that point. We state below these results without proof.

1. For all values of n (a constant), $\underset{x\to a}{\text{Lt}}\, \frac{x^n-a^n}{x-a}=na^{n-1}$

Examples. (i) $\underset{x\to 3}{\text{Lt}}\, \frac{x^4-81}{x-3}=\frac{x^4-3^4}{x-3}=4.3^{4-1}$

$=4.3^3=4.27=108$ [Here $a=3$ and $n=4$]

(ii) $\underset{x\to 2}{\text{Lt}}\, \frac{x^{\frac{2}{3}}-2^{\frac{2}{3}}}{x-2}=\frac{2}{3}2^{\frac{2}{3}-1}=\frac{2}{3}2^{-\frac{1}{3}}=\frac{2}{3.\sqrt[3]{2}}$

2. $\underset{x\to 0}{\text{Lt}}\, \frac{\sin x}{x}=1$ where x is in radians. Note that the denominator is a pure number.

3. $\underset{x\to 0}{\text{Lt}}\, \frac{e^x-1}{x}=1$. 4. $\underset{x\to 0}{\text{Lt}}\, \frac{\log_e(1+x)}{x}=1$.

5. $\underset{x\to 0}{\text{Lt}}\, x\sin\left(\frac{1}{x}\right)=0$.

§ 3·7. Limit Theorems.

The following Theorems regarding the limit of the sum, product etc. of two or more functions are very much necessary. Their proof is out of syllabus; but their statements and applications are included. It is essential for the students to remember these theorems. Let us state the theorems.

If $\underset{x\to a}{\text{Lt}}\ f(x)$ and $\underset{x\to a}{\text{Lt}}\ g(x)$ both exist.

Then (1) $\underset{x\to a}{\text{Lt}}\ \{k\,f(x)\}=k.\underset{x\to a}{\text{Lt}}\ \{f(x)\}$ where k is a constant.

(2) $\underset{x\to a}{\text{Lt}}\ \{f(x)\pm g(x)\}=\underset{x\to a}{\text{Lt}}\ f(x)\pm\underset{x\to a}{\text{Lt}}\ g(x)$

(3) $\underset{x\to a}{\text{Lt}}\ \{f(x).g(x)\}=\underset{x\to a}{\text{Lt}}\ f(x).\ \underset{x\to a}{\text{Lt}}\ g(x).$

(4) $\underset{x\to a}{\text{Lt}}\left\{\frac{f(x)}{g(x)}\right\}=\frac{\underset{x\to a}{\text{Lt}}\ f(x)}{\underset{x\to a}{\text{Lt}}\ g(x)}$ if $\underset{x\to a}{\text{Lt}}\ g(x)\neq0.$

(5) $\underset{x\to a}{\text{Lt}}\ f\{g(x)\}=f\{\underset{x\to a}{\text{Lt}}\ g(x)\}$ if $f(x)$ be a continuous function.

Example 1. $\underset{x\to2}{\text{Lt}}\ \{5x^2\}=5.\underset{x\to2}{\text{Lt}}\ x^2=5.4=20.$

Ex. 2. $\underset{x\to3}{\text{Lt}}\left(\frac{5}{x-3}\right)\neq5\underset{x\to3}{\text{Lt}}\left(\frac{1}{x-3}\right)$ as $\underset{x\to3}{\text{Lt}}\ \frac{1}{x-3}$ does not exist

Ex. 3. $\underset{x\to0}{\text{Lt}}\left(x^3+\frac{\sin x}{x}\right)=\underset{x\to0}{\text{Lt}}\left(x^3\right)+\underset{x\to0}{\text{Lt}}\left(\frac{\sin x}{x}\right)=0+1=1.$

Ex. 4. Though $\frac{x^2}{x-2}-\frac{4}{x-2}=\frac{x^2-4}{x-2}$, yet $\underset{x\to2}{\text{Lt}}\left(\frac{x^2}{x-2}\right)-\underset{x\to2}{\text{Lt}}\ \frac{4}{x-2}\neq\underset{x\to2}{\text{Lt}}\ \frac{x^2-4}{x-2}$ as each limit on the left does not exist.

Ex. 5. $\underset{x\to0}{\text{Lt}}\ (x\sin x)=\underset{x\to0}{\text{Lt}}\ x\ \underset{x\to0}{\text{Lt}}\ \sin x=0.0=0$

Ex. 6. $\underset{x\to0}{\text{Lt}}\ \frac{\sin x}{x}=\underset{x\to0}{\text{Lt}}\left(\sin x\,.\,\frac{1}{x}\right)\neq\underset{x\to0}{\text{Lt}}\ \sin x.\ \underset{x\to0}{\text{Lt}}\left(\frac{1}{x}\right)$ as $\underset{x\to0}{\text{Lt}}\left(\frac{1}{x}\right)$ does not exist.

Ex. 7. $\underset{x\to2}{\text{Lt}}\ \frac{x^2-7x+12}{x^2-4x+3}=\frac{\underset{x\to2}{\text{Lt}}\ (x^2-7x+12)}{\underset{x\to2}{\text{Lt}}\ (x^2-4x+3)}$ [Theorem (iv)]

$$=\frac{\underset{x\to2}{\text{Lt}}\ (x^2)-7\underset{x\to2}{\text{Lt}}\ (x)+\underset{x\to2}{\text{Lt}}\ (12)}{\underset{x\to2}{\text{Lt}}\ (x^2)-4\underset{x\to2}{\text{Lt}}\ (x)+\underset{x\to2}{\text{Lt}}\ (3)}=\frac{4-7.2+12}{4-4.2+3}=\frac{2}{-1}=-2.$$

Ex. 8. $\underset{x\to2}{\text{Lt}}\ \frac{x^2-4}{x-2}\neq\frac{\underset{x\to2}{\text{Lt}}\ (x^2-4)}{\underset{x\to2}{\text{Lt}}\ (x-2)}$, as the limit in the denominator is 0. (See Theorem IV)

Ex. 9. $\underset{x\to1}{\text{Lt}}\ e^x=e^{\underset{x\to1}{\text{Lt}}\ x}=e^1=e$

Ex. 10. $\underset{x\to3}{\text{Lt}}\ \sin(x^2)=\sin\left(\underset{x\to3}{\text{Lt}}\ x^2\right)=\sin 9.$

The functions in Ex 9 and Ex.10 are of the form $f\{g(x)\}$.

In Ex. 9. $f(x)=e^x$ and $g(x)=x.$

In Ex. 10. $f(x)=\sin x$ and $g(x)=x^2$

Here e^x and $\sin x$ are two continuous functions.

§ 3·8. Meaning of $x\to\infty$ and $x\to-\infty$.

In this section we shall show that if $x\to\infty$, then $y=\frac{1}{x}\to0+$ and if $x\to-\infty$, then $y=\frac{1}{x}\to0-$.

First let $x\to\infty$ and $y=\frac{1}{x}$.

Notice the following table carefully.

x	1	10	100	1000	100000000	10^{101}
$y=\frac{1}{x}$	1	·1	·01	·001	·00000001	10^{-101}

From the table we find that as the positive values of x increase, the value of $y=\frac{1}{x}$ decreases (always remaining positive); $x\to\infty$ means that the value of x can be made as large as one desires and can be made greater than any positive number large at pleasure. So in the above table by making the value of x sufficiently large, the value of $y=\frac{1}{x}$ can be made smaller than any positive number however small. Also as x increases it remains positive and so $\frac{1}{x}$ (as small as it may be) always remains positive. So as $x\to\infty$, $y=\frac{1}{x}\to0+$.

Again $x \to -\infty$ means x will always remain negative but its numerical value can be made larger than any positive number, large at pleasure. So when $x \to -\infty$, then one can prepare the following table.

x	-1	-10	-100	-1000	-100000000	-10^{100}
$y=\frac{1}{x}$	-1	$-\cdot 1$	$-\cdot 01$	$-\cdot 001$	$-\cdot 00000001$	-10^{-100}

From the table we find that though negative, as the numerical value of x increases, the numerical value of $y=\frac{1}{x}$ decreases. And though negative, making the numerical value of x sufficiently large, we can make the numerical value of $y=\frac{1}{x}$ smaller than any positive number as small as we like. Again as x is negative, so $\frac{1}{x}$ is always negative. So, as $x \to \infty$, $y=\frac{1}{x} \to 0-$.

In the above dicussion we have found that $\underset{x \to \infty}{\mathrm{Lt}} \frac{1}{x} = 0+$ and $\underset{x \to -\infty}{\mathrm{Lt}} \frac{1}{x} = 0-$.

For this, when $x \to \infty$ and $x \to -\infty$, generally we put $x=\frac{1}{y}$ or $y=\frac{1}{x}$ and then $y \to 0+$ or, $y \to 0-$

Example 1. $\underset{x \to \infty}{\mathrm{Lt}} \dfrac{x^2+x+1}{x^3+x^2-x+2}$

$$= \underset{y \to 0+}{\mathrm{Lt}} \frac{\frac{1}{y^2}+\frac{1}{y}+1}{\frac{1}{y^3}+\frac{1}{y^2}-\frac{1}{y}+2} \text{ [taking } x=\tfrac{1}{y} \text{ and as } x \to \infty, \text{ then } y \to 0+]$$

$$= \underset{y \to 0+}{\mathrm{Lt}} \frac{y^3(1+y+y^2)}{y^2(1+y-y^2+2y^3)} = \underset{y \to 0+}{\mathrm{Lt}}\; y \; \frac{\underset{y \to 0+}{\mathrm{Lt}}\; (1+y+y^2)}{\underset{y \to 0+}{\mathrm{Lt}}\; (1+y-y^2+2y^3)}$$

$= 0.\frac{1}{1} = 0.$

Ex. 2. $\underset{x \to \infty}{\mathrm{Lt}} \dfrac{2x^2-5x+2}{x^2+3x+2}$

$$= \underset{y \to 0+}{\mathrm{Lt}} \frac{\frac{2}{y^2}-\frac{5}{y}+2}{\frac{1}{y^2}+\frac{3}{y}+2} \quad \text{[Taking } x=\tfrac{1}{y} \text{ and as } x \to \infty, \text{ then } y \to 0+]$$

$$= \underset{y \to 0+}{\mathrm{Lt}} \frac{(2-5y+2y^2)}{(1+3y+y^2)} = \frac{\underset{y \to 0+}{\mathrm{Lt}} \; (2-5y+2y^2)}{\underset{y \to 0+}{\mathrm{Lt}} \; (1+3y+y^2)} = \frac{2}{1} = 2$$

[Note. In the "Integral calculus" portion of this book instead of repeatedly writing $\frac{1}{n}=h\to 0+$, we have written $h\to 0$ when $n\to\infty$].

Ex. 3. Show that if x be a natural number

$$\underset{x\to\infty}{\text{Lt}}\ \frac{1+2+3+\cdots\cdots+x}{x}\ \text{does not exist.}$$

Given limit$=\underset{x\to\infty}{\text{Lt}}\ \frac{x(x+1)}{2x}=\underset{x\to\infty}{\text{Lt}}\ \frac{x+1}{2}\to\infty$

Hence the given limit does not exist.

Note. Here the numerator could not made independent of x and as $x\to\infty$, then $(x+1)\to\infty$.

Examples 3

Ex. 1. Determine the following limits.

(i) $\underset{x\to 4}{\text{Lt}}\ \frac{x^2-3x+2}{x^2-4x+3}$ (ii) $\underset{x\to 2}{\text{Lt}}\ \frac{x^2-3x+2}{x^2-4x+3}$

(iii) $\underset{x\to 1}{\text{Lt}}\ \frac{x^2-3x+2}{x^2-4x+3}$ [H. S. 1982]

(i) $$\underset{x\to 4}{\text{Lt}}\ \frac{x^2-3x+2}{x^2-4x+3}=\frac{\underset{x\to 4}{\text{Lt}}\ (x^2-3x+2)}{\underset{x\to 4}{\text{Lt}}\ (x^2-4x+3)}=\frac{4^2-3.4+2}{4^2-4.4+3}=\frac{6}{3}=2$$

(ii) $$\underset{x\to 2}{\text{Lt}}\ \frac{x^2-3x+2}{x^2-4x+3}=\frac{\underset{x\to 2}{\text{Lt}}\ (x^2-3x+2)}{\underset{x\to 2}{\text{Lt}}\ (x^2-4x+3)}=\frac{2^2-3.2+2}{4^2-4.4+3}=0.$$

(iii) $$\underset{x\to 1}{\text{Lt}}\ \frac{x^2-3x+2}{x^2-4x+3}=\underset{x\to 1}{\text{Lt}}\ \frac{(x-1)(x-2)}{(x-1)(x-3)}=\underset{x\to 1}{\text{Lt}}\ \frac{x-2}{x-3}$$

$$=\frac{\underset{x\to 1}{\text{Lt}}\ (x-2)}{\underset{x\to 1}{\text{Lt}}\ (x-3)}=\frac{1-2}{1-3}=\frac{-1}{-2}=\frac{1}{2}.$$

Note: In Examples (i) and (ii) when $x=4$, or $x=2$, the denominator does not become 0. So, we put $x=4$ and $x=2$ in the expressions and evaluate the limits. In Ex. (iii) when $x=1$, the denominator becomes 0; so we cannot put $x=1$ at the very out set. Factorising the denominator we find $(x-1)$ is a factor and its presence makes the denominator 0. So our next effort is to get rid of this factor $(x-1)$; we now factorise the numerator also and find that $(x-1)$ is also a factor of the numerator. So, we cancel the common factor $(x-1)$ from both the numerator and denominator and get rid of $(x-1)$ in the denominator. Now the given limit be comes $\underset{x\to 1}{\text{Lt}}\ \frac{x-2}{x-3}$. Here $x=1$ does not make the denominator 0. So the limit is evaluated by puting $x=1$.

Remember that $x\to 1$ means x does not assume the value 1. So in $\underset{x\to 1}{\text{Lt}}\ x^2$, $x\neq 1$. But in this case the limiting value of x^2 is $1^2=1$.

Let us now take another example.

$\underset{x\to 3}{\text{Lt}}\ \frac{x^2-3x+2}{x^2-4x+3}$ In this case due to the Presence of the factor $(x-3)$ the denominator becomes 0 when $x=3$. But $(x-3)$ is not a factor of the numerator. So, we cannot get rid of $(x-3)$ from the denominator in this case and so the limit cannot be evaluated.

Ex. 2. Evaluate the limit.

$$\underset{x\to 0}{\text{Lt}}\ \frac{a_0x^m+a_1x^{m-1}+a_2x^{m-2}+\cdots+a_m}{b_0x^n+b_1x^{n-2}+b_2x^{n-2}+\cdots+b_n}\ [b_n\neq 0]$$

$$\underset{x\to 0}{\text{Lt}}\ \frac{a_0x^m+a_1x^{m-2}+a_2x^{m-2}+\cdots+a_n}{b_0x^n+b_1x^{n-2}b_x{}^{n-2}+\ \cdots\ +b_n}$$

$$=\frac{\underset{x\to 0}{\text{Lt}}\ (a_0x^m+a_1x^{m-1}+a_2x^{m-2}+\cdots+a_m)}{\underset{x\to 0}{\text{Lt}}\ (b_0x^m+b_1x^{m-1}+b_2x^{m-2}+\cdots+b_m)}=\frac{a_m}{b_m}$$

Ex. 3. Evaluate the following limits.

(i) $\underset{x\to 0}{\text{Lt}}\ \frac{1}{x}\{\sqrt{1+x}-\sqrt{1-x}\}$ [H. S. 1978]

(ii) $\underset{x\to 0}{\text{Lt}}\ \frac{1-\sqrt{1+x^2}}{x^2}$ [H. S. 1980]

(iii) $\underset{x\to 0}{\text{Lt}}\ \frac{\sqrt{1+ax}-\sqrt{1-ax}}{x}$ [H. S. 1980]

(iv) $\underset{x\to 0}{\text{Lt}}\ \frac{\sqrt{1+ax}-\sqrt{1-bx}}{x}$ (v) $\underset{x\to 4}{\text{Lt}}\ \frac{\sqrt{x}-2}{x-4}$ [H. S. 1985]

(vi) $\underset{x\to a}{\text{Lt}}\ \frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}(a>b)$ [H. S. 1984]

(vii) $\underset{x\to 0}{\text{Lt}}\ \frac{x}{\sqrt{1+x}-1}$.

(i) $\underset{x\to 0}{\text{Lt}}\ \frac{1}{x}\{\sqrt{1+x}-\sqrt{1-x}\}$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})}=\underset{x\to 0}{\text{Lt}}\ \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{2}{\sqrt{1+x}+\sqrt{1-x}}=\frac{2}{2}=1.$$

(ii) $\underset{x\to 0}{\text{Lt}}\ \frac{1-\sqrt{1-x^2}}{x^2}=\underset{x\to 0}{\text{Lt}}\ \frac{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})}{x^2(1+\sqrt{1-x^2})}$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{1-(1-x^2)}{x^2(1+\sqrt{1-x^2})}=\underset{x\to 0}{\text{Lt}}\ \frac{x^2}{x^2(1+\sqrt{1-x^2})}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{1}{1+\sqrt{1-x^2}}=\frac{1}{2}.$$

(iii) $\underset{x\to 0}{\text{Lt}}\ \frac{\sqrt{1+ax}-\sqrt{1-ax}}{x}$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{(\sqrt{1+ax}-\sqrt{1-ax})(\sqrt{1+ax}+\sqrt{1+ax})}{x(\sqrt{1+ax}+\sqrt{1-ax})}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{(1+ax)-(1-ax)}{x(\sqrt{1+ax}+\sqrt{1-ax})}=\underset{x\to 0}{\text{Lt}}\ \frac{2ax}{x(\sqrt{1+ax}+\sqrt{1-ax})}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{2}{\sqrt{1+ax}+\sqrt{1-ax}}=\frac{2}{2}=1,$$

(iv) $\underset{x\to 0}{\text{Lt}}\ \frac{\sqrt{1+ax}-\sqrt{1-bx}}{x}$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{(\sqrt{1+ax}-\sqrt{1-bx})(\sqrt{1+ax}+\sqrt{1-bx})}{x(\sqrt{1+ax}+\sqrt{1-bx})}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{(1+ax)-(1-bx)}{x(\sqrt{1+ax}+\sqrt{1-bx})}=\underset{x\to 0}{\text{Lt}}\ \frac{(a+b)x}{x(\sqrt{1+ax}+\sqrt{1-bx})}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{a+b}{\sqrt{1+ax}+\sqrt{1-ax}}=\frac{a+b}{2}.$$

(v) $\underset{x\to 4}{\text{Lt}}\ \frac{\sqrt{x}-2}{x-4}=\underset{x\to 4}{\text{Lt}}\ \frac{\sqrt{x}-2}{(\sqrt{x}+2)(\sqrt{x}-2)}=\underset{x\to 4}{\text{Lt}}\ \frac{1}{\sqrt{x}+2}$

$$=\frac{1}{\sqrt{4}+2}=\frac{1}{2+2}=\tfrac{1}{4}.$$

(vi) $$\operatorname*{Lt}_{x\to a}\frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$$

$$=\operatorname*{Lt}_{x\to a}\frac{(\sqrt{x-b}-\sqrt{a-b})(\sqrt{x-b}+\sqrt{a-b})}{(x^2-a^2)(\sqrt{x-b}+\sqrt{a-b})}$$

$$=\operatorname*{Lt}_{x\to a}\frac{(x-b)-(a-b)}{(x+a)(x-a)(\sqrt{x-b}+\sqrt{a-b})}$$

$$=\operatorname*{Lt}_{x\to a}\frac{x-a}{(x+a)(x-a)(\sqrt{x-b}+\sqrt{a-b})}$$

$$=\operatorname*{Lt}_{x\to a}\frac{1}{(x+a)(\sqrt{x-b}+\sqrt{a-b})}=\frac{1}{2a.2\sqrt{a-b}}=\frac{1}{4a\sqrt{a-b}}.$$

(vii) $$\operatorname*{Lt}_{x\to 0}\frac{x}{\sqrt{1+x}-1}=\operatorname*{Lt}_{x\to 0}\frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}$$

$$=\operatorname*{Lt}_{x\to 0}\frac{x(\sqrt{1+x}+1)}{1+x-1}=\operatorname*{Lt}_{x\to 0}\frac{x(\sqrt{1+x}+1)}{x}=\operatorname*{Lt}_{x\to 0}(\sqrt{1+x}+1)=2$$

Ex. 4. Evaluate the following limits.

(i) $$\operatorname*{Lt}_{x\to 1}\frac{x^2-1}{\sqrt{3x+1}-\sqrt{5x-1}}$$ [H. S. 1979]

(ii) $$\operatorname*{Lt}_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$$ [H. S. 1980]

(iii) $$\operatorname*{Lt}_{x\to 3}\frac{\sqrt{x-3}+\sqrt{x}-\sqrt{3}}{\sqrt{x^2-9}}$$ [Joint Entrace 1985]

(i) $$\operatorname*{Lt}_{x\to 1}\frac{x^2-1}{\sqrt{3x+1}-\sqrt{5x-1}}$$

$$=\operatorname*{Lt}_{x\to 1}\frac{(x^2-1)(\sqrt{3x+1}+\sqrt{5x-1})}{(\sqrt{3x+1}+\sqrt{5x-1})(\sqrt{3x+1}-\sqrt{5x-1})}$$

$$=\operatorname*{Lt}_{x\to 1}\frac{(x+1)(x-1)(\sqrt{3x+1}+\sqrt{5x-1})}{(3x+1)-(5x-1)}$$

$$=\operatorname*{Lt}_{x\to 1}\frac{(x+1)(x-1)(\sqrt{3x+1}+\sqrt{5x-1})}{-2(x-1)}$$

$$=\operatorname*{Lt}_{x\to 1}\frac{(x+1)(\sqrt{3x+1}+\sqrt{5x-1})}{-2}$$

$$=\frac{(1+1)(\sqrt{3+1}+\sqrt{5+1})}{-2}=\frac{2(2+2)}{-2}=-4.$$

(ii) $\underset{x\to 3}{\text{Lt}} \dfrac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$

$$=\underset{x\to 3}{\text{Lt}} \frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{(\sqrt{x-2}-\sqrt{4-x})(\sqrt{x-2}+\sqrt{4-x}}$$

$$=\underset{x\to 3}{\text{Lt}} \frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{(x-2)-(4-x)}$$

$$=\underset{x\to 3}{\text{Lt}} \frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{2(x-3)}$$

$$=\underset{x\to 3}{\text{Lt}} \frac{(\sqrt{x-2}+\sqrt{4-x})}{2}=\frac{\sqrt{3-2}+\sqrt{4-3}}{2}=\frac{\sqrt{1}+\sqrt{1}}{2}=1$$

(iii) $\underset{x\to 3}{\text{Lt}} \dfrac{\sqrt{x-3}+\sqrt{x}-\sqrt{3}}{\sqrt{x^2-9}}$

$$=\underset{x\to 3}{\text{Lt}} \frac{\sqrt{x-3}}{\sqrt{x^2-9}}+\underset{x\to 3}{\text{Lt}} \frac{\sqrt{x}-\sqrt{3}}{\sqrt{x^2-9}}$$

$$=\underset{x\to 3}{\text{Lt}} \frac{\sqrt{x-3}}{\sqrt{x-3}\sqrt{x+3}}+\underset{x\to 3}{\text{Lt}} \frac{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}{(\sqrt{x}+\sqrt{3}\sqrt{x-3}\sqrt{x+3})}$$

$$=\underset{x\to 3}{\text{Lt}} \frac{1}{\sqrt{x+3}}+\underset{x\to 3}{\text{Lt}} \frac{(\sqrt{x}+\sqrt{3})(\sqrt{x}-\sqrt{3})}{(\sqrt{x}+\sqrt{3})\sqrt{x-3}\sqrt{x+3}}$$

$$=\frac{1}{\sqrt{3+3}}+\underset{x\to 3}{\text{Lt}} \frac{(x-3)}{(\sqrt{x}+\sqrt{3})\sqrt{x-3}\sqrt{x+3}}$$

$$=\frac{1}{\sqrt{6}}+\underset{x\to 3}{\text{Lt}} \frac{\sqrt{x-3}}{(\sqrt{x}+\sqrt{3})(\sqrt{x+3})}$$

$$=\frac{1}{\sqrt{6}}+\frac{\sqrt{3-3}}{(\sqrt{3}+\sqrt{3})(\sqrt{3+3})}=\frac{1}{\sqrt{6}}+\frac{0}{2\sqrt{3}\sqrt{6}}=\frac{1}{\sqrt{6}}$$

Ex. 5. Evaluate the following limits.

(i) $\underset{x\to 0}{\text{Lt}} \dfrac{x^2}{x}$ [Joint Entrance 1981] (ii) $\underset{x\to 2}{\text{Lt}} \dfrac{x^2-4}{x-2}$

(iii) $\underset{x\to a}{\text{Lt}} \dfrac{x^3-a^3}{x^2-a^2}$ (iv) $\underset{x\to a}{\text{Lt}} \dfrac{x^m-a^m}{x^n-a^n}$ (v) $\underset{x\to a}{\text{Lt}} \dfrac{x^{\frac{6}{11}}-a^{\frac{6}{11}}}{x^{\frac{2}{7}}-a^{\frac{2}{7}}}$

(vi) $\underset{x\to 4}{\text{Lt}} \dfrac{x^4-256}{x-4}$ (vii) $\underset{x\to 1}{\text{Lt}} \dfrac{x^{\frac{5}{2}}-1}{x^{\frac{7}{2}}-1}$

(i) $\underset{x\to 0}{\text{Lt}} \dfrac{x^2}{x} = \underset{x\to 0}{\text{Lt}} (x) = 0$

(ii) $\underset{x\to 2}{\text{Lt}} \dfrac{x^2-4}{x-2} = \underset{x\to 2}{\text{Lt}} \dfrac{(x+2)(x-2)}{x-2} = \underset{x\to 2}{\text{Lt}} (x+2) = 4$

(iii) $\underset{x\to a}{\text{Lt}} \dfrac{x^3-a^3}{x^2-a^2} = \underset{x\to a}{\text{Lt}} \dfrac{(x-a)(x^2+xa+a^2)}{(x-a)(x+a)}$

$$= \underset{x\to a}{\text{Lt}} \frac{x^2+xa+a^2}{x+a} = \frac{3a^2}{2a} = \tfrac{3}{2}a.$$

(iv) $\underset{x\to a}{\text{Lt}} \dfrac{x^m-a^m}{x^n-a^n} = \underset{x\to a}{\text{Lt}} \dfrac{\dfrac{x^m-a^m}{x-a}}{\dfrac{x^n-a^n}{x-a}}$

$$= \frac{\underset{x\to a}{\text{Lt}} \dfrac{x^m-a^m}{x-a}}{\underset{x\to a}{\text{Lt}} \dfrac{x^n-a^n}{x-a}} = \frac{ma^{m-1}}{na^{n-1}} = \frac{m}{n}a^{m-n}$$

(v) $\underset{x\to a}{\text{Lt}} \dfrac{x^{\frac{6}{11}}-a^{\frac{6}{11}}}{x^{\frac{2}{7}}-a^{\frac{2}{7}}}$

$$= \underset{x\to a}{\text{Lt}} \frac{\dfrac{x^{\frac{6}{11}}-a^{\frac{6}{11}}}{x-a}}{\dfrac{x^{\frac{2}{7}}-a^{\frac{2}{7}}}{x-a}} = \frac{\underset{x\to a}{\text{Lt}} \dfrac{x^{\frac{6}{11}}-a^{\frac{6}{11}}}{x-a}}{\underset{x\to a}{\text{Lt}} \dfrac{x^{\frac{2}{7}}-a^{\frac{2}{7}}}{x-a}}$$

$$= \frac{\frac{6}{11}a^{\frac{6}{11}-1}}{\frac{2}{7}a^{\frac{2}{7}-1}} = \tfrac{42}{22}a^{\frac{6}{11}-\frac{2}{7}} = \tfrac{21}{11}a^{\frac{20}{77}}$$

(vi) $\underset{x\to 4}{\text{Lt}} \dfrac{x^4-256}{x-4}$

$$= \underset{x\to 4}{\text{Lt}} \frac{x^4-(4)^4}{x-4} = 4.4^{4-1} = 4.4^3 = 4.64 = 256.$$

(vii) $$\underset{x\to1}{\mathrm{Lt}}\ \frac{x^{\frac{5}{2}}-1}{x^{\frac{7}{2}}-1}=\underset{x\to1}{\mathrm{Lt}}\ \frac{\dfrac{x^{\frac{5}{2}}-1}{x-1}}{\dfrac{x^{\frac{7}{2}}-1}{x-1}}$$

$$=\frac{\underset{x\to1}{\mathrm{Lt}}\ \dfrac{x^{\frac{5}{2}}-1}{x-1}}{\underset{x\to1}{\mathrm{Lt}}\ \dfrac{x^{\frac{7}{2}}-1}{x-1}}=\frac{\frac{5}{2}.1^{\frac{5}{2}-1}}{\frac{7}{2}.1^{\frac{7}{2}-1}}=\frac{5}{7}$$

Ex. 6. Evaluate the following limits.

(i) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{\sin 5x}{x}$ (ii) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{\sin 7x}{3x}$ (iii) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{\sin mx}{nx}$

iv) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{\sin 3x}{\sin 2x}$ [H. S. 1982, 1986]

(v) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{1-\cos x}{x^2}$ [H. S. 1978]

(vi) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{\sin x(1-\cos x)}{x^3}$ [H. S. 1984]

(vii) $\underset{x\to a}{\mathrm{Lt}}\ \dfrac{1-\cos(x-a)}{(x-a)^2}$ [H. S. 1987]

(viii) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{2\sin x-\sin 2x}{x^3}$ [H. S. 1983]

(ix) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{\sin^{-1}x}{x}$ [H. S. 1984] (x) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{x}{\tan^{-1}x}$

(xi) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{\tan x-\sin x}{x^3}$ (xii) $\underset{x\to0}{\mathrm{Lt}}\ \dfrac{x-\sin 2x}{x-\sin 3x}$

(i) $$\underset{x\to0}{\mathrm{Lt}}\ \frac{\sin 5x}{x}=\underset{x\to0}{\mathrm{Lt}}\left(\frac{\sin 5x}{5x}.5\right)=5\underset{5x\to0}{\mathrm{Lt}}\ \frac{\sin 5x}{5x}=5.1=5.$$

(ii) $$\underset{x\to0}{\mathrm{Lt}}\ \frac{\sin 7x}{3x}=\frac{1}{3}\underset{x\to0}{\mathrm{Lt}}\ \frac{\sin 7x}{x}=\frac{1}{3}\underset{x\to0}{\mathrm{Lt}}\left(\frac{\sin 7x}{7x}.7\right)$$

$$=\frac{7}{3}.\underset{x\to0}{\mathrm{Lt}}\ \frac{\sin 7x}{7x}=\frac{7}{3}.1=\frac{7}{3}.$$

(iii) $$\underset{x\to 0}{\text{Lt}}\ \frac{\sin mx}{nx}=\frac{1}{n}\ \underset{x\to 0}{\text{Lt}}\left(\frac{\sin mx}{mx}.m\right)$$

$$=\frac{m}{n}\ \underset{x\to 0}{\text{Lt}}\ \frac{\sin mx}{mx}=\frac{m}{n}.1=\frac{m}{n}.$$

(iv) $$\underset{x\to 0}{\text{Lt}}\ \frac{\sin 3x}{\sin 2x}=\underset{x\to 0}{\text{Lt}}\ \frac{\dfrac{\sin 3x}{3x}.3}{\dfrac{\sin 2x}{2x}.2}$$

$$=\tfrac{3}{2}\underset{x\to 0}{\text{Lt}}\left(\frac{\dfrac{\sin 3x}{3x}}{\dfrac{\sin 2x}{2x}}\right)=\tfrac{3}{2}.\frac{\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin 3x}{3x}}{\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin 2x}{2x}}=\tfrac{3}{2}.\tfrac{1}{1}=\tfrac{3}{2}$$

(v) $$\underset{x\to 0}{\text{Lt}}\ \frac{1-\cos x}{x^2}=\underset{x\to 0}{\text{Lt}}\ \frac{2\sin^2\dfrac{x}{2}}{x^2}=2\underset{x\to 0}{\text{Lt}}\ \frac{\sin^2\dfrac{x}{2}}{\dfrac{x^2}{4}.4}$$

$$=\tfrac{2}{4}\underset{x\to 0}{\text{Lt}}\ \frac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}.\underset{x\to 0}{\text{Lt}}\ \frac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}=\tfrac{1}{2}.1.1=\tfrac{1}{2}.$$

(vi) $$\underset{x\to 0}{\text{Lt}}\ \frac{\sin x(1-\cos x)}{x^3}=\underset{x\to 0}{\text{Lt}}\ \frac{\sin x\ .\ 2\sin^2\dfrac{x}{2}}{x^3}$$

$$=2\underset{x\to 0}{\text{Lt}}\left\{\frac{\sin x}{x}\ \frac{\sin\dfrac{x}{2}}{\dfrac{x}{2}.2}.\frac{\sin\dfrac{x}{2}}{\dfrac{x}{2}.2}\right\}$$

$$=\tfrac{2}{4}\underset{x\to 0}{\text{Lt}}\ \frac{\sin x}{x}.\underset{x\to 0}{\text{Lt}}\ \frac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}.\underset{x\to 0}{\text{Lt}}\ \frac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}=\tfrac{1}{2}.1.1=\tfrac{1}{2}.$$

(vii) $$\underset{x\to a}{\text{Lt}}\ \frac{1-\cos(x-a)}{(x-a)^2}=\underset{x\to a}{\text{Lt}}\frac{2\sin^2\dfrac{(x-a)}{2}}{(x-a)^2}$$

$$=2\underset{x\to a}{\text{Lt}}\ \frac{\sin^2\left(\dfrac{x-a}{2}\right)}{\left(\dfrac{x-a}{2}\right)^2.4}$$

$$=\frac{2}{4}\left\{\underset{(x-a)\to 0}{\text{Lt}}\frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}\ \underset{(x-a)\to 0}{\text{Lt}}\ \frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}\right\}$$

[when, $x\to a$, then $(x-a)\to 0$]

$=\frac{1}{2}.1.1=\frac{1}{2}.$

(viii) $\underset{x\to 0}{\text{Lt}}\ \frac{2\sin x-\sin 2x}{x^3}=\underset{x-0}{\text{Lt}}\ \frac{2\sin x-2\sin x\cos x}{x^3}$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{2\sin x(1-\cos x)}{x^3}=\underset{x\to 0}{2\text{Lt}}\cdot\frac{\sin x}{x}\ \underset{x\to 0}{\text{Lt}}\frac{2\sin^2\frac{x}{2}}{x^2}$$

$$=\underset{x\to 0}{4\text{Lt}}\ \frac{\sin x}{x}\cdot\underset{x\to 0}{\text{Lt}}\ \frac{\sin\frac{x}{2}}{\frac{x}{2}.2}\cdot\underset{x\to 0}{\text{Lt}}\frac{\sin\frac{x}{2}}{\frac{x}{2}\cdot 2}$$

$$=\frac{4}{4}\underset{x\to 0}{\text{Lt}}\ \frac{\sin x}{x}\cdot\underset{x\to 0}{\text{Lt}}\ \frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\underset{x\to 0}{\text{Lt}}\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

$=1.1.1.1=1.$

(ix) Let $\sin^{-1}x=\theta$ $\therefore$ $\sin\theta=x$ and when $x\to 0$, then $\sin\theta\to 0$ or $\theta\to 0$.

So, $\underset{x\to 0}{\text{Lt}}\ \frac{\sin^{-1}x}{x}=\underset{x\to 0}{\text{Lt}}\ \frac{\theta}{\sin\theta}$

$$=\underset{\to 0}{\text{Lt}}\ \frac{1}{\frac{\sin\theta}{\theta}}=\frac{1}{\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin\theta}{\theta}}=\frac{1}{1}=1$$

(x) Let $\tan^{-1}x=\theta$ $\therefore$ $\tan\theta=x$ and as $x\to 0$, $\tan\theta\to 0$ and so θ will approach the value 0.

$\therefore$ $\underset{x\to 0}{\text{Lt}}\ \frac{x}{\tan^{-1}x}=\underset{x\to 0}{\text{Lt}}\ \frac{\tan\theta}{\theta}$

$$=\underset{\theta\to 0}{\text{Lt}}\left(\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}\right)=\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin\theta}{\theta}\cdot\underset{\theta\to 0}{\text{Lt}}\ \frac{1}{\cos\theta}$$

$=1.1=1.$

(xi) $$\underset{x\to0}{\mathrm{Lt}}\frac{\tan x-\sin x}{x^3}=\underset{x\to0}{\mathrm{Lt}}\left\{\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}\right\}$$

$$=\underset{x\to0}{\mathrm{Lt}}\left\{\frac{\sin x}{x^3}\left(\frac{1}{\cos x}-1\right)\right\}=\underset{x\to0}{\mathrm{Lt}}\left\{\frac{\sin x}{x^3}\cdot\frac{1-\cos x}{\cos x}\right\}$$

$$=\underset{x\to0}{\mathrm{Lt}}\left\{\frac{\sin x}{x^3}\cdot\frac{2\sin^2\frac{x}{2}}{\cos x}\right\}=2\underset{x\to0}{\mathrm{Lt}}\frac{\sin x}{x}\cdot\underset{x\to0}{\mathrm{Lt}}\frac{\sin^2\frac{x}{2}}{x^2}\times$$

$$\underset{x\to0}{\mathrm{Lt}}\frac{1}{\cos x}=2\times1\times\underset{x\to0}{\mathrm{Lt}}\frac{\sin\frac{x}{2}}{\frac{x}{2}\times2}\cdot\underset{x\to0}{\mathrm{Lt}}\frac{\sin\frac{x}{2}}{\frac{x}{2}\times2}$$

$$=\frac{2}{4}\underset{x\to0}{\mathrm{Lt}}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\underset{x\to0}{\mathrm{Lt}}\frac{\sin\frac{x}{2}}{\frac{x}{2}}=\frac{2}{4}.1.1=\frac{1}{2}$$

(xii) $$\underset{x\to0}{\mathrm{Lt}}\frac{x-\sin 2x}{x-\sin 3x}=\underset{x\to0}{\mathrm{Lt}}\left\{\frac{1-\frac{\sin 2x}{x}}{1-\frac{\sin 3x}{x}}\right\}$$

(Dividing the numerator and denominator by x)

$$=\frac{\underset{x\to0}{\mathrm{Lt}}\left(1-\frac{\sin 2x}{2x}.2\right)}{\underset{x\to0}{\mathrm{Lt}}\left(1-\frac{\sin 3x}{3x}.3\right)}=\frac{\underset{x\to0}{\mathrm{Lt}}\,1-2\underset{x\to0}{\mathrm{Lt}}\frac{\sin 2x}{2x}}{\underset{x\to0}{\mathrm{Lt}}\,1-3\underset{x\to0}{\mathrm{Lt}}\frac{\sin 3x}{3}}$$

$$=\frac{1-2}{1-3}=\frac{-1}{-2}=\frac{1}{2}$$

Ex 7. Evaluate.

(i) $\underset{x\to\frac{\pi}{2}}{\mathrm{Lt}}\frac{2x-\pi}{\cos x}$ [H. S. 1981
I. I. T. 1973

(ii) $\underset{x\to\pi}{\mathrm{Lt}}\frac{\sin x}{\pi-x}$ [H. S. 1983]

(iii) $\underset{x\to\frac{\pi}{2}}{\mathrm{Lt}}\left(\frac{\pi}{2}-x\right)\tan x$ [H. S. 1987]

(iv) $\underset{x\to 1}{\text{Lt}}\ (1-x)\tan\frac{\pi x}{2}$ [Joint Entrance 1988 ; I.I.T'78, 84]

(i) Let $\frac{\pi}{2}-x=y$; ∴ when $x\to\frac{\pi}{2}$, then $y=\left(\frac{\pi}{2}-x\right)\to 0.$

Now, $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \frac{2x-\pi}{\cos x}=\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \frac{2\left(x-\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}-x\right)}$

$=-2\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \frac{\pi-x}{\sin\left(\frac{\pi}{2}-x\right)}$

$=-2\underset{y\to 0}{\text{Lt}}\ \frac{y}{\sin y}=-2.\frac{1}{\underset{y\to 0}{\text{Lt}}\ \frac{\sin y}{y}}=-2\frac{1}{1}=-2$

(ii) Let $\pi-x=y$ ∴ when $x\to\pi$, then $y=(\pi-x)\to 0$

Now, $\underset{x\to\pi}{\text{Lt}}\ \frac{\sin x}{\pi-x}=\underset{(\pi-x)\to 0}{\text{Lt}}\ \frac{\sin(\pi-x)}{\pi-x}$

$=\underset{y\to 0}{\text{Lt}}\ \frac{\sin y}{y}=1$

(iii) Let, $\frac{\pi}{2}-x=y$ ∴ when $x\to\frac{\pi}{2}$

then, $\frac{\pi}{2}-x=y\to 0$

Now. $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \left(\frac{\pi}{2}-x\right)\tan x=\underset{x\to\frac{\pi}{2}}{\text{Lt}}\left\{\left(\frac{\pi}{2}-x\right)\frac{\sin x}{\cos x}\right\}$

$=\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \sin x.\ \underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \frac{\frac{\pi}{2}-x}{\sin\left(\frac{\pi}{2}-x\right)}$

$=\sin\frac{\pi}{2}\ \underset{y\to 0}{\text{Lt}}\ \frac{y}{\sin y}=1.\frac{1}{\underset{y\to 0}{\text{Lt}}\ \frac{\sin y}{y}}=1.\ \frac{1}{1}=1.$

(iv) $\underset{x\to 1}{\text{Lt}}\ (1-x)\tan\left(\dfrac{\pi x}{2}\right)$

$$=\underset{x\to 1}{\text{Lt}}\ (1-x)\tan\left\{\frac{\pi}{2}-\frac{\pi}{2}(1-x)\right\}$$

$$=\underset{x\to 1}{\text{Lt}}\ (1-x)\cot\frac{\pi}{2}(1-x)$$

$$=\underset{x\to 1}{\text{Lt}}\ \frac{(1-x)\cos\frac{\pi}{2}(1-x)}{\sin\frac{\pi}{2}(1-x)}$$

$$=\underset{(1-x)\to 0}{\text{Lt}}\ \frac{(1-x)}{\sin\frac{\pi}{2}(1-x)}\ \underset{x\to 0}{\text{Lt}}\ \cos\frac{\pi}{2}(1-x)$$

$$=\underset{(1-x)\to 0}{\text{Lt}}\ \frac{1}{\dfrac{\sin\frac{\pi}{2}(1-x)}{\frac{\pi}{2}(1-x)}\cdot\dfrac{\pi}{2}}\ .\ 1.$$

$$=\frac{1}{\frac{\pi}{2}}\ \frac{1}{\underset{(1-x)\to 0}{\text{Lt}}\ \dfrac{\sin\frac{\pi}{2}(1-x)}{\frac{\pi}{2}(1-x)}}=\frac{2}{\pi}\cdot\frac{1}{1}=\frac{2}{\pi}.$$

Ex. 8. Evaluate :—

(i) $\underset{h\to 0}{\text{Lt}}\ \dfrac{\sin(x+h)-\sin x}{h}$ [H. S. 1979]

(ii) $\underset{h\to 0}{\text{Lt}}\ \dfrac{\cos(x+h)-\cos x}{h}$ [H. S. 1981]

(iii) $\underset{h\to 0}{\text{Lt}}\ \dfrac{\tan(x+h)-\tan x}{h}$ [H. S. 1980]

(i) $\underset{h\to 0}{\text{Lt}}\ \dfrac{\sin(x+h)-\sin x}{h}=\underset{h\to 0}{\text{Lt}}\ \dfrac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$

$$=\underset{h\to 0}{\text{Lt}}\ \cos\left(x+\frac{h}{2}\right)\ \underset{h\to 0}{\text{Lt}}\ \frac{\sin\frac{h}{2}}{\frac{h}{2}}=\cos x\ .\ 1=\cos x$$

(ii) $\underset{h\to 0}{\text{Lt}} \dfrac{\cos(x+h)-\cos x}{h}=\underset{h\to 0}{\text{Lt}} \dfrac{2\sin\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h}$

$$=-\underset{h\to 0}{\text{Lt}} \sin\left(x+\frac{h}{2}\right).\ \underset{h\to 0}{\text{Lt}} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$=-\sin x\ 1=-\sin x.$$

(iii) $\underset{h\to 0}{\text{Lt}} \dfrac{\tan(x+h)-\tan x}{h}$

$$=\underset{h\to 0}{\text{Lt}} \frac{\dfrac{\sin(x+h)}{\cos(x+h)}-\dfrac{\sin x}{\cos x}}{h}$$

$$=\underset{h\to 0}{\text{Lt}} \frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$=\underset{h\to 0}{\text{Lt}} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$=\underset{h\to 0}{\text{Lt}} \frac{\sin h}{h}.\ \underset{h\to 0}{\text{Lt}} \frac{1}{\cos(x+h)\cos x}$$

$$=1.\ \frac{1}{\cos x.\cos x}=\sec^2 x.$$

Ex. 9. Evaluate :—

(i) $\underset{h\to 0}{\text{Lt}} \dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$ [I. I. T. 1980]

(ii) $\underset{x\to a}{\text{Lt}} \dfrac{x\sin a-a\sin x}{x-a}.$

(i) $\underset{h\to 0}{\text{Lt}} \dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

$$=\underset{h\to 0}{\text{Lt}} \frac{(a^2+2ah+h^2)\sin(a+h)-a^2\sin a}{h}$$

$$=\underset{h\to 0}{\text{Lt}} \frac{a^2\sin(a+h)-a^2\sin a+(2ah+h^2)\sin(a+h)}{h}$$

$$=\underset{h\to 0}{\text{Lt}} \left[\frac{a^2\{\sin(a+h)-\sin a\}}{h}+\frac{h(2a+h)\sin(a+h)}{h}\right]$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{a^2.2\cos\left(a+\frac{h}{2}\right)\sin\frac{h}{2}}{h}+\underset{h\to 0}{\text{Lt}}\ (2a+h)\sin(a+h)$$

$$=a^2\underset{h\to 0}{\text{Lt}}\cos\left(a+\frac{h}{2}\right)\underset{h\to 0}{\text{Lt}}\ \frac{\sin\frac{h}{2}}{\frac{h}{2}}+\underset{h\to 0}{\text{Lt}}\ (2a+h)\underset{h\to 0}{\text{Lt}}\ \sin(a+h)$$

$$=a^2.\cos a.1+2a.\sin a=a^2\cos a+2a\sin a.$$

(ii) $\underset{x\to a}{\text{Lt}}\ \dfrac{x\sin a-a\sin x}{x-a}$

$$=\underset{x\to a}{\text{Lt}}\ \frac{x\sin a-a\sin a+a\sin a-a\sin x}{x-a}$$

$$=\underset{x\to a}{\text{Lt}}\ \frac{(x-a)\sin a-a(\sin x-\sin a)}{x-a}$$

$$=\underset{x\to a}{\text{Lt}}\ \frac{(x-a)\sin a}{x-a}-a\underset{x\to a}{\text{Lt}}\ \frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a}$$

$$=\underset{x\to a}{\text{Lt}}\ \sin a-a\underset{x\to a}{\text{Lt}}\cos\frac{x+a}{2}\underset{(x-a)\to 0}{\text{Lt}}\ \frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}$$

$$=\sin a-a\cos a.1=\sin a-a\cos a.$$

Ex. 10. Evaluate :—

(i) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{ax}-1}{x}$ (ii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{x^2}-1}{x}$

(iii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{\alpha x}-e^{\beta x}}{x}$

(i) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{ax}-1}{x}=\underset{x\to 0}{\text{Lt}}\left(\dfrac{e^{ax}-1}{ax}.a\right)$

$$=a\underset{x\to 0}{\text{Lt}}\ \frac{e^{ax}-1}{ax}=a.1=a.$$

(ii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{x^2}-1}{x}=\underset{x\to 0}{\text{Lt}}\left(\dfrac{e^{x^2}-1}{x^2}.x\right)$

$$=\underset{x^2\to 0}{\text{Lt}}\ \frac{e^{x^2}-1}{x^2}.\underset{x\to 0}{\text{Lt}}\ x=1.0=0$$

(iii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{\alpha x}-e^{\beta x}}{x}=\underset{x\to 0}{\text{Lt}}\ \dfrac{(e^{\alpha x}-1)-(e^{\beta x}-1)}{x}$

$=\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{\alpha x}-1}{x}-\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{\beta x}-1}{x}$

$=\underset{x\to 0}{\text{Lt}}\ \left(\dfrac{e^{\alpha x}-1}{\alpha x}.\alpha\right)-\underset{x\to 0}{\text{Lt}}\left(\dfrac{e^{\beta x}-1}{\beta x}.\beta\right)$

$=\alpha.\ \underset{x\to 0}{\text{Lt}}\ \dfrac{e^{\alpha x}-1}{\alpha x}-\beta.\ \underset{x\to 0}{\text{Lt}}\ \dfrac{e^{\beta x}-1}{\beta x}$

$=\alpha.1-\beta.1=\alpha-\beta.$

Ex. 11. Show that

(i) $\underset{x\to 0}{\text{Lt}}\ \dfrac{a^x-1}{x}=\log_e a$

(ii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{a^x-b^x}{x}=\log_e \dfrac{a}{b}$

(i) $\underset{x\to 0}{\text{Lt}}\ \dfrac{a^x-1}{x}=\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{\log a^x}-1}{x}$

$=\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{x\log a}-1}{x}=\underset{x\to 0}{\text{Lt}}\ \left(\dfrac{e^{x\log a}-1}{x\log_e a}.\log_e a\right)$

$=\log_e a\ \underset{x\to 0}{\text{Lt}}\ \dfrac{e^{x\log a}-1}{x\log_e a}$

$=\log_e a.1=\log_e a.$

(ii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{a^x-b^x}{x}=\underset{x\to 0}{\text{Lt}}\ \dfrac{(a^x-1)-(b^x-1)}{x}$

$=\underset{x\to 0}{\text{Lt}}\ \dfrac{a^x-1}{x}-\underset{x\to 0}{\text{Lt}}\ \dfrac{b^x-1}{x}$

$=\log_e a-\log_e b=\log_e \dfrac{a}{b}.$

Ex. 12. Evaluate :

(i) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\log(1+3x)}{x}$ (ii) $(1+x)^{\frac{1}{x}}$

(i) $\underset{x\to0}{\text{Lt}}\left\{\dfrac{\log(1+3x)}{3x}.\,3\right\}=3\underset{x\to0}{\text{Lt}}\dfrac{\log(1+3x)}{3x}=3.1=3.$

(ii) $\underset{x\to0}{\text{Lt}}\log(1+x)^{\frac{1}{x}}=\underset{x\to0}{\text{Lt}}\dfrac{1}{x}\log(1+x)=1.$

$\therefore\ \log\underset{x\to0}{\text{Lt}}(1+x)^{\frac{1}{x}}=1$

or, $\underset{x\to0}{\text{Lt}}(1+x)^{\frac{1}{x}}=e^1=e.$

Ex. 13. Evaluate the following limits.

(i) $\underset{x\to0}{\text{Lt}}\dfrac{e^{\sin x}-1}{x}$ [H. S. 1983]

(ii) $\underset{x\to0}{\text{Lt}}\dfrac{e^{x^2}-1}{\sin^2 x}$ [H. S. 1985]

(iii) $\underset{x\to0}{\text{Lt}}\dfrac{e^x-e^{-x}}{x}$ [H. S. 1987]

(iv) $\underset{x\to0}{\text{Lt}}\dfrac{\sin x}{\log_e(1+x)^{\frac{1}{2}}}$ [H. S. 1988]

(v) $\underset{x\to0}{\text{Lt}}\dfrac{2^x-1}{(1+x)^{\frac{1}{2}}-1}$ [I. I. T. 1982]

i) $\underset{x\to0}{\text{Lt}}\dfrac{e^{\sin x}-1}{x}=\underset{x\to0}{\text{Lt}}\left(\dfrac{e^{\sin x}-1}{\sin x}\cdot\dfrac{\sin x}{x}\right)$

$=\underset{x\to0}{\text{Lt}}\dfrac{e^{\sin x}-1}{\sin x}.\ \underset{x\to0}{\text{Lt}}\dfrac{\sin x}{x}=\underset{x\to0}{\text{Lt}}\dfrac{e^{\sin x}-1}{\sin x}\cdot1$

$=\underset{x\to0}{\text{Lt}}\dfrac{e^{\sin x}-1}{\sin x}.$

Now when, $x\to0$, then $\sin x\to0$

$\therefore$ given limit$=\underset{\sin x\to0}{\text{Lt}}\dfrac{e^{\sin x}-1}{\sin x}=1$

(ii) $$\underset{x\to 0}{\text{Lt}}\ \frac{e^{x^2}-1}{\sin^2 x}=\underset{x\to 0}{\text{Lt}}\ \left\{\frac{\dfrac{e^{x^2}-1}{x^2}}{\dfrac{\sin^2 x}{x^2}}\right\}$$

$$=\frac{\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{x^2}-1}{x^2}}{\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin x}{x}\ \underset{x\to 0}{\text{Lt}}\ \dfrac{\sin x}{x}}=\frac{1}{1\,.\,1}=1.$$

(iii) $$\underset{x\to 0}{\text{Lt}}\ \frac{e^{x}-e^{-x}}{x}=\underset{x\to 0}{\text{Lt}}\ \frac{(e^{x}-1)-(e^{-x}-1)}{x}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{e^{x}-1}{x}-\underset{x\to 0}{\text{Lt}}\ \frac{e^{-x}-1}{x}=\underset{x\to 0}{\text{Lt}}\ \frac{e^{x}-1}{x}+\underset{x\to 0}{\text{Lt}}\ \frac{e^{-x}-1}{-x}$$

$$=1+1=2.$$

(iv) $$\underset{x\to 0}{\text{Lt}}\ \frac{\sin x}{\log_e (1+x)^{\frac{1}{2}}}=\underset{x\to 0}{\text{Lt}}\ \frac{\dfrac{\sin x}{x}}{\dfrac{1}{2}\dfrac{\log_e (1+x)}{x}}$$

$$=\frac{\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin x}{x}}{\dfrac{1}{2}\underset{x\to 0}{\text{Lt}}\ \dfrac{\log (1+x)}{x}}=\frac{1}{\frac{1}{2}\,.\,1}=2.$$

(v) $$\underset{x\to 0}{\text{Lt}}\ \frac{2^{x}-1}{(1+x)^{\frac{1}{2}}-1}=\underset{x\to 0}{\text{Lt}}\ \frac{\dfrac{2^{x}-1}{x}}{\dfrac{\sqrt{1+x}-1}{x}}$$

$$=\frac{\underset{x\to 0}{\text{Lt}}\ \dfrac{2^{x}-1}{x}}{\underset{x\to 0}{\text{Lt}}\ \dfrac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}}=\frac{\log_e 2}{\underset{x\to 0}{\text{Lt}}\ \dfrac{1+x-1}{x(\sqrt{1+x}+1)}}$$

$$=\frac{\log_e 2}{\underset{x\to 0}{\text{Lt}}\ \dfrac{x}{x(\sqrt{1+x}+1)}}=\frac{\log_e 2}{\underset{x\to 0}{\text{Lt}}\ \dfrac{1}{\sqrt{1+x}+1}}=\frac{\log_e 2}{\frac{1}{2}}=2\log_c 2$$

$$=\log_e 2^2=\log_e 4.$$

Ex. 14. Evaluate :—

(i) $\displaystyle \mathop{\mathrm{Lt}}_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x}$ (ii) $\displaystyle \mathop{\mathrm{Lt}}_{x\to \infty} \frac{\sin x}{x}$

(i) $$\mathop{\mathrm{Lt}}_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x} = \mathop{\mathrm{Lt}}_{x\to 0} \left\{\frac{x \sin\left(\frac{1}{x}\right)}{\frac{\sin x}{x}}\right\}$$

$$= \frac{\mathop{\mathrm{Lt}}\limits_{x\to 0} x \sin\left(\frac{1}{x}\right)}{\mathop{\mathrm{Lt}}\limits_{x\to 0} \frac{\sin x}{x}} = \frac{0}{1} = 0.$$

(ii) Let $x = \frac{1}{y}$ $\therefore$ when $x \to \infty$ then $y \to 0+$

Now $$\mathop{\mathrm{Lt}}_{x\to \infty} \frac{\sin x}{x} = \mathop{\mathrm{Lt}}_{y\to 0+} \frac{\sin\left(\frac{1}{y}\right)}{\frac{1}{y}} = \mathop{\mathrm{Lt}}_{y\to 0} y \sin\left(\frac{1}{y}\right) = 0.$$

Ex. 15. (i) $$\mathop{\mathrm{Lt}}_{x\to 2} \frac{x^2-4}{x-2} = \mathop{\mathrm{Lt}}_{x\to 2} \frac{x^2}{x-4} - \mathop{\mathrm{Lt}}_{x\to 2} \frac{4}{x-2}$$

(ii) $$\mathop{\mathrm{Lt}}_{x\to 2} \frac{x^2-4}{x-2} = \mathop{\mathrm{Lt}}_{x\to 2} (x^2-4) . \mathop{\mathrm{Lt}}_{x\to 2} \frac{1}{x-2}$$

Are the two statements correct ?

(i) None of the limits $\displaystyle \mathop{\mathrm{Lt}}_{x\to 2} \frac{x^2}{x-2}$ and $\displaystyle \mathop{\mathrm{Lt}}_{x\to 2} \frac{4}{x-4}$

exist and so, $$\mathop{\mathrm{Lt}}_{x\to 2} \frac{x^2}{x-2} - \mathop{\mathrm{Lt}}_{x\to 2} \frac{4}{x-2} \neq \mathop{\mathrm{Lt}}_{x\to 2} \left(\frac{x^2}{x-2} - \frac{4}{x-2}\right)$$

$$= \mathop{\mathrm{Lt}}_{x\to 2} \frac{x^2-4}{x-2}$$

As $\displaystyle \mathop{\mathrm{Lt}}_{x\to 2} \frac{1}{x-2}$ does not exist, so

$$\mathop{\mathrm{Lt}}_{x\to 2} (x^2-4) \times \mathop{\mathrm{Lt}}_{x\to 2} \frac{1}{x-2} \neq \mathop{\mathrm{Lt}}_{x\to 2} \left\{(x^2-4) . \frac{1}{x-2}\right\}$$

$$= \mathop{\mathrm{Lt}}_{x\to 2} \frac{x^2-4}{x-2}$$

Ex. 16. (i) $f(x)=x$ when $x \geqslant 0$

$= -x$ when $x < 0$

Determine $\underset{x\to 0}{\text{Lt}}\ f(x)$ if it exists.

(ii) Examine the existence of $\underset{x\to 0}{\text{Lt}}\ f(x)=\underset{x\to 0}{\text{Lt}}\ \dfrac{|x|}{x}$

(iii) $f(x)=[x]$, where $[x]$ is the greatest integer less than or equal to x.

Determine $\underset{x\to 2}{\text{Lt}}\ [x]$

(iv) $f(x)=\sin x,\ x \neq n\pi\ [x=0, \pm 1, \pm 2, \pm 3, \cdots\cdots]$

$= 2$ (otherwise)

and $g(x)=x^2+1, \quad x \neq 0, 2.$

$= 4 \qquad x=0$

$= 5 \qquad x=2.$

Determine $\underset{x\to 0}{\text{Lt}}\ g[f(x)]$ If it exists I.I.T. 1986]

(v) $G(x) = -\sqrt{25-x^2}$

Evaluate $\underset{x\to 1}{\text{Lt}}\ \dfrac{G(x)-G(1)}{x-1}$ [e. f. I.I.T 1983]

(vi) A function $f(x)$ is defined as follows.

$f(x)=x^2$ when $x<1$

$= 3$ when $x=1$

$= x^2+2$ when $x>1$

Examine the existence of $\underset{x\to 1}{\text{Lt}}\ f(x)$

(vii) $f(x)=\dfrac{\sin [x]}{[x]},\ [x] \neq 0$

$= 0, \qquad [x]=0.$

where $[x]$ is the Greatest integer less than or equal to x

Examine whether $\underset{x\to 0}{\text{Lt}}\ f(x)$ exists or not. [I. I. T. 1985]

(viii) Evaluate $\underset{x\to 1}{\text{Lt}}\left[x^2+\sqrt{x-1}\right]$

if it exists.

(i) $\underset{x\to 0+}{\text{Lt}} f(x)=\underset{x\to 0+}{\text{Lt}} x=0$

$\underset{x\to 0-}{\text{Lt}} f(x)=\underset{x\to 0-}{\text{Lt}} (-x)=0.$

$\therefore \underset{x\to 0+}{\text{Lt}} f(x)=\underset{x\to 0-}{\text{Lt}} f(x)=0.$

$\therefore \underset{x\to 0}{\text{Lt}} f(x)=0.$

(ii) $\underset{x\to 0+}{\text{Lt}} \frac{|x|}{x}=\underset{x\to 0+}{\text{Lt}} \frac{x}{x}=\underset{x\to 0+}{\text{Lt}} (1)=1$

$\underset{x\to 0-}{\text{Lt}} \frac{|x|}{x}=\underset{x\to 0-}{\text{Lt}} \frac{-x}{x}=\underset{x\to 0-}{\text{Lt}} (-1)=-1$

$\therefore \underset{x\to 0+}{\text{Lt}} \frac{|x|}{x}\neq\underset{x\to 0-}{\text{Lt}} \frac{|x|}{x}$

$\therefore \underset{x\to 0}{\text{Lt}} \frac{|x|}{x}$ does not exist.

(iii) $\underset{x\to 2-}{\text{Lt}} [x]=\underset{x\to 2-}{\text{Lt}} (1)=1$

$\underset{x\to 2+}{\text{Lt}} [x]=\underset{y\to 2+}{\text{Lt}} (2)=2$

[Note. when $x\to 2-$ then $x<2$, so $[x]=1$

when $x\to 2+$ then $x>2$; so $[x]=2$]

$\therefore \underset{x\to 2-}{\text{Lt}} [x]\neq\underset{x\to 2+}{\text{Lt}} [x]$

$\therefore \underset{x\to 2}{\text{Lt}} [x]$ does not exist

(iv) $\underset{x\to 0}{\text{Lt}} g[f(x)]=\underset{x\to 0}{\text{Lt}} [\{f(x)\}^2+1]$

$=\underset{x\to 0}{\text{Lt}} [\sin^2 x+1]=1.$

(v) $\underset{x\to 1}{\text{Lt}} \dfrac{G(x)-G(1)}{x-1}$

$$=\underset{x\to 1}{\text{Lt}} \frac{-\sqrt{25-x^2}+\sqrt{25-1}}{x-1}$$

$$=\underset{x\to 1}{\text{Lt}} \frac{(\sqrt{24}-\sqrt{25-x^2})(\sqrt{24}+\sqrt{25-x^2})}{(\sqrt{24}+\sqrt{25-x^2})(x-1)}$$

$$=\underset{x\to 1}{\text{Lt}} \frac{24-25+x^2}{(\sqrt{24}+\sqrt{25-x^2})(x-1)}$$

$$=\underset{x\to 1}{\text{Lt}} \frac{x^2-1}{(\sqrt{24}+\sqrt{25-x^2})(x-1)}=\underset{x\to 1}{\text{Lt}} \frac{x+1}{\sqrt{24}+\sqrt{25-x^2}}$$

$$=\frac{2}{\sqrt{24}+\sqrt{24}}=\frac{2}{2\sqrt{24}}=\frac{1}{\sqrt{24}}.$$

(vi) $\underset{x\to 1-}{\text{Lt}} f(x)=\underset{x\to 1-}{\text{Lt}} x^2=1.$

$\underset{x\to 1+}{\text{Lt}} f(x)=\underset{x\to 1+}{\text{Lt}} (x^2+2)=3.$

$\therefore \underset{x\to 1-}{\text{Lt}} f(x)\neq \underset{x\to 1+}{\text{Lt}} f(x).$

$\therefore \underset{x\to 1}{\text{Lt}} f(x)$ does not exist.

(vii) when $x\to 0-$, $x<0$ but is very close to 0.

$\therefore$ when $x\to 0-$, $[x]=-1$

when $x\to 0+$, $x>0$ but is very close to 0.

$\therefore$ when $x\to 0+$, $[x]=0$.

Now $\underset{x\to 0-}{\text{Lt}} f(x)=\underset{x\to 0-}{\text{Lt}} \dfrac{\sin [x]}{[x]}=\underset{x\to 0}{\text{Lt}} \dfrac{\sin(-1)}{-1}$

$$=\underset{x\to 0}{\text{Lt}} \frac{-\sin 1}{-1}=\sin 1$$

Also $\underset{x\to 0+}{\text{Lt}} f(x)=\underset{x\to 0+}{\text{Lt}} \dfrac{\sin [x]}{[x]}=\underset{x\to 0+}{\text{Lt}} \dfrac{\sin 0}{0}$

which does not exist

$\therefore \lim\limits_{x\to 0} \dfrac{\sin [x]}{[x]}$ does not exist

Note : Here evaluation of $\lim\limits_{x\to 0-} f(x)$ is not necessary.

(viii) $x\to 1-$ means that the value of x is very close to 1 but less than 1. So when $x\to 1-$, then $(x-1)$ is always negative and $\sqrt{x-1}$ is imaginary. So, $x^2+\sqrt{x-1}$ is imaginary.

$\therefore \lim\limits_{x\to 1-} (x^2+\sqrt{x-1})$ does not exist.

i.e., $\lim\limits_{x\to 1} (x^2+\sqrt{x-1})$ does not exist.

Ex. 17. Evaluate the following limits.

(i) $\lim\limits_{x\to\infty}\left\{\sqrt{x^4-x^2+2}-x^2\right\}$

(ii) $\lim\limits_{x\to\infty} \dfrac{1^2+2^2+3^2+\cdots+x^2}{x^3}$

[x is a natural number]

(iii) $\lim\limits_{x\to\infty}\left\{\dfrac{1}{1-x^2}+\dfrac{2}{1-x^2}+\cdots+\dfrac{x}{1-x^2}\right\}$, x is a natural number.

[I. I. T. 1984]

(iv) $\lim\limits_{x\to\infty}\left[\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\cdots+\dfrac{1}{x(x+1)}\right]$

(v) $\lim\limits_{x\to-\infty}\left[\dfrac{x^4\sin\left(\frac{1}{x}\right)+x^2}{1+(|x|)^2}\right]$

(vi) $\lim\limits_{x\to\pi} \dfrac{1}{\pi-x}$.

(i) $\lim\limits_{x\to\infty}\left\{\sqrt{x^4-x^2+2}-x^2\right\}$

$$=\lim_{x\to\infty} \frac{\{\sqrt{x^4-x^2+2}-x^2\}\{\sqrt{x^4-x^2+2}+x^2\}}{\sqrt{x^4-x^2+2}+x^2}$$

$$=\lim_{x\to\infty}\frac{x^4-x^2+2-x^4}{\sqrt{x^4-x^2+2}+x^2}$$

$$=\lim_{x\to\infty}\frac{x^2\left(-1+\frac{2}{x^2}\right)}{x^2\left(\sqrt{1-\frac{1}{x^2}+\frac{2}{x^4}}+1\right)}$$

$$=\lim_{y\to 0}\frac{-1+2y^2}{\sqrt{1-y^2+2y^4}+1}$$

$\left[\text{Let } y=\frac{1}{x};\quad \text{so when } x\to\infty, \text{ then } y\to 0+\right]$

$$=\frac{-1}{\sqrt{1}+1}=-\frac{1}{2}.$$

(ii) $\lim_{x\to\infty}\frac{1^2+2^2+3^2+\cdots+x^2}{x^3}$

$$=\lim_{x\to\infty}\frac{x(x+1)(2x+1)}{6x^3}=\lim_{x\to\infty}\frac{2x^2\left(1+\frac{1}{x}\right)\left(1+\frac{1}{2x}\right)}{6x^2}$$

$$=\lim_{\frac{1}{x}\to 0+}\frac{\left(1+\frac{1}{x}\right)\left(1+\frac{1}{2x}\right)}{3}=\tfrac{1}{3}.$$

(iii) $\lim_{x\to\infty}\left[\frac{1}{1-x^2}+\frac{2}{1-x^2}+\cdots+\frac{x}{1-x^2}\right]$

$$=\lim_{x\to\infty}\frac{1+2+3+\cdots+x}{1-x^2}$$

$$=\lim_{x\to\infty}\frac{x(x+1)}{2(1-x^2)}=\lim_{x\to\infty}\frac{x^2\left(1+\frac{1}{x}\right)}{-2x^2\left(1-\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{1}{-2\left(1+\frac{1}{x}\right)}=-\tfrac{1}{2}$$

$\left[\text{when } x\to\infty, \text{ then } \frac{1}{x}\to 0+\right]$

(iv) $\underset{x\to\infty}{\text{Lt}}\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{x(x+1)}\right]$

$=\underset{x\to\infty}{\text{Lt}}\left[1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{x}-\frac{1}{x+1}\right]$

$=\underset{x\to\infty}{\text{Lt}}\left[1-\frac{1}{x+1}\right]=\underset{x\to\infty}{\text{Lt}}\ \frac{x}{x+1}=\underset{x\to\infty}{\text{Lt}}\ \frac{x}{x\left(1+\frac{1}{x}\right)}$

$=\underset{\frac{1}{x}\to 0+}{Lt}\ \frac{1}{\left(1+\frac{1}{x}\right)}=1$ $\left[\text{as when } x\to\infty,\ \text{then } \frac{1}{x}\to 0+\right]$

(v) $\underset{x\to-\infty}{Lt}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{1+(\,|\,x\,|\,)^3}\right]$

$=\underset{x\to-\infty}{Lt}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{1-x^3}\right]$

($\because$ $x\to-\infty$, $\therefore$ $x<0$ and $|\,x\,|=-x$)

$=\underset{y\to 0-}{Lt}\left[\frac{\frac{1}{y^4}\sin(y)+\frac{1}{y^2}}{1-\frac{1}{y^3}}\right]$ $\left(x=\frac{1}{y}\text{ (say), when } x\to-\infty,\ \text{then } y\to 0-\right)$

$=\underset{y\to 0-}{Lt}\left[\frac{\sin y+y^2}{y(y^3-1)}\right]$

$=\underset{y\to 0-}{Lt}\left\{\frac{\sin y}{y}\cdot\frac{1}{y^3-1}\right\}+\underset{y\to 0}{Lt}\left\{\frac{y^2}{y(y^3-1)}\right\}$

$=\underset{y\to 0-}{Lt}\ \frac{\sin y}{y}\cdot\underset{y\to 0-}{Lt}\left(\frac{1}{y^3-1}\right)+\underset{y\to 0}{Lt}\ \frac{y}{y^3-1}=1.(-1)+0=-1.$

(vi) Let $\pi-x=y$ or $y=\frac{1}{\pi-x}$

$\therefore$ when $x\to\pi-$ then $y=\frac{1}{\pi-x}\to\infty.$

when $x\to\pi+$then, $y=\frac{1}{\pi-x}\to-\infty$

$$\therefore \underset{x\to\pi-}{Lt} \frac{1}{\pi-x}=\underset{y\to\infty}{Lt} y=\infty.$$

$$\underset{x\to\pi+}{Lt} \frac{1}{\pi-x}=\underset{y\to-\infty}{Lt} y=-\infty$$

$\therefore \underset{x\to\pi}{Lt} \dfrac{1}{\pi-x}$ does not exist.

Ex. 18. show that

(i) $\underset{h\to 0}{Lt} \dfrac{(a+h)^n-a^n}{h}==na^{n-1}$

(ii) $\underset{h\to 0}{Lt} \dfrac{e^{a+h}-e^a}{e}=e^a$

(iii) $\underset{h\to 0}{Lt} \dfrac{\log(a+h)-\log a}{h}=\dfrac{1}{a}.$

(i) Let $a+h=x$ $\therefore$ $h=x-a$ and as $h\to 0$,

then $(x-a)\to 0$ or, $x\to a$.

So, given limit$=\underset{x\to a}{Lt} \dfrac{x^n-a^n}{x-a}=na^{n-1}$

(ii) $\underset{h\to 0}{Lt} \dfrac{e^{a+h}-e^a}{h}=\underset{h\to 0}{Lt} \dfrac{e^a(e^h-1)}{h}$

$$=e^a \underset{h\to 0}{Lt} \frac{e^h-1}{h}=e^a\ 1=e^a.$$

(iii) $\dfrac{\log(a+h)-\log a}{h}=\underset{h\to 0}{Lt} \dfrac{\log\left(\dfrac{a+h}{a}\right)}{h}$

$$=\underset{h\to 0}{Lt} \frac{\log\left(1+\frac{h}{a}\right)}{\frac{h}{a}\cdot a}=\frac{1}{a} \underset{\frac{h}{a}\to 0}{Lt} \frac{\log\left(1+\frac{h}{a}\right)}{\frac{h}{a}}=\frac{1}{a}.\ 1=\frac{1}{a}$$

Ex. 19. For all values of x determine the value of $f(x)$,

where $f(x)=\underset{x\to\infty}{Lt} \dfrac{1}{1+x^n}$.

when $-1<x\leqslant 1$ i. e., $|x|\leqslant 1$, then $0\leqslant x^{2n}\leqslant 1$

$$\therefore \underset{x\to\infty}{Lt}\ x^{2n}=0$$

and $\underset{x\to\infty}{Lt}\ \dfrac{1}{1+x^{2n}}=\dfrac{1}{1}=1.$

when $|x|=1$, i. e. $x=-1$ or, 1, then $x^{2n}=(x^2)^n$ $=\{(-1)^2\}^n$ or, $\{(1)^2\}^n=1$ or, 1 (however large n may be)

$$\therefore \underset{x\to\infty}{Lt}\ \frac{1}{1+x^{2n}}=\underset{x\to\infty}{Lt}\ \frac{1}{1+1}=\tfrac{1}{2}.$$

when $|x|>1$ i. e. $x<-1$ or, $x>1$,

Then x^2 is positive and >1. So as the value of x will increase, So the value of x^{2n} will also increase. So $y=(1+x^{2n})\to\infty$ i. e.

$$\underset{x\to\infty}{Lt}\ \frac{1}{1+x^{2n}}=\underset{y\to\infty}{Lt}\ \frac{1}{y}=\underset{\frac{1}{y}\to 0+}{Lt}\ \left(\frac{1}{y}\right)=0.$$

$\therefore\ f(x)=1,\ \tfrac{1}{2}$ or, 0 according as $|x|<1, =1$ or, $1>1$

Exercise 3

Evaluate the limits :—

1. (i) $\underset{x\to2}{Lt}\ (4x^3+3x^2-5x+7)$

(ii) $\underset{x\to3}{Lt}\ \dfrac{2x+3}{x^2+3}$ (iii) $\underset{x\to-1}{Lt}\ (x^2+4)(2x+1)$

(iv) $\underset{x\to0}{Lt}\ \dfrac{ax+b}{cx+d}\ (d\neq0)$

2. (i) $\underset{x\to0}{Lt}\ \dfrac{x}{x}$ (ii) $\underset{x\to0}{Lt}\ \dfrac{x^3}{x^2}$ (iii) $\underset{x\to4}{Lt}\ \dfrac{x^2-16}{x-4}$

(iv) $\underset{x\to1}{Lt}\ \dfrac{x^3-1}{x-1}$ (v) $\underset{x\to-\frac{1}{2}}{Lt}\ \dfrac{4x^2-1}{2x+1}$

3. (i) $\underset{x\to2}{Lt}\ \dfrac{x^2-7x+12}{x^2+x-2}$ (ii) $\underset{x\to3}{Lt}\ \dfrac{x^2-7x+12}{x^2+x-2}$

(iii) $\underset{x\to-2}{Lt}\ \dfrac{x^2-x-6}{x^2+x-2}$

4. (i) $\underset{x\to 0}{Lt}\ \dfrac{ax^2+bx+c}{px^2+qx+r}\ (r\neq 0)$

(ii) $\underset{x\to \infty}{Lt}\ \dfrac{ax^2+bx+c}{px^2+qx+r}\ (p\neq 0)$

5. (i) $\underset{h\to 0}{Lt}\ \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ (ii) $\underset{h\to 0}{Lt}\left\{\dfrac{1}{h}\left(\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}\right)\right\}$

(iii) $\underset{x\to 0}{Lt}\ \dfrac{\sqrt{1+x}-1}{x}$ (iv) $\underset{x\to 0}{Lt}\ \dfrac{a-\sqrt{a^2-x^2}}{x^2}$

(v) $\underset{x\to 0}{Lt}\ \dfrac{\sqrt{1+2x}-\sqrt{1-3x}}{x}$.

6. (i) $\underset{x\to 0}{Lt}\ \dfrac{\sqrt{1+x^3}-\sqrt{1+x}}{\sqrt{1+x^4}-\sqrt{1+x}}$ (ii) $\dfrac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$

(iii) $\underset{x\to 0}{Lt}\ \dfrac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^4}-\sqrt{1+x}}$

7. (i) $\underset{x\to 0}{Lt}\ \dfrac{x}{\sqrt[3]{x+1}-1}$ (ii) $\underset{x\to 4}{Lt}\ \dfrac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}}$

8. (i) $\underset{x\to \frac{1}{5}}{Lt}\ \dfrac{2-10x}{3-15x}$ (ii) $\underset{x\to 3}{Lt}\ \dfrac{x^2-3x}{x-3}$

(iii) $\underset{u\to 2}{Lt}\left[\dfrac{1}{u-2}-\dfrac{4}{u^2-4}\right]$

9. (i) $\underset{x\to 3}{Lt}\ \dfrac{x^3-125}{x-5}$ (ii) $\underset{x\to 3}{Lt}\ \dfrac{x^3-125}{x^4-625}$

(iii) $\underset{x\to 1}{Lt}\ \dfrac{x^m-1}{x^n-1}$ (iv) $\underset{x\to a}{Lt}\ \dfrac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x^{\frac{4}{7}}-a^{\frac{4}{7}}}$

(v) $\underset{x\to 1}{Lt}\ \dfrac{x^2+2x+1}{x^{-1}+1}$ (vi) $\underset{x\to -2}{Lt}\ \dfrac{x^3+8}{x^5+32}$

(vii) $\underset{x\to 0}{Lt}\ \dfrac{(1+x)^5-1}{x}$ (viii) $\underset{x\to -2}{Lt}\ \dfrac{x^2-4}{x+2}$ [C. U.]

(ix) $\underset{h\to 0}{Lt}\ \dfrac{(x+h)^6-x^6}{h}$ (x) $\underset{x\to 1}{Lt}\left\{\dfrac{2}{1-x^2}+\dfrac{1}{x-1}\right\}$

(xi) $\underset{h\to 0}{Lt} \dfrac{\{a(x+h)^2+b(x+h)+c\}-(ax^2+bx+c)}{h}$

10. (i) $\underset{x\to 0}{Lt} \dfrac{x}{\sqrt{1+x}-1}$ (ii) $\underset{x\to 0}{Lt} \dfrac{x}{2-\sqrt{4-x}}$

(iii) $\underset{x\to 1}{Lt} \dfrac{\sqrt{3+x}+\sqrt{5-x}}{x^2-1}$ [Tripura 1986]

(iv) $\underset{x\to 0}{Lt} \dfrac{x^2}{a-\sqrt{a^2-x^2}}$ (v) $\underset{x\to 0}{Lt} \dfrac{3-\sqrt{9-x^2}}{x^2}$

11. (i) $\underset{x\to 0}{Lt} \dfrac{\sin x^\circ}{x}$ (ii) $\underset{x\to 0}{Lt} \dfrac{\tan x}{x}$

(iii) $\underset{x\to 0}{Lt} \dfrac{\tan x^0}{x}$ (iv) $\underset{x\to 0}{Lt} \dfrac{\tan^{-1}x}{x}$

12. (i) $\underset{x\to 0}{Lt} \dfrac{\sin 2x}{3x}$ (ii) $\underset{x\to 0}{Lt} \dfrac{\sin 4x}{4}$ (iii) $\underset{x\to 0}{Lt} \dfrac{\sin x}{3x}$

(iv) $\underset{x\to 0}{Lt} \dfrac{\sin \frac{x}{3}}{x}$ [Tripura '78]

(v) $\underset{x\to 0}{Lt} \dfrac{\sin \frac{x}{2}}{x}$ [Tripura '79]

(vi) $\underset{x\to 0}{Lt} \dfrac{x-\sin 3x}{x-\sin 4x}$

13. (i) $\underset{x\to 0}{Lt} \dfrac{1-\cos 2x}{x^2}$ (ii) $\underset{x\to 0}{Lt} \dfrac{1-\cos 2x}{x}$

(iii) $\underset{x\to 0}{Lt} \dfrac{\operatorname{cosec} x-\cot x}{x}$ (iv) $\underset{x\to 0}{Lt} \dfrac{x^3}{1-\cos x}$

(v) $\underset{x\to 0}{Lt} \dfrac{\sin 4x}{\sin 3x}$ (vi) $\underset{\theta\to 0}{\mathrm{Lt}} \tan 4\theta \operatorname{cosec} 2\theta$

(vii) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{x(\cos ax-\cos bx)}{x^3}$ (viii) $\underset{x\to 0}{Lt} \dfrac{(\cos 4x-\cos 6x)}{x^2}$

(ix) $\underset{x\to a}{Lt} \dfrac{\sin(x-a)}{x-a}$ (x) $\underset{x\to 0}{Lt} \dfrac{\sin ax}{\tan bx}$

(xi) $\underset{x\to 0}{Lt} x \operatorname{cosec} x$ (xii) $\underset{x\to \frac{\pi}{4}}{Lt} \dfrac{1-\tan x}{1-\cot x}$

(xiii) $\underset{x\to 0}{Lt} \dfrac{\sin^2 x \cos x}{x^2}$

14. (i) $\underset{x\to\frac{\pi}{2}}{Lt} \dfrac{\cos x}{\frac{\pi}{2}-x}$ (ii) $\underset{x\to\frac{\pi}{4}}{Lt} \dfrac{\cos 2x}{x-\frac{\pi}{4}}$

(iii) $\underset{x\to\frac{\pi}{2}}{Lt} \dfrac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}$ (iv) $\underset{x\to\frac{\pi}{2}}{Lt} (\sec x-\tan x)$

(v) $\underset{x\to\frac{\pi}{4}}{Lt} \dfrac{\sin x-\cos x}{\frac{\pi}{4}-x}$ (vi) $\underset{x\to\pi}{Lt} \dfrac{x^2+1}{\cos x}$

(vii) $\underset{x\to\frac{\pi}{2}}{Lt} \dfrac{1 \quad \sin^3 x}{\cos^2 x}$ (ix) $\underset{\theta\to\pi}{Lt} \dfrac{\cos\theta+\cos 2\theta}{\tan^2\theta}$

(x) $\underset{x\to\pi}{Lt} \dfrac{1+\cos x}{\tan^2 x}$ (xi) $\underset{\theta\to\frac{\pi}{2}}{Lt} \dfrac{\cot^2\theta-3}{\operatorname{cosec}\theta-2}$

15. (i) $\underset{x\to 0}{Lt} \dfrac{e^{5x}-1}{x}$ (ii) $\underset{x\to 0}{Lt} \dfrac{e^x-1}{2x}$ (iii) $\underset{x\to 0}{Lim} \dfrac{e^{mx}-1}{nx}$.

(iv) $\underset{x\to 0}{Lt} \dfrac{ae^x+be^{-x}}{a+b}$ (v) $\underset{x\to 0}{Lt} \dfrac{e^{\tan x}-1}{x}$

16. (i) $\underset{x\to 0}{Lt} \dfrac{\log(1+ax)}{x}$ (ii) $\underset{x\to 0}{Lt} \dfrac{\log(1+2x)}{\sin 3x}$

(iii) $\underset{x\to 0}{Lt} \dfrac{\log(1+3x)}{\tan 2x}$ (iv) $\underset{x\to 0}{Lim} (1+2x)^{\frac{1}{x}}$ (v) $\underset{x\to 0}{Lim} (1+ax)^{\frac{1}{x}}$

(vi) $\underset{x\to 1}{Lim} \dfrac{\log x}{x-1}$ (vii) $\underset{x\to 0}{Lim} \dfrac{2^x-3^x}{x}$

17. (i) $\underset{x\to 0}{Lt} \dfrac{\log(1+\sin x)}{x}$ (ii) $\underset{x\to 0}{Lt} \dfrac{\log(1+ax)}{\sin bx}$

(iii) $\underset{x\to 0}{Lim} \dfrac{\sin\log(1+x)}{\log(1+\sin x)}$

18. (i) $\underset{x\to 0}{Lt}\, x\cos\left(\dfrac{1}{x}\right)$ (ii) $\underset{x\to 0}{Lt} \dfrac{x^2\cos\left(\frac{1}{x}\right)}{\sin x}$

19. A function $f(x)$ is defined as follows :
$f(x)=1$, 0 or -1 according as $x>0$, $=0$, or <0.

Find $\underset{x\to 0}{Lim} f(x)$.

20. $f(x)=2x+1$ when $x\geqslant 1$
$=2x-1$ when $x<1$

comment on the existence of $\underset{x\to 1}{Lt} f(x)$.

21. A function $f(x)$ is defined as follows :
$f(x)=x$ when $0\leqslant x<1$
$=2$ when $x=1$
$=2-x$ when $1<x\leqslant 2$.

Determine $\underset{x\to 1}{Lt} f(x)$.

22. A function $f(x)$ is defined as follows :
$f(x)=x^2$ when $x<1$
$=2{\cdot}5$ when $x=1$
$=x^2+2$ when $x>1$.

Does $\underset{x\to 1}{Lt} f(x)$ exist ?

23. A function $f(x)$ is defined as follows :
$f(x)=-x$ when $x\leqslant 0$
$=x$ when $0<x<1$
$=2-x$ when $x\geqslant 1$.

Determine $\underset{x\to 0}{Lt} f(x)$ and $\underset{x\to 0}{Lt} f(x)$.

24. Show that

(i) $\underset{x\to\infty}{Lt} e^{\frac{1}{x}}=1$ (ii) $\underset{n\to\infty}{Lt} \left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}\right)=2$

(iii) $\underset{n\to\infty}{Lt} \dfrac{1^3+2^3+3^3+\cdots+n^3}{n^4}=\dfrac{1}{4}$.

(iv) $\underset{n\to\infty}{Lt} \dfrac{1.2+2.3+3.4+\cdots+n(n+1)}{n^3}=\dfrac{1}{3}$.

25. Show that $\underset{n\to\infty}{Lt}\ \dfrac{n^2+n+1}{2n+3}$ does not exist.

26. Prove that (i) $\underset{x\to\infty}{Lt}\ \dfrac{ae^x+be^{-x}}{e^x+e^{-x}}=a$

(ii) $\underset{x\to-\infty}{Lt}\ \dfrac{ae^x+be^{-x}}{e^x+e^{-x}}=b.$

27. If n be a positive integer and $a_0, a_1, a_2, \cdots, a_n$ be constants, show that

$$\underset{x\to\infty}{Lt}\ (a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n)=\infty$$

28. Prove that $\underset{x\to\infty}{Lt}\ \dfrac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}=0.$

29. If x be a constant, evaluate $\underset{x\to\infty}{Lt}\ \dfrac{x^n}{n\,!}$.

3 . If $f(x)=\dfrac{\sqrt{x^4+1}-2x^2-1}{x^2}$, show that

(i) $\underset{x\to0}{Lt}\ f(x)=-2$ (ii) $\underset{x\to\infty}{Lt}\ f(x)=-1.$

31. Evaluate : $\underset{n\to\infty}{Lt}\ \dfrac{1}{1+\sin^2 nx}$

32. Evaluate the limit $\underset{x\to\infty}{Lt}\ \dfrac{x^n-1}{x^n+1}$

33. Show that (i) $\underset{x\to0}{Lt}\ (1+x)^{\frac{1}{x}}=e$ (ii) $\underset{x\to\infty}{Lt}\ \left(1+\dfrac{1}{x}\right)^x=e$

34. Evaluate the limits

(i) $\underset{x\to1}{Lt}\ \dfrac{\cos\left(\dfrac{\pi x}{2}\right)}{1-\sqrt{x}}$ (ii) $\underset{x\to1}{Lt}\ \dfrac{\sqrt{x}-1-\sqrt{x-1}}{\sqrt{x^2-1}}$

35. Evaluate the limits :

(i) $\underset{x\to2}{Lt}\ \dfrac{x^3\log(x-1)}{x^2-4}$ (ii) $\underset{x\to0}{Lim}\ \dfrac{\sin x^3(1-\cos x^3)}{x^9}$

36. Evaluate :

(i) $\lim\limits_{x\to 0} \dfrac{12^x - 4^x - 3^x + 1}{x^2}$ (ii) $\lim\limits_{x\to a} \dfrac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$

(iii) $\mathop{Lt}\limits_{x\to\infty} \sqrt{\dfrac{x - \sin x}{x + \cos^2 x}}$ [I. I. T. 1979]

37. If m and n be integers and $a_0 \neq 0$, $b_0 \neq 0$, then prove that

$$\lim_{x\to\infty} \frac{a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n}$$

$=0$ when $m<n$, $=\dfrac{a_0}{b_0}$ when $m=n$; $=\infty$ when $m>n$.

38. Prove that, $\lim\limits_{x\to\infty} \dfrac{x^n g(x) + h(x)}{x^n + 1}$

$=h(x)$ when $0<x<1$; $=\frac{1}{2}\{h(x)+g(x)\}$ when $x=1$.
$=g(x)$ when $x>1$

CHAPTER FOUR

CONTINUITY

§ 4·1. We have discussed about continuity of a function while discussing graphs of a function in chapter two. There we have said that if the graph of a function $f(x)$ is broken at the point $\{x, f(x)\}$ corresponding to some x, then the function is said to be discontinuous at the point ; if the function be not broken at the point, then the function is continuous at the point. If the graph of a function is not broken any where, then the function is continuous every here. In the next section we proceed to discuss continuity of a function mathematically.

§ 4·2. Analytical definition of the continuity of a function at a point.

If $\underset{x \to a}{Lt} f(x) = f(a)$, then the function $f(x)$ is continuous at the point $x=a$. Hence the conditions of continuity of a function at a point $x=a$ are

(i) $\underset{x \to a}{Lt} f(x)$ must have a finite value

(ii) $f(a)$ must possess a finite value

(ii) $\underset{x \to a}{Lt} f(x) = f(a)$

Hence if any one of the above three conditions fail, the function will be discontinuous at the point $x=a$.

Again the condition for the existence of $\underset{x \to a}{Lt} f(x)$ is that both of $\underset{x \to a-}{Lt} f(x)$ and $\underset{x \to a+}{Lt} f(x)$ must exist finitely and $\underset{x \to a-}{Lt} f(x) = \underset{x \to a+}{Lt} f(x)$.

So if any of these two limits does not exist, then the function is discontinuous at the point.

Example 1. A function $f(x)$ is defined as follows :

$$f(x)=2x-1 \text{ when } x\geqslant 1$$
$$=x \text{ when } x<1.$$

Let us discuss the continuity of the function at $x=1$.

$\underset{x\to 1+}{Lt} f(x)=\underset{x\to 1+}{Lt}(2x-1)=1$ [when $x>1$, then $f(x)=2x-1$]

$\underset{x\to 1-}{Lt} f(x)=\underset{x\to 1-}{Lt} x=1$ [when $x<1$, then $f(x)=x$]

$\therefore \underset{x\to 1-}{Lt} f(x)=\underset{x\to 1+}{Lt} f(x)=1$

So, $\underset{x\to 1}{Lt} f(x)=1.$

Also $f(1)=2.1-1=2-1=1$

$\therefore \underset{x\to 1}{Lt} f(x)=f(1)$ and the function is continuous at $x=1$.

Ex. 2. A function $f(x)$ is defined in the following manner.

$$f(x)=x^2 \text{ when } x>2$$
$$=3x \text{ when } x=2$$
$$=3x-2 \text{ when } x<2.$$

Examine the continuity of the function at $x=2$.

Here $\underset{x\to 2+}{Lt} f(x)=\underset{x\to 2+}{Lt} x^2=4.$

$\underset{x\to 2-}{Lt} f(x)=\underset{x\to 2-}{Lt}(3x-2)=6-2=4$

$\therefore \underset{x\to 2+}{Lt} f(x)=\underset{x\to 2-}{Lt} f(x)=4.$

$\therefore \underset{x\to 2}{Lt} f(x)=4.$

But $f(2)=3.2=6.$

So, $\underset{x\to 2}{Lt} f(x)\neq f(2).$

Hence the function is not continuous i.e., discontinuous at $x=2$.

Ex. 3. A function $f(x)$ is defined in the following manner.

$$f(x)=3x+2 \text{ when } x\leqslant 1$$
$$=x^2 \text{ when } x>1.$$

Discuss the continuity of the function at $x=1$.

Here $\underset{x\to 1-}{Lt} f(x)=\underset{x\to 1-}{Lt}(3.1+2)=5$

$$\underset{x\to 1+}{Lt} f(x)=\underset{x\to 1+}{Lt}(x^2)=1$$

$$\therefore \quad \underset{x\to 1-}{Lt} f(x)\neq \underset{x\to 1+}{Lt} f(x)$$

So, $\underset{x\to 1}{Lt} f(x)$ does not exist and so the function is discontinuous at $x=1$

Ex. 4. Discuss the continuity of the function $f(x)=\dfrac{\sin x}{x}$ at the point $x=0$.

Here $f(0)$ is undefined i.e., does not exist and any question of continuity does not arise.

Note. In Example 1, $\underset{x\to 1}{Lt} f(x)$ and $f(1)$ both exist and they are equal and so $f(x)$ is continuous at $x=1$. In ex. 2. both $\underset{x\to 2}{Lt} f(x)$ and $f(2)$ exist ; but the function is discontinuous at $x=2$ due to the inequality of $\underset{x\to 2}{Lt} f(x)$ and $f(2)$.

In ex. 3. $\underset{x\to 1}{Lt} f(x)$ does not exist but $f(1)$ exists. In ex. 4 $f(0)$ does not exist but $\underset{x\to 0}{Lt} f(x)$ exists $\left(\text{you know } \underset{x\to 0}{Lt} \dfrac{\sin x}{x}=1\right)$.

§ 4·3. Continuity of a function in an interval.

$a\leqslant x\leqslant b$ is a closed interval. If c be an interior point of the interval, i.e., if $a<c<b$, then the continuity of $f(x)$ at $x=c$ demands

$$\underset{x\to c-}{Lt} f(x)=\underset{x\to c+}{Lt} f(x)=f(c)$$

But for the two end points a and b we shall consider only one sided limit. For the left end point a, $\underset{x\to a+}{Lt} f(x)=f(a)$ and for the end point b, $\underset{x\to b-}{Lt} f(x)=f(b)$ are to be ensured for continuity. In these cases the other two limits, viz., the left hand limit and the right hand limit are not defined. The function will be continuous in the interval $a\leqslant x\leqslant b$, if $f(x)$ is continuous at every point of the interval.

§ 4·4. Geometrical Discussions.

In chapter two we have said that a function is said to be continuous in an interval if its graph in the interval is continuous i.e., the graph has no break in the interval. We now discuss the converse proposition i.e., if a function is continuous in an interval, then its graph in the interval is continuous.

Let the function $f(x)$ be continuous in an interval i.e., it is continuous at every point of the interval. Let $a\leqslant c<b$ i.e., c is an interior point of the interval $a\leqslant x\leqslant b$. So the function $f(x)$ is continuous at $x=c$. $\underset{x\to c}{Lt} f(x)=f c)$ and $f(c)$ possesses a finite value. So $P\{c, f(c)$ is an interior point of the graph of the function and with centre c, there exists a small interval $c-\delta<x<c+\delta$ where $\delta>0$ is small at pleasure such that in this interval the difference $|f(x)-f(c)|$ of $f(x)$ and $f(c)$ is less than every preassigned positive number however small. So on both sides of P the graph has infinite number of points whose distance from P is less than every positive number however small.

[In every interval, however small it may be, there exists an infinite number of points.]

So in moving from one side of P to the other along the graph one need not lift his pencil or chalk i.e., the graph is not broken at the point. With similar reasonings it can be shown that the graph is not broken in any interior point of the graph. For the extreme points A and B (A and B are points of the graph corresponding to $x=a$ and $x=b$) if one moves from the right and left respectively along the graph towards the points, then it can be shown by the same reasonings that one shall not have to lift his pencil or chalk. So, the graph is not broken at these points also. So, the graph is not broken at any point of the graph i.e., the graph is continuous.

§ 4·5. Some important properties of continuous functions.

1. If two functions $f(x)$ and $g(x)$ be continuous at $x=a$, then the sum, difference, product or quotient of the two functions are also continuous at $x=a$. In case of quotient the denominator $f(a)$ or $g(a)$ must not vanish.

2. If $f(x)$ is continuous in an interval and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one point c in $a<x<b$, such that $f(c)=0$.

§ 4·6. Removable discontinuity.

When $\underset{x\to a}{Lt} f(x)$ exists but $f(x)$ is discontinuous at $x=a$, then two cases may arise ; viz.,

(i) $f(a)$ does not exist.

or (ii) $f(a)$ exists but $\underset{x\to a}{Lt} f(x) \neq f(a)$.

If we define the function such that $f(a)=\underset{x\to a}{Lt} f(x)$, then the function will be continuous at $x=a$. In this case the discontinuity of $f(x)$ at $x=a$ is called removable discontinuity. But if $\underset{x\to a}{Lt} f(x)$ does not exist, then this type of removal is not possible.

The function $f(x)=\frac{\sin x}{x}$ is not continuous at $x=0$ as in this case $f(0)$ is undefined. The discontinuity can be removed by redefining the function as

$f(x)=\frac{\sin x}{x}$ when $x\neq 0$; $f(x)=1$ when $x=0$. For this new definition,

$\underset{x\to 0}{Lt} f(x)=\underset{x\to 0}{Lt}\frac{\sin x}{x}=1=f(0)$ and $f(x)$ is continuous at $x=0$ and the discontinuity is removed.

Examples 4

Example 1. Show that every constant is a continuous function of x for every value of the variable x.

Let c be a constant and a be any given value of x. Here $\underset{x\to a}{\text{Lt}}\, c=c$ [as the value of c is independent of x]. Also if $f(x)=c$, then $f(c)=c$, as the value of c remains c what ever be the value of x.

$$\therefore \quad \underset{x\to a}{\text{Lt}}\, f(x)=c=f(a).$$

Hence 'c' i.e., every constant is continuous for every value of x.

Ex. 2. If n be any positive integer, show that x^n is continuous at $x=a$.

$$\underset{x\to a}{\text{Lt}}\, x^n=\underset{x\to a}{\text{Lt}}\,(x.x.x\cdots\text{to } n \text{ factors})$$

$$=\underset{x\to a}{\text{Lt}}\, x.\ \underset{x\to a}{\text{Lt}}\, x.\ \underset{x\to a}{\text{Lt}}\, x\cdots\text{to } n \text{ factors}$$

$$=a.a.a\cdots\text{to } n \text{ factors.}$$

$$=a^n=f(a).$$

So, at $x=a$, x^n is continuous.

Note. Here we have given an intuitive proof of $\underset{x\to a}{\text{Lt}}\, x^n=a^n$.

Ex. 3. Show that the polynomial

$f(x)=(a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n)$ is continuous for all values of x.

As the given expression is a polynomial, so n is a positive integer and $a_0, a_1, a_2, \cdots, a_n$ are constants. Let $x=c$, be any value of x.

Now $\underset{x\to c}{\text{Lt}}\, f(x)=\underset{x\to c}{\text{Lt}}\,(a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n)$

$$=\underset{x\to c}{\text{Lt}}\,(a_0x^n)+\underset{x\to c}{\text{Lt}}\,(a_1x^{n-1})+\underset{x\to c}{\text{Lt}}\,(a_2x^{n-1})+\cdots+\underset{x\to c}{\text{Lt}}\,(a_n).$$

$$=a_0\underset{x\to c}{\text{Lt}}\, x^n+a_1\underset{x\to c}{\text{Lt}}\, x^{n-1}+a_2\underset{x\to c}{\text{Lt}}\, a_2x^{n-2}+\cdots+\underset{x\to c}{\text{Lt}}\, a_n$$

$$=a_0c^n+a_1c^{n-1}+a_2c^{n-2}+\cdots+a_n=f(c)$$

Hence the polynomial $f(x)$ is continuous at $x=c$. As c is any value of x, so $f(x)$ is continuous for all values of x.

Ex. 4. Find the points of discontinuity of the function

$$f(x)=\frac{1}{x^2-3x+2}.$$

$$f(x)=\frac{1}{x^2-3x+2}=\frac{1}{(x-1)(x-2)}.$$

So at the points $x=1$ and $x=2$, $f(x)$ is undefined and so the function is discontinuous at these points.

Now if $a\neq 1$ or $a\neq 2$, then

$$\underset{x\to a}{Lt}\left(\frac{1}{x^2-3x+2}\right)=\frac{1}{\underset{x\to a}{Lt}\,(x^2-3x+2)}=\frac{1}{a^2-3a+2}=f(a)$$

So, $f(x)$ is continuous at $x=a(\neq 1$ or $2)$. Hence $f(x)$ is continuous at all points other that $x=1$ or 2.

or, in other words $x=1$ and $x=2$ are the two points of discontinuity of the function $f(x)$.

Ex. 5. $f(x)=2x$ when $x\geqslant 1$

$=3x^2-1$ when $x<1$.

Show that the function $f(x)$ is continuous at $x=1$.

$$\underset{x\to 1+}{Lt} f(x)=\underset{x\to 1+}{Lt}(2x)=2$$

$$\underset{x\to 1-}{Lt} f(x)=\underset{x\to 1-}{Lt}(3x^2-1)=3-1=2.$$

$$\therefore\ \underset{x\to 1+}{Lt} f(x)=\underset{x\to 1-}{Lt} f(x)=2$$

$$\therefore\ \underset{x\to 1}{Lt} f(x) \text{ exists and equals } 2.$$

Also $f(1)=2.1=2$

$$\underset{x\to 1}{Lt} f(x)=f(1).$$

Hence the function $f(x)$ is continuous at $x=1$.

Ex. 6. $f(x)=x^2$ when $x \geqslant 2$

$=2x+1$ when $x<2$.

Show that the function is discontinuous at $x=2$.

$$\underset{x\to 2+}{Lt} f(x)=\underset{x\to 2+}{Lt} (x^2)=2^2=4.$$

$$\underset{x\to 2-}{Lt} f(x)=\underset{x\to 2-}{Lt} (2x+1)=2.2+1=5.$$

$$\therefore \underset{x\to 2+}{Lt} f(x)\neq \underset{x\to 2-}{Lt} f(x)$$

So, $\underset{x\to 2}{Lt} f(x)$ does not exist.

Hence the function is discontinuous at $x=2$.

Ex. 7. A function $f(x)$ is defined as

$f(x)=2x+3$ when $x<0$

$=1$ when $x=0$

$=x^2+3$ when $x>0$.

Examine the continuity of the function at $x=0$.

$$\underset{x\to 0-}{Lt} f(x)=\underset{x\to 0-}{Lt} (2x+3)=3.$$

$$\underset{x\to 0+}{Lt} f(x)=\underset{x\to 0+}{Lt} (x^2+3)=3.$$

$$\therefore \underset{x\to 0-}{Lt} f(x)=\underset{x\to 0+}{Lt} f(x)=3.$$

$\therefore \underset{x\to 0}{Lt} f(x)$ exists and equals 3.

But $f(0)=1$.

$$\therefore \underset{x\to 0}{Lt} f(x)\neq f(0).$$

Hence the function $f(x)$ is discontinuous at $x=0$.

Ex. 8. The function $f(x)$ is defined below.

$f(x)=x+2$ when $x<0$

$=x^2+2$ when $0\leqslant x<2$

$=x-4$ when $2\leqslant x$.

Examine the continuity of the function at $x=0$ and $x=2$.

$$\underset{x\to 0-}{Lt}\ (x)=\underset{x\to 0-}{Lt}\ (x+2)=2$$

$$\underset{x\to 0+}{Lt}\ f(x)=\underset{x\to 0+}{Lt}\ (x^2+2)=2$$

$$\therefore\ \underset{x\to 0-}{Lt}\ f(x)=\underset{x\to 0+}{Lt}\ f(x)=2.$$

$$\therefore\ \underset{x\to 0}{Lt}\ f(x) \text{ exists and } \underset{x\to 0}{Lt}\ f(x)=2.$$

Also $f(0)=2$. $\therefore\ \underset{x\to 0}{Lt}\ f(x)=f(0)$.

So, the function is continuous at $x=0$,

$$\underset{x\to 2-}{Lt}\ f(x)=\underset{x\to 2-}{Lt}\ (x^2+2)=6.$$

$$\underset{x\to 2+}{Lt}\ f(x)=\underset{x\to 2+}{Lt}\ (x-4)=2-4=-2.$$

$$\therefore\ \underset{x\to 2-}{Lt}\ f(x)\neq\underset{x\to 2+}{Lt}\ f(x)$$

So, $\underset{x\to 2}{Lt}\ f(x)$ does not exist and the function is discontinuous at $x=2$.

Note. When $x<0$, then also $x<2$.

But $x\to 2-$ means x is very close to 2 and less than 2. So when $x\to 2-$, we take $f(x)=x^2+2$ and not $(x+2)$.

Ex. 9. A function $f(x)$ is defined as follows :

$$\begin{aligned} f(x)&=-x,\ x\leqslant 0\\ &=x,\ 0\leqslant x<1\\ &=2-x,\ x\geqslant 1. \end{aligned}$$

Show that the function $f(x)$ is continuous at $x=0$ and $x=1$.

[C. U. ; State Council W. Bengal 1985]

$$\underset{x\to 0-}{Lt}\ f(x)=\underset{x\to 0-}{Lt}\ (-x)=0$$

$$\underset{x\to 0+}{Lt}\ f(x)=\underset{x\to 0+}{Lt}\ (x)=0$$

$\therefore \underset{x\to 0-}{Lt} f(x)=\underset{x\to 0+}{Lt} f(x)=0$

$\therefore \underset{x\to 0}{Lt} f(x)$ exists and $\underset{x\to 0}{Lt} f(x)=0.$

Also $f(0)=0$

$\therefore \underset{x\to 0}{Lt} f(x)=f(0)$ and the function is continuous at $x=0$.

$\underset{x\to 1-}{Lt} f(x)=\underset{x\to 1-}{Lt} (x)=1$

$\underset{x\to 1+}{Lt} f(x)=\underset{x\to 1+}{Lt} (2-x)=2-1=1.$

$\therefore \underset{x\to 1-}{Lt} f(x)=\underset{x\to 1+}{Lt} f(x)=1$

$\therefore \underset{x\to 1}{Lt} f(x)$ exists and $\underset{x\to 1}{Lt} f(x)=1$

Also $f(1)=2-1=1$

$\therefore \underset{x\to 1}{Lt} f(x)=f(1).$

So, the function is continuous at $x=1$.

Hence the function is continuous at $x=0$ and $x=1$.

Ex. 10. $f(x)=1+|\sin x|$; show that the function $f(x)$ is continuous at $x=0$.

when $x\to 0-$, then $x<0$ and is very close to 0.

So, when $x\to 0-$, $\sin x<0$ $\therefore$ $|\sin x|=-\sin x$.

$\therefore \underset{x\to 0-}{Lt} f(x)=1-\sin x=1-\sin 0=1$

when $x\to 0+$, then $x>0$ and is very close to 0.

$\therefore$ when $x\to 0+$, then $\sin x>0$ and $|\sin x|=\sin x$.

$\therefore \underset{x\to 0+}{Lt} f(x)=\underset{x\to 0+}{Lt} \{1+|\sin x|\}$

$=\underset{x\to 0+}{Lt}(1+\sin x)=1.$

$\therefore \underset{x\to 0-}{Lt} f(x)=\underset{x\to 0+}{Lt} f(x)=1$

$\therefore \quad \underset{x \to 0}{Lt} f(x)$ exists and equals 1.

Also $f(0)=1+|\sin 0|=1+0=0.$.

$\therefore \quad \underset{x \to 0}{Lt} f(x)=f(0).$

Hence the function $f(x)$ is continuous at $x=0$.

Ex. 11. $f(x)=\sin x, x \neq n\pi [n=0, \pm 1, \pm 2, \cdots]$

$\qquad =2, x=n\pi [n=0, \pm 1, \pm 2, \cdots]$

$g(x)=x^2+1, x \neq 0, 2$

$\qquad =4, x=0$

$\qquad =5, x=2$

Show that the function $g[f(x)]$ is discontinuous at $x=0$.

[c.f. I. I. T. 1986]

$\underset{x \to 0}{Lt} g[f(x)]=\underset{x \to 0}{Lt} g(\sin x)$

$[\because x \to 0, \text{ so } x \neq 0 \quad \therefore f(x)=\sin x]$

$=\underset{x \to 0}{Lt} (\sin^2 x+1)=1.$

Again $g[f(0)]=g(2)=5$

$\therefore \quad \underset{x \to 0}{Lt} g[f(x)] \neq g[f(0)]$

Hence the function $g[f(x)]$ is discontinuous at $x=0$.

Ex. 12. $f(x)=\dfrac{x^2}{2}, 0 \leqslant x<1$

$\qquad =2x^2-3x+\frac{3}{2}, 1 \leqslant x \leqslant 2.$

Examine the continuity of $f(x)$ at $x=1$. [c.f. I. I. T. 1983]

$\underset{x \to 1-}{Lt} f(x)=\underset{x \to 1-}{Lt} \left(\dfrac{x^2}{2}\right)=\frac{1}{2}$

$\underset{x \to 1+}{Lt} f(x)=\underset{x \to 1+}{Lt} (2x^2-3x+\frac{3}{2})=2-3+\frac{3}{2}=\frac{1}{2}$

$\therefore \quad \underset{x \to 1-}{Lt} f(x)=\underset{x \to 1+}{Lt} f(x)=\frac{1}{2}.$

$\therefore \underset{x \to 1}{Lt} f(x)$ exists and $\underset{x \to 1}{Lt} f(x) = \frac{1}{2}$

Also $f(1) = 2.1^2 - 3.1 + \frac{3}{2} = \frac{1}{2}$

$\therefore \underset{x \to 1}{Lt} f(x) = f(1)$ and the function is continuous at $x = 1$.

Ex. 13. The function $f(x) = \dfrac{x^2 - 9}{x - 3}$ is undefined at $x = 3$. What value should be assigned to $f(3)$, so that the discontinuity of $f(x)$ at $x = 3$ can be removed.

As $f(3)$ is undefined and so $f(x)$ is discontinuous at $x = 3$.

Now $\underset{x \to 3}{Lt} \dfrac{x^2 - 9}{x - 3} = \underset{x \to 3}{Lt} (x + 3) = 6$.

So, if we assign the value 6 to $f(3)$, i.e., redefine the function as.

$$f(x) = \frac{x^2 - 9}{x - 3} \quad \text{when } x \neq 3$$

$$= 6 \quad \text{when } x = 3,$$

then $\underset{x \to 3}{Lt} f(x) = 6 = f(3)$

and the function becomes continuous at $x = 3$.

Hence the value 6 should be assigned to $f(3)$ in order to remove the discontinuity of the function.

Ex. 14. $f(x) = x + 1$, when $x \leqslant 1$

$= 3 - ax^2$, when $x > 1$.

For what value of a will $f(x)$ be continuous ? [H. S. 1983]

$\underset{x \to 1-}{Lt} f(x) = \underset{x \to 1-}{Lt} (x + 1) = 2$

$\underset{x \to 1+}{Lt} f(x) = \underset{x \to 1}{Lt} (3 - ax^2) = 3 - a$.

So, $\underset{x \to 1-}{Lt} f(x)$ and $\underset{x \to 1+}{Lt} f(x)$ will be equal if $2 = 3 - a$ or $a = 1$ and in this case

$\underset{x \to 1}{Lt} f(x) = 2$. Also $f(1) = 3 - 1^2 = 2$

$\therefore \underset{x\to 1}{Lt} f(x)=f(1)$ and the function is continuous.

So the required value of a is 1.

Ex. 15. The function $f(x)=\dfrac{x^2-1}{x^3-1}$ is undefined at the point $x=1$; what should be the value of $f(1)$ such that $f(x)$ may be continuous at $x=1$. Give arguments. [H. S. 1986]

$$\underset{x\to 1}{Lt} f(x = \underset{x\to 1}{Lt} \frac{x^2-1}{x^3-1}=\underset{x\to 1}{Lt} \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}$$

$$=\underset{x\to 1}{Lt} \frac{x+1}{x^2+x+1}=\frac{\underset{x\to 1}{Lt}(x+1)}{\underset{x\to 1}{Lt}(x^2+x+1)}=\tfrac{2}{3}.$$

So if the value $\frac{2}{3}$ is assigned to $f(1)$,

then $\underset{x\to 1}{Lt} f(x)$ will be equal to $f(1)$ and the function will become continuous.

Ex. 16. Without using graph paper draw the graph of the function $\sin\frac{1}{x}$. Discuss the limit and continuity of the function at $x=0$. [c. f. Joint Entrance 1984]

First let $x\to 0+$

In this case if n assumes positive integral values large at pleasure, when x assumes the values $\frac{2}{n\pi}$, then $\sin\left(\frac{1}{x}\right)$ takes the values 1, 0, -1, 0, etc. in succession and also intermediate values. So $\sin\left(\frac{1}{x}\right)$ does not approach any finite value. In fact, $\sin\frac{1}{x}$ oscillates between the values -1 and $+1$

$$\text{So. } \underset{x\to 0+}{Lt} \sin\left(\frac{1}{x}\right) \text{ does not exist.}$$

Again when $x\to 0-$, then x is negative and let $x=-y$ $(y>0)$

$\therefore \sin\left(\frac{1}{x}\right)=\sin\left(-\frac{1}{y}\right)=-\sin\left(\frac{1}{y}\right)$. So in this case

$$\underset{x\to 0-}{Lt} \sin\left(\frac{1}{x}\right)=\underset{y\to 0+}{Lt}\left\{-\sin\left(\frac{1}{y}\right)\right\}=-\underset{y\to 0+}{Lt} \sin\left(\frac{1}{y}\right)$$

(as in case of $x\to 0+$) does not exist.

So $\underset{x\to 0}{Lt}\ \sin\left(\frac{1}{x}\right)$ does not exist. Also $\sin\left(\frac{1}{x}\right)$ is itself undefined at $x=0$. So from all points of view the function is discontinuous at $x\to 0$. We draw below the graph of $\sin\left(\frac{1}{x}\right)$

Notice that the graph is broken at $x=0$

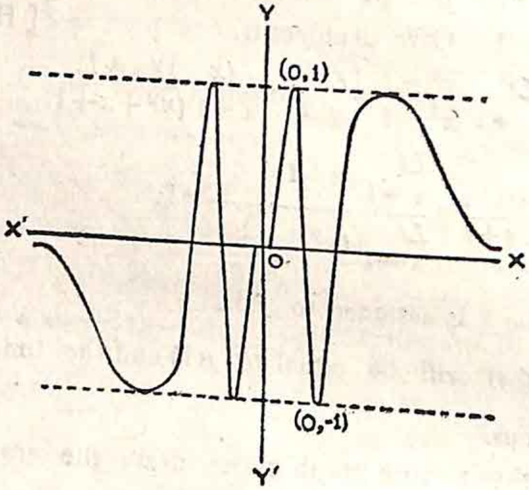


fig. 4·1

Ex. 17. $f(x)=x\sin\left(\frac{1}{x}\right)$ when $x\neq 0$.

$=0$ when $x=0$.

Discuss the continuity of the function at $x=0$.

$$\left|x\sin\left(\frac{1}{x}\right)-0\right|=\left|x\sin\left(\frac{1}{x}\right)\right|=|x|\left|\sin\left(\frac{1}{x}\right)\right|\leqslant x$$

as $\left|\sin\frac{1}{x}\right|\leqslant 1$

So, making x close to 0, as close as one likes i,e., making $|x|\ (\geqslant 0)$, small at pleasures.

$\left|x\sin\left(\frac{1}{x}\right)-0\right|$ can be made less than any preassigned positive number as small as it may be

$$\therefore\ \underset{x\to 0}{Lt}\left(x\sin\frac{1}{x}\right)=0$$

Also $f(0)=0$ $\quad\therefore\ \underset{x\to 0}{Lt}\left\{x\sin\left(\frac{1}{x}\right)\right\}=f(0).$

So, $f(x)$ is continuous at $x=0$.

Ex. 18. Determine the values of a, b, c for which the function

$$f(x)=\frac{\sin(a+1)x+\sin x}{x} \quad \text{for } x<0$$

$$=c \quad \text{for } x=0$$

$$=\frac{(x+bx^2)^{\frac{1}{2}}-x^{\frac{1}{2}}}{bx^{\frac{3}{2}}} \quad \text{for } x>0$$

is continuous at $x=0$.

$$\underset{x\to0-}{Lt}\ f(x)=\underset{x\to0-}{Lt}\ \frac{\sin(a+1)x+\sin x}{x}$$

$$=\underset{x\to0-}{Lt}\ \frac{2\sin\left(\frac{a+2}{2}\right)x\cos\frac{a}{2}x}{x}$$

$$=\underset{x\to0-}{Lt}\left\{\frac{\sin\left(\frac{a+2}{2}\right)x}{\left(\frac{a+2}{2}\right)x}(a+2)\,.\cos\frac{a}{2}x\right\}$$

$$=(a+2)\underset{x\to0-}{Lt}\ \frac{\sin\left(\frac{a+2}{2}\right)x}{\left(\frac{a+2}{2}\right)x}\cdot\underset{x\to0}{Lt}\ \cos\left(\frac{a}{2}x\right)$$

$$=(a+2).1.1=a+2.$$

$$\underset{x\to0+}{Lt}\ f(x)=\underset{x\to0+}{Lt}\ \frac{(x+bx^2)^{\frac{1}{2}}-x^{\frac{1}{2}}}{bx^{\frac{3}{2}}}$$

$$=\underset{x\to0+}{Lt}\ \frac{\{(x+bx^2)^{\frac{1}{2}}-x^{\frac{1}{2}}\}\{(x+bx^2)^{\frac{1}{2}}+x^{\frac{1}{2}}\}}{bx^{\frac{3}{2}}\{(x+bx\ )^{\frac{1}{2}}+x^{\frac{1}{2}}\}}$$

$$=\underset{x\to0+}{Lt}\ \frac{(x+bx^2)-x}{bx^{\frac{3}{2}}\{(x+bx^2)^{\frac{1}{2}}+x^{\frac{1}{2}}\}}$$

$$=\underset{x\to0+}{Lt}\ \frac{bx^2}{bx^{\frac{3}{2}}x^{\frac{1}{2}}\{(1+bx)^{\frac{1}{2}}+1\}}$$

$$=\underset{x\to0+}{Lt}\ \frac{1}{(1+bx)^{\frac{1}{2}}+1}=\frac{1}{2}$$

So if $f(x)$ is continuous at $x=0$, then

$$\underset{x\to0-}{Lt}\ f(x)=\underset{x\to0+}{Lt}\ f(x)$$

or, $a+2=\frac{1}{2}$ $\therefore$ $a=-\frac{3}{2}$.

$\therefore$ $\underset{x\to 0}{Lt} f(x)=\frac{1}{2}$

Also $f(0)=c$

If $f(x)$ is continuous at $x=0$, then

$\underset{x\to 0}{Lt} f(x)=f(0)$ $\therefore$ $\frac{1}{2}=c$.

$\therefore$ $a=-\frac{3}{2}$, $c=\frac{1}{2}$ and b can have any value other than 0 (when $b=0$, then $f(0)$ is undefined)

Ex. 19. $[x]$ denotes the greatest integer less than or equal to x and x is any integer. Show that at $x=n$, the function $f(x)=[x]+[-x)$ is discontinuous. Is this discontinuity removable ?

$\underset{x\to n-}{Lt} f(x)=\underset{x\to n-}{Lt} \{[x]+[-x]\}=\underset{x\to n-}{Lt} \{(n-1)+(-n)\}$
$=\underset{x\to n-}{Lt} (-1)=-1.$

$\underset{x\to n+}{Lt} f(x)=\underset{x\to n+}{Lt} \{[x]+[-x]\}=\underset{x\to n+}{Lt} \{n+(-n-1)\}$
$=\underset{x\to n+}{Lt} (-1)=-1.$

Again $f(n)=[n]+[-n]=n+(-n)=0$

$\therefore$ $\underset{x\to n}{Lt} f(x)\neq f(n)$ when $\neq 0$ is an integer.

$\therefore$ The function $f(x)$ is discontinuous at $x=(\neq 0)$. The discontinuity is removable by defining $f(x)=-1$ when n is an integer $\neq 0$.

Exercise 4

1. A function $f(x)$ is defined below.

$f(x)=x-1$ when $x>0$
$=-\frac{1}{2}$ when $x=0$
$=x+1$ when $x<0$.

Determine whether the function is continuous at $x=0$ or not.

2. A function $f(x)$ is defined as follows :

$f(x)=x$ when $x<1$
$=1+x$, when $x>1$
$=\frac{3}{2}$ when $x=1$

Examine the continuity of the function at $x=1$

3. Determine whether the function $|x|$ is continuous or not at $x=0$.

4. $f(x)=[x]$, where $[x]$ denotes the greatest integer less than or equal to x, Show that function is discontinuous at every integral value of x.

5. Prove that the discontinuity of the function $f\,x=\frac{|x|}{x}$ at $x=0$ is not removable.

6. Prove that the function $\frac{\sin x}{x}$ is discontinuous at $x=0$ and the discontinuity will be removed if $f(0)$ is defined as 1.

7. The function $f(x)=\frac{x^2-16}{x-4}$ is discontinuous at $x=4$. What value should be assiged to $f(4)$, so that the discontinuity will be removed ?

8. Find the points of discontinuity of the following functions.

(i) $\frac{x^2-5x+6}{x^2-3x+2}$ (ii) $\frac{\tan x}{x}$ (iii) $\cot x$

9. Which of the following functions are continuous at $x=0$ and which are not ?

(i) $f(x)=\frac{x^3+5x^2}{\sin x}$ when $x\neq 0$, $f(0)=0$.

(ii) $f(x)=\frac{x^2+5x^2}{\sin x}$ when $x\neq 0$, $f(0)=1$

(iii) $f(x)=\frac{x^3+5x^2+x}{\sin x}$ when $x\neq 0$; $f(0)=0$

(iv) $f(x)=\frac{x^3+5x^2+x}{\sin x}$ when $x\neq 0$; $f(0)=1$

10. The difinition of the function $f(x)$ is as follows.

$f(x)=0$ when $x^2>1$
$=2$ when $x^2<1$
$=\frac{1}{2}$ when $x^2=1$

Show that, though the function has a finite value for every x, yet it is discontinuous at $x=1$ and $x=-1$

Draw the graph of the function.

11. The function $f(x)$ is defined as follows :

$f(x)=3+2x$ when $x\leqslant 0$

$=3-2x$ when $x>0$,

Examine the continuity of the function at $x=0$

12. $f(x)=|x-4|-2$.

Examine the continuity of the function at $x=0$. Also draw a neat graph of the function.

13. The definition of a function $f(x)$ is as follows :

$f(x)=x^2$ when $x<1$

$=2{\cdot}5$ when $x=1$

$=x^2+2$ when $x>1$

Show that the function is discontinuous at $x=1$.

14. A function $f(x)$ is defined as follows :

$f(x)=x^2$ when $x\geqslant 2$

$=3x+c$ when $x<2$ (c is a constant)

If the function is continuous at $x=2$, Show that $c+2=0$.

15. $f(x)=x\cos\left(\frac{1}{x}\right)$ when $x\neq 0$

what should be the value of $f(0)$ so that the function will be continuous at $x=0$.

16. A function $f(x)$ is defined as follows :

$f(x=\frac{x^2}{a}-a$ when $x<a$

$=0$ when $x=a$.

$=a-\frac{a^2}{x}$ when $x>a$.

Show that the function is continuous at $x=a$

17, $f(x)=\sin x\cos\left(\frac{1}{x}\right)$ when $x\neq 0$

$=0$ when $x=0$.

Show that the function is continuous at $x=0$.

18. Let $f(x+y)=f(x)+f(y)$ for all x and y. If $f(x)$ is continuous at $x=0$, then show that $f(x)$ is continuous at all x.

[I. I. T, 1981]

19. A function $f(x)$ is defined as follows.

$$f(x) = -2\sin x \text{ when } -\pi \leqslant x \leqslant -\frac{\pi}{2}$$

$$= a\sin x + b \text{ when } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$= \cos x \text{ when } \frac{\pi}{2} \leqslant x \leqslant \pi$$

If the function is continuous in the interval $-\pi \leqslant x \leqslant \pi$, find the values of a and b.

20. Let $f(x)$ be a continuous function and $g(x)$ be a discontinuous function. Prove that $f(x)+g(x)$ is a discontinuous function,

21. The function $f(x)=\dfrac{\log(1+ax)-\log(1-bx)}{x}$ is undefined at $x=0$, Show that if $f(0)$ is defined as $f(0)=a+b$, then the function will be continuous at $x=0$ [c. f. I. I. T. 1983]

22. The definition of the function $f(x)$ is given below.

$f(x)=\frac{5}{2}-x$ when $x<2$, $f(x)=1$ when $x=2$

and $f(x)=x-\frac{3}{2}$, when $x>2$. Is $f(x)$ continuous at $x=2$? Give reasons. [Joint Entrance 1986]

23. Exmine whether the function $f(x)=|x-1|+|x+1|$ is continuous at $x=1$. [Joint Entrance 1988]

CHAPTER FIVE

DERIVATIVES

§ 5·1 **Derivative.** Determination of the derivative of a function or differentiation is the main operation discussed in Differential calculus. We have in the previous chapters discussed about Real Numbers, Functions, Limit and continuity. We shall now discuss about the determination of the derivatives of functions of real variables with respect to the independent variable. Again differentiation of a function is a particular type of limit operation and whether this particular limit exists or not depends on the continuity of the function. The relation between continuity and differentiability will be discussed in § 5·2. But before that we must discuss what is meant by derivative of a function.

Definition. If $\underset{h\to 0}{Lt} \dfrac{f(a+h)-f(a)}{h}$ exists, then the value of this limit is defined as the **derivative** or **differential coefficient** of $f(x)$ with respect to x at $x=a$. If the limit does not exist, the derivative also does not exist and the function is not differentiable at $x=a$.

Above, we have defined the derivative of a function at any point $x=a$. Here 'a' is a constant and 'h' is a variable, generally instead of determining the derivative of a function at a particular point $x=a$, we determine the derivative at x which is denoted as $f'(x)$ (Evidently the value of $f(x)$ depends on the value of x and is thus itself a function of x). To determine the derivative of $f(x)$ at $x=a$, we first find $f'(x)$ and then put $x=a$ in $f'(x)$. This is denoted by $f'(a)$. If the value of $f'(a)$ is finite, then $f'(a)$ exists, otherwise it does dot.

We have already mentioned that in the definition

$$\underset{h\to 0}{Lt} \frac{f(a+h)-f(a)}{h} \text{ of } f'(a)$$

a is constant and h is a variable. Similarly in the definition

$$\underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$$

of the derivative of a function $f(x)$ with respect to x at any point x. x is a constant and h is a variable. For, we actually we determine the derivatives of a function at particular points and

$$\underset{h\to 0}{Lt}\ \frac{f(x+h)-f(x)}{h}$$

is the generalised expression of $\underset{h\to 0}{Lt}\ \frac{f(a+h)-f(a)}{h}$, the derivative of $f(x)$ at $x=a$. The process of determination of the derivative is called differentiation.

As we use the symbols '+', '−', '×', '÷', $\underset{x\to}{Lt}$ for the operations of additinn, subtraction, multiplication, division and Limit respectively, so we must have a symbol for the opperation of differentiation with respect to x. $\frac{\text{'}d\text{'}}{dx}$ is the symbol used in case of the operation of differentiation with respect to x. So, $f'(x)=\frac{d}{dx}\left\{f(x)\right\}$ The derivative of x^2 is $\frac{d}{dx}(x^2)$. The derivative of y with respect to x is $\frac{d}{dx}(x)=\frac{dy}{dx}$. The students must be particular in understanding that the operation is $\frac{d}{dx}$ and not $\frac{dy}{dx}$ [Experience has shown that students, due to lack of proper understanding frequently refer determination of derivative of x^2 as determination of $\frac{dy}{dx}$ of x^2 and likewise]

A function may possess a derivative with respect to x for particular values of x or may not possess the derivative at some other points. [In pronouncing or writing derivative of a function with respect to x, we do not always use the phrase "with respect to x" ; if this phrase is not mentioned, one must understand that the differentiation is with respect to the independent variable of the function] Again at different points the values of the derivatives of a function (when they exist) are generally different. Follow the following example.

Example. Determine from the definition the derivatives of the function $f(x)=\frac{1}{x}$ at the points $x=1$, $x=2$ and at $x=0$.

Here the derivative of $f(x)$ at $x=1$

$$=\underset{h\to 0}{Lt}\frac{f(1+h)-f(1)}{h}=\underset{h\to 0}{Lt}\frac{\frac{1}{1+h}-1}{h}=\underset{h\to 0}{Lt}\frac{1-1-h}{h(1+h)}$$

$$=\underset{h\to 0}{Lt}\frac{-h}{h(1+h)}=\underset{h\to 0}{Lt}\frac{-1}{1+h}=\frac{-1}{1}=-1$$

$$f(2)=\underset{h\to 0}{Lt}\frac{f(2+h)-f(2)}{h}=\underset{h\to 0}{Lt}\frac{\frac{1}{2+h}-\frac{1}{2}}{h}$$

$$=\underset{h\to 0}{Lt}\frac{2-2-h}{2(2+h).h}=\underset{h\to 0}{Lt}\frac{-h}{2(2+h)h}=\underset{h\to 0}{Lt}\frac{-1}{2(2+h)}$$

$$=-\tfrac{1}{4}.$$

$$f'(0)=\underset{h\to 0}{Lt}\frac{f(0+h)-f(0)}{h}=\underset{h\to 0}{Lt}\frac{f(h)-f(0)}{h}$$

But as $f(0)$ is undefined, so the limit does not exist i.e., $f'(0)$ does not exist

Note 1. In the next few sections we shall determine formulas for finding derivatives of different types of functions. These formulas are derived, directly from the definition. To determine the derivative of a function from the definition is also referred to as finding the derivative *ab-initio* or from the first principle.

Note 2. In the above example we have found that the derivative $\frac{1}{x}$ does not exist as the function is undefined at $x=0$. This may tempt the students to conclude that if the derivative of a function at a point does not exist then the function does not exist at the point. But this is not the case. In the next section it will be shown that the function $|x|$ has a definite value at $x=0$ but its derivative at the point does not exist.

3. The students are once again cautioned that $\frac{dy}{dx}=\frac{d}{dx}(y)$ and $\frac{dy}{dx}\neq dy\div dx$. This has been discussed in chapter one of the "Application of Calculus" portion.

4. $\underset{x\to a}{Lt}\frac{f(x)-f(a)}{x-a}=f'(a)$.

Proof. Let $x=a+h$ $\therefore$ $x-a=h$ and when $x\to a$, then $h\to 0$.

So, $\underset{x\to a}{Lt}\frac{f(x)-f(a)}{x-a}=\underset{h\to 0}{Lt}\frac{f(a+h)-f(a)}{h}=f'(a)$.

5. The students are naturally interested to know the application of a subject which they learn. As yet we have not mentioned about applications of differentiation. Derivatives have wide applications in the determination of the instantaneous rate of change of a function at a point, finding the equations of tangents and normals to a curve at a point, determination of approximate value of a function at a point, determination of the maximum and minimum values and in dynamics. These topics have been discussed "in the application of calculus" portion of this Book. But these are small fragments of the wide range of applications of calculus.

§ 5·2 Relation between differentiability and continuity of a function.

Theorem. If a function is differentiable at a point, then it is continuous at the point.

The function $f(x)$ is differentiable at the point $x=a$. To prove that the function is continuous at the point.

Proof. $f(a+h)-f(a)=\frac{f(a+h)-f(a)}{h}\times h$

$$\therefore \underset{h\to 0}{Lt}\{f(a+h)-f(a)\}=\underset{h\to 0}{Lt}\left\{\frac{f(a+h)-f(a)}{h}\times h\right\}$$

$$=\underset{h\to 0}{Lt}\frac{f(a+h)-f(a)}{h}\times\underset{h\to 0}{Lt}h=f'(a)\times 0=0.$$

[as $f(x)$ is differentiable at $x=a$, So $\underset{h\to 0}{Lt}\dfrac{f(a+h)-f(a)}{h}$ tends to a finite limit $f'(a)$]

$$\therefore \underset{h\to 0}{Lt} f(a+h)=f(a)$$

Hence $f(x)$ is continuous at $x=a$.

The converse of the above theorem is not always true. If a function is continuous at a point $x=a$, it is not neccessarily differentiable at $x=a$. The function $|x|$ is continuous at $x=a$ but is not differentiable at $x=a$.

$\underset{x\to 0}{Lt} |x|=0=|0|$. So $|x|$ is continuous at $x=0$.

Let us now consider the derivative of $|x|$ at $x=0$.

$$\frac{d}{dx}(|x|)=\underset{h\to 0}{Lt}\frac{|0+h|-|0|}{h}=\underset{h\to 0}{Lt}\frac{|h|-0}{h}=\underset{h\to 0}{Lt}\frac{|h|}{h}$$

Now $\underset{h\to 0+}{Lt}\dfrac{|h|}{h}=\underset{h\to 0+}{Lt}\dfrac{h}{h}=\underset{h\to 0+}{Lt}(1)=1.$

and $\underset{h\to 0-}{Lt}\dfrac{|h|}{h}=\underset{h\to 0-}{Lt}\dfrac{-h}{h}=\underset{h\to 0-}{Lt}(-1)=-1.$

$$\therefore \underset{h\to 0-}{Lt}\frac{|h|}{h}\neq\underset{h\to 0+}{Lt}\frac{|h|}{h}$$

i.e., $\underset{h\to 0}{Lt}\dfrac{|0+h|-|0|}{h}$ does not extst., i.e.,

$|x|$ is not differentiable at $x=0$.

Note. $\underset{h\to 0-}{Lt}\dfrac{f(a+h)-f(a)}{h}$ and $\underset{h\to 0+}{Lt}\dfrac{f(a+h)-f(a)}{h}$

are respectively called the left hand and right hand derivatives of $f(x)$ at $x=a$. If a function is differentiable at a point $x=a$, then the left hand and right hand derivatives of the function must exist finitely at the point and must be equal. The common value of the two derivatives is the value of the derivative of the function at the point.

§ 5·3 Derivatives of elementary functions.

(i) $f(x)=x^n$ where n is a real constant.

Here $f'(x)=\dfrac{d}{dx}(x^n)=\underset{h\to 0}{It}\;\dfrac{f(x+h)-f(x)}{h}$

$$=\underset{h\to 0}{Lt}\;\frac{(x+h)^n-x^n}{h}=\underset{y\to x}{Lt}\;\frac{y^n-x^n}{y-x}=nx^{n-1}$$

[Where $y=x+h$ or $h=y-x$ and as $h\to 0$, then $y\to x$.]

Note. The formula $\dfrac{d}{dx}(x^n)=nx^{n-1}$ can be remembered in the following manner.

Decrease the index of x by 1 and multiply the new power of x by the original index of x to get the derivative. For example to find $\dfrac{d}{dx}(x^5)$ first decrease the index of 5 by 1 i.e. make the index 4 and get x^4. Multiply x^4 by the original index i.e., 5. So $5x^4$ is the derivative of x^5.

(ii) Derivatives of Trigonometric functions.

(a) $f(x)=\sin x$.

$$f'(x)=\underset{h\to 0}{Lt}\;\frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lt}\;\frac{\sin(x+h)-\sin x}{h}$$

$$=\underset{h\to 0}{Lt}\;\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}=\underset{h\to 0}{Lt}\;\cos\left(x+\frac{h}{2}\right)\underset{h\to 0}{Lt}\;\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$=\cos x.1=\cos x.$

$\therefore\ \dfrac{d}{x}(\sin x)=\cos x$

(b) $f(x)=\cos x$.

$$f'(x)=\underset{h\to 0}{Lt}\;\frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lt}\;\frac{\cos(x+h)-\cos x}{h}$$

$$=\underset{h\to 0}{Lt}\;\frac{2\sin\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h}=\underset{h\to 0}{Lt}\;\frac{-\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\underset{h\to 0}{Lt}\ \sin\left(x+\frac{h}{2}\right)\ \underset{h\to 0}{Lt}\ \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\sin x.1 = -\sin x$$

$$\therefore\ \frac{d}{dx}(\cos x) = -\sin x.$$

(*c*) $f(x)=\tan x$

$$f'(x) = \underset{h\to 0}{Lt}\ \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\ \frac{\tan(x+h)-\tan x}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} = \underset{h\to 0}{Lt}\ \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$= \underset{h\to 0}{Lt}\ \frac{\sin h}{h\cos(x+h)\cos x} = \underset{h\to 0}{Lt}\ \frac{\sin h}{h}\cdot\underset{h\to 0}{Lt}\ \frac{1}{\cos(x+h)\cos x}$$

$$= 1.\frac{1}{\cos^2 x} = \sec^2 x.$$

$$\therefore\ \frac{d}{dx}(\tan x) = \sec^2 x.$$

(*d*) $f(x)=\cot x.$

$$f'(x) = \underset{h\to 0}{Lt}\ \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\ \frac{\cot(x+h)-\cot x}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{\cos(x+h)\sin x - \cos x\sin(x+h)}{h\sin(x+h)\sin x}$$

$$= \underset{h\to 0}{Lt}\ \frac{\sin(x-x-h)}{h\sin(x+h)\sin x} = \underset{h\to 0}{Lt}\ \frac{\sin(-h)}{h\sin(x+h)\sin x}$$

$$= -\underset{h\to 0}{Lt}\ \frac{\sin h}{h,\sin(x+h)\sin x}$$

$$= -\underset{h\to 0}{Lt}\ \frac{\sin h.}{h}\ \underset{h\to 0}{Lt}\ \frac{1}{\sin(x+h)\sin x} = -1.\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x.$$

$$\therefore\ \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$$

(e) $f(x)=\sec x$.

$$f'(x)=\underset{h\to 0}{Lt}\ \frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lt}\ \frac{\sec(x+h)-\sec x}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{\dfrac{1}{\cos(x+h)}-\dfrac{1}{\cos x}}{h}=\underset{h\to 0}{Lt}\ \frac{\cos x-\cos(x+h)}{h\cos(x+h)\cos x}$$

$$=\underset{h\to 0}{Lt}\ \frac{2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h\cos(x+h)\cos x}=\underset{h\to 0}{Lt}\ \frac{\sin\left(x+\frac{h}{2}\right)}{\cos(x+h)\cos x}$$

$$\underset{h\to 0}{Lt}\ \frac{\sin\frac{h}{2}}{\frac{h}{2}}=\frac{\sin x}{\cos x.\cos x}.1=\sec x\tan x.$$

$$\therefore\quad \frac{d}{dx}(\sec x)=\sec x\tan x$$

(f) $f(x)=\operatorname{cosec} x$.

$$f'(x)=\underset{h\to 0}{Lt}\ \frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lt}\ \frac{\operatorname{cosec}(x+h)-\operatorname{cosec} x}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{\dfrac{1}{\sin(x+h)}-\dfrac{1}{\sin x}}{h}=\underset{h\to 0}{Lt}\ \frac{\sin x-\sin(x+h)}{h\sin(x+h)\sin x}$$

$$=\underset{h\to 0}{Lt}\ \frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h\sin(x+h)\sin x}=-\underset{h\to 0}{Lt}\ \frac{\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{\frac{h}{2}\sin(x+h)\sin x}$$

$$=-\underset{h\to 0}{Lt}\ \frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h)\sin x}\cdot\underset{h\to 0}{Lt}\ \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$=-\frac{\cos x}{\sin x\sin x}.\,1=-\operatorname{cosec} x\cot x.$$

$$\therefore\quad \frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x.$$

Note. Above we find that derivatives of some of the trigonometric functions are positive and some are negative. The following may be useful as a good aid to memory.

Trigonometric functions which begin with co (such as cos, cosec and cot, will have their derivatives negative. Derivatives of the other trigonometric functions viz sin x, tan x and sec x are positive.

(iii) (*a*) $f(x)=e^x$.

$$f'(x)=\underset{h\to 0}{Lt}\ \frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lt}\ \frac{e^{x+h}-e^x}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{e^x(e^h-1)}{h}=e^x\underset{h\to 0}{Lt}\ \frac{e^h-1}{h}$$

$=e^x.1=e^x$ [Here x is a constant]

$$\therefore\ \frac{d}{dx}(e^x)=e^x.$$

(*b*) $f(x)=a^x$

$$f'(x)=\underset{h\to 0}{Lt}\ \frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lt}\ \frac{a^{x+h}-a^x}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{a^x(a^h-1)}{h}=a^x\underset{h\to 0}{Lt}\ \frac{e^{h\log a}-1}{h}$$

$$=a^x\underset{h\to 0}{Lt}\ \left(\frac{e^{h\log a}-1}{h\log a}\times\log a\right)$$

$$=a^x\log a\times\underset{h\to 0}{Lt}\ \frac{e^{h\log a}-1}{h\log a}=a^x\log_e a\times 1=a^x\log_e a$$

$$\therefore\ \frac{d}{dx}(a^x)=a^x\log_e a$$

(iv) $f(x)=\log x$.

$$f'(x)=\underset{h\to 0}{Lt}\ \frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lt}\ \frac{\log(x+h)-\log x}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{\log\left(\frac{x+h}{x}\right)}{h}=\underset{h\to 0}{Lt}\ \frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}\cdot x}=\frac{1}{x}\cdot\underset{h\to 0}{Lt}\ \frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$$

$$=\frac{1}{x}\cdot 1=\frac{1}{x}.$$

$$\therefore \quad \frac{d}{dx}(\log x)=\frac{1}{x}.$$

Note. In this section we have found the derivatives of elementary functions other then the inverse circular functions. They have been discussed in Examples 5A and in § 5·7

We now give a list of the derivatives of the four types of elementary functions discussed in this section.

1. $\frac{d}{dx}(x^n)=nx^{n-1}$ [for all real values of n]

2. (*a*) $\frac{d}{dx}(\sin x)=\cos x$; (*b*) $\frac{d}{dx}(\cos x)=-\sin x$.

(*c*) $\frac{d}{dx}(\tan x)=\sec^2 x$; (*d*) $\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$

(*e*) $\frac{d}{dx}(\sec x)=\sec x \tan x$; (*f*) $\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x \cot x$

3. (*a*) $\frac{d}{dx}(e^x)=e^x$; (*b*) $\frac{d}{dx}(a^x)=a^x \log {}_e^a$

4. $\frac{d}{dx}(\log x)=\frac{1}{x}$.

§ 5·4. Fundamental Theorems on derivatives.

Theorem 1. The derivative of a constant is 0.

Proof. Let c be any constant. As for every value of x, the value of c is c i.e., c has a value, so c can be taken as a function of x. Let $c=f(x)$.

$$\therefore \quad \frac{d}{dx}(c)=\underset{h\to 0}{Lt}\ \frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lt}\ \frac{c-c}{h}=\underset{h\to 0}{Lt}\ 0=0.$$

Theorem 2. If c be a constant, then $\frac{d}{dx}\{c\,f(x)\}$

$=c\frac{d}{dx}\{f(x)\}$, at all those points where $f(x)$ is differentiable.

Proof. Let $\phi(x)=c\,f(x)$ and $f(x)$ is differentiable at x.

Then $\frac{d}{dx}\{c\,f(x)\}=\frac{d}{dx}\{\phi(x)\}=\underset{h\to0}{Lt}\ \frac{\phi(x+h)-\phi(x)}{h}$

$=\underset{h\to0}{Lt}\ \frac{c\,f(x+h)-c\,f(x)}{h}=\underset{h\to0}{Lt}\ \left[c\left\{\frac{f(x+h)-f(x)}{h}\right\}\right]$

$=c\underset{h\to0}{Lt}\ \left\{\frac{f(x+h)-f(x)}{h}\right\}$ $\Big[$As $f(x)$ is differentiable at x, so $\underset{h\to0}{Lt}\ \frac{f(x+h)-f(x)}{h}$ exists.$\Big]$

$=c\frac{d}{dx}\{f(x)\}=c\,f'(x).$

Theorem 3. If $f(x)$ and $g(x)$ be both differentiable at x, then $\frac{d}{dx}\{f(x)\pm g(x)\}=\frac{d}{dx}\{f(x)\}\pm\frac{d}{dx}\{g(x)\}$.

Proof. Let $F(x)=f(x)\pm g(x)$

$\therefore\ \frac{d}{dx}\{f(x)\pm g(x)\}=\frac{d}{dx}\{F(x)\}=\underset{h\to0}{Lt}\ \frac{F(x+h)-F(x)}{h}$

$=\underset{h\to0}{Lt}\ \frac{\{f(x+h)\pm g(x+h)\}-\{f(x)\pm g(x)\}}{h}$

$=\underset{h\to0}{Lt}\ \left\{\frac{f(x+h)-f(x)}{h}\pm\frac{g(x+h)-g(x)}{h}\right\}$

$=\underset{h\to0}{Lt}\ \frac{f(x+h)-f(x)}{h}\pm\underset{h\to0}{Lt}\ \frac{g(x+h)-g(x)}{h}$

[$\because$ $f(x)$ and $g(x)$ are both differentiable at x, So $\underset{h\to0}{Lt}\ \frac{f(x+h)-f(x)}{h}$ and $\underset{h\to0}{Lt}\ \frac{g(x+h)-g(x)}{h}$ both exist]

$=\frac{d}{dx}\{f(x)\}\pm\frac{d}{dx}\{g(x)\}=f'(x)\pm g'(x)$

Corollary. If $f_1(x)$, $f_2(x)$, $f_3(x),\ldots\ldots, f_n(x)$ are n[n is a finite positive integer] functions of x differentiable at x, then

(1) $\frac{d}{dx}\{\pm f_1(x)\pm f_2(x)\pm f_3(x)\pm\cdots\cdots\pm f_n(x)\}$

$=\pm\frac{d}{dx}\{f_1(x)\}\pm\frac{d}{dx}\{f_2(x)\}\pm\frac{d}{dx}\{f_3(x)\}\pm\cdots\cdots\pm\frac{d}{dx}\{f_n(x)\}$

(2) $$\frac{d}{dx}\{\pm c_1 f_1(x) \pm c_2 f_2(x) \pm c_3 f_3(x) \pm \cdots\cdots \pm c_n f_n(x)\}$$

$$= \pm c_1 \frac{d}{dx}\{f_1(x)\} \pm c_2 \frac{d}{dx}\{f_2(x)\} \pm c_3 \frac{d}{dx}\{f_3(x)\} \pm \cdots\cdots \pm c_n \frac{d}{dx}\{f_n(x)\}.$$

Theorem 4. If $f(x)$ and $g(x)$ be two differentiable functions of x, then

$$\frac{d}{dx}\{f(x)g(x)\} = g(x)\frac{d}{dx}\{f(x)\} + f(x)\frac{d}{dx}\{g(x)\}.$$

Proof. Let $F(x) = f(x)g(x)$.

$$\therefore \quad \frac{d}{dx}\{f(x)g(x)\} = \frac{d}{dx}\{F(x)\} = \underset{h\to 0}{Lt} \frac{F(x+h) - F(x)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \underset{h\to 0}{Lt} \left[\left\{\frac{f(x+h) - f(x)}{h}\right\} g(x+h) + f(x)\left\{\frac{g(x+h) - g(x)}{h}\right\}\right]$$

$$= \underset{h\to 0}{Lt} \frac{f(x+h) - f(x)}{h} \underset{h\to 0}{Lt}\, g(x+h) + \underset{h\to 0}{Lt}\, f(x) \underset{h\to 0}{Lt} \frac{g(x+h) - g(x)}{h}$$

[As $f(x)$ and $g(x)$ are differentiable, so $g(x)$ is continuous and all the limits exist]

$$= \frac{d}{dx}\{f(x)\}g(x) + f(x)\frac{d}{dx}\{g(x)\} = g(x)f'(x) + f(x)g'(x)$$

Theorem 5. If $f(x)$ and $g(x)$ be two differentiable functions and $g(x) \neq 0$, then

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\dfrac{d}{dx}\{f(x)\} - f(x)\dfrac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$$

Proof : Let $F(x)=\dfrac{f(x)}{g(x)}$

$$\therefore \quad \frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\}=\frac{d}{dx}\left\{F(x)\right\}=\underset{h\to 0}{Lt}\ \frac{F(x+h)-F(x)}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{\dfrac{f(x+h)}{g(x+h)}-\dfrac{f(x)}{g(x)}}{h}=\underset{h\to 0}{Lt}\ \frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x+h)g(x)}$$

$$=\underset{h\to 0}{Lt}\ \frac{f(x+h)\,g(x)-f(x)\,g(x)+f(x)\,g(x)-f(x)\,g(x+h)}{h\,g(x+h)\,g(x)}$$

$$\underset{h\to 0}{Lt}\ \frac{\{f(x+h)-f(x)\}\,g(x)-f(x)\{g(x+h)-g(x)\}}{h\,g(x+h)g(x)}$$

$$=\frac{\underset{h\to 0}{Lt}\ \dfrac{f(x+h)-f(x)}{h}\ \underset{h\to 0}{Lt}\ g(x)-\underset{h\to 0}{Lt}\ f(x)\underset{h\to 0}{Lt}\left\{\dfrac{g(x+h)-g(x)}{h}\right\}}{\underset{h\to 0}{Lt}\left\{g(x+h)\,g(x)\right\}}$$

[As $f(x)$ and $g(x)$ are differentiable and so $g(x)$ is continuous, hence all the limits exist separately]

$$=\frac{\left\{\dfrac{d}{dx}\,f(x)\right\}g(x)-f(x)\,\dfrac{d}{dx}\left\{g(x)\right\}}{g(x)\,.\,g(x)}$$

$$=\frac{f'(x)\,g(x)-f(x)\,g'(x)}{\{g(x)\}^2}$$

Note. This formula is remembered as derivative of quotient.

$$=\frac{(\text{ denominator}\times\text{derivative of the numerator })-(\text{ numerator}\times\text{derivative of the denominator })}{(\text{ denominator })^2}$$

EXAMPLES 5A

Example 1. Find the derivatives of the following functions from the definition.

(i) $y=x$ (ii) $y=x^2$ [H. S. '80] (iii) $y=x^6$ [H.S. '78]
(iv) $y=-2x$ [H. S. '79] (v) $y=x^3+2x$ [H. S. '83]
(vi) $y=x^{28}$ [Joint Entrance 1980]

(i) Let $y = f(x)$.

$$\therefore \quad \frac{dy}{dx} = f'(x) = \underset{h\to 0}{\text{Lt}} \; \frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \; \frac{x+h-x}{h} = \underset{h\to 0}{\text{Lt}} \; \frac{h}{h} = \underset{h\to 0}{\text{Lt}} \; (1) = 1.$$

(ii) Let $y = f(x)$

$$\therefore \quad \frac{dy}{dx} = f'(x) = \underset{h\to 0}{\text{Lt}} \; \frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \; \frac{(x+h)^2 - x^2}{h} = \underset{h\to 0}{\text{Lt}} \; \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \; \frac{2hx + h^2}{h} = \underset{h\to 0}{\text{Lt}} \; \frac{h(2x+h)}{h} = \underset{h\to 0}{\text{Lt}} \; (2x+h) = 2x.$$

(iii) Let $y = x^6$

$$\therefore \quad \frac{dy}{dx} = \underset{h\to 0}{\text{Lt}} \; \frac{(x+h)^6 - x^6}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \; \frac{x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6 - x^6}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \; \frac{6x^5 + 15x^4h^2 + 20x^3h^3 + 15x^2h^3 + 6xh^5 + h^6}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \; \frac{h(6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \; (6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5)$$

$$= 6x^5.$$

(iv) $$\frac{dy}{dx} = \underset{h\to 0}{\text{Lt}} \; \frac{-2(x+h) - (2x)}{h} = \underset{h\to 0}{\text{Lt}} \; \frac{-2x - 2h + 2x}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \; \frac{-2h}{h} = \underset{h\to 0}{\text{Lt}} \; (-2) = -2.$$

(v) $$\frac{dy}{dx} = \underset{h\to 0}{\text{Lt}} \; \frac{(x+h)^3 + 2(x+h) - x^3 - 2x}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \; \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x}{h}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{3x^2h+3xh^2+h^3+2h}{h}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{h(3x^2+3xh+h^2+2)}{h}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ (3x^2+3xh+h^2+2)$$

$$=3x^2+2.$$

(vi) $$\frac{dy}{dx}=\underset{h\to 0}{\mathrm{Lt}}\ \frac{(x+h)^{28}-x^{28}}{h}$$

$$=\underset{y\to x}{\mathrm{Lt}}\ \frac{y^{28}-x^{28}}{y-x}\qquad\left[\begin{array}{l}\text{where } y=x+h\,;\ \therefore\ h=y-x \text{ and}\\ \text{when } h\to 0, \text{ then } y\to x\end{array}\right]$$

$$=28x^{27}.$$

Ex. 2. Find the derivatives of the following functions from definition.

(i) $y=\sqrt{x}$ at $x=2$ [Joint Entrance '81]

(ii) $x^{-\frac{3}{2}}$ at $x=3$ [Joint Entrance '82]

(i) At $x=2$, $$\frac{dy}{dx}=\underset{h\to 0}{\mathrm{Lt}}\ \frac{\sqrt{2+h}-\sqrt{2}}{h}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{(\sqrt{2+h}-\sqrt{2})(\sqrt{2+h}+\sqrt{2})}{h(\sqrt{2+h}+\sqrt{2})}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{2+h-2}{h(\sqrt{2+h}+\sqrt{2})}=\underset{h\to 0}{\mathrm{Lt}}\ \frac{h}{h(\sqrt{2+h}+\sqrt{2})}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{1}{\sqrt{2+h}+\sqrt{2}}=\frac{1}{2\sqrt{2}}$$

(ii) Let $f(x)=x^{-\frac{3}{2}}$

$\therefore$ At $x=3$

$$f'(x)=\underset{h\to 0}{\mathrm{Lt}}\ \frac{f(3+h)-f(3)}{h}=\underset{h\to 0}{\mathrm{Lt}}\ \frac{(3+h)^{-\frac{3}{2}}-3^{-\frac{3}{2}}}{h}$$

$$\underset{h\to 0}{\mathrm{Lt}}\ \frac{\dfrac{1}{(3+h)^{\frac{3}{2}}}-\dfrac{1}{3^{\frac{3}{2}}}}{h}=\underset{h\to 0}{\mathrm{Lt}}\ \frac{3^{\frac{3}{2}}-(3+h)^{\frac{3}{2}}}{h\{(3+h)^{\frac{3}{2}}.3^{\frac{3}{2}}\}}$$

$$= -\underset{h\to 0}{\mathrm{Lt}} \frac{\{(3+h)^{\frac{3}{2}} - 3^{\frac{3}{2}}\}\{(3+h)^{\frac{3}{2}} + 3^{\frac{3}{2}}\}}{h\{(3+h)^{\frac{3}{2}}.3^{\frac{3}{2}}\}\{(3+h)^{\frac{3}{2}} + 3^{\frac{3}{2}}\}}$$

$$= -\underset{h\to 0}{\mathrm{Lt}} \frac{(3+h)^3 - 3^3}{h\{(3+h)^3.3^{\frac{3}{2}}\}\{(3+h)^{\frac{3}{2}} + 3^{\frac{3}{2}}\}}$$

$$= -\underset{h\to 0}{\mathrm{Lt}} \frac{27 + 27h + 9h^2 + h^3 - 27}{h\{(3+h)^{\frac{3}{2}}\ 3^{\frac{3}{2}}\}\{(3+h)^{\frac{3}{2}} + 3^{\frac{3}{2}}\}}$$

$$= -\underset{h\to 0}{\mathrm{Lt}} \frac{h(27 + 9h + h^2)}{h\{(3+h)^{\frac{3}{2}}.3^{\frac{3}{2}}\}\{(3+h)^{\frac{3}{2}} + 3^{\frac{3}{2}}\}}$$

$$= -\underset{h\to 0}{\mathrm{Lt}} \frac{27 + 9h + h^2}{\{(3+h)^{\frac{3}{2}}.3^{\frac{3}{2}}\}\{(3+h)^{\frac{3}{2}} + 3^{\frac{3}{2}}\}}$$

$$= -\frac{27}{3^{\frac{3}{2}}.3^{\frac{3}{2}}.2.3^{\frac{3}{2}}} = -\frac{1}{6\sqrt{3}}$$

Ex. 3. Find $f'(x)$ when

(i) $f(x) = \sqrt{x}$ [Joint Entrance 1986]

(ii) $f(x) = \dfrac{1}{\sqrt{x}}$

(i) $$f'(x) = \underset{h\to 0}{\mathrm{Lt}} \frac{f(x+h) - f(x)}{h} = \underset{h\to 0}{\mathrm{Lt}} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \underset{h\to 0}{\mathrm{Lt}} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \underset{h\to 0}{\mathrm{Lt}} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(ii) $$f'(x) = \underset{h\to 0}{\mathrm{Lt}} \frac{f(x+h) - f(x)}{h} = \underset{h\to 0}{\mathrm{Lt}} \frac{\dfrac{1}{\sqrt{x+h}} - \dfrac{1}{\sqrt{x}}}{h}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{\sqrt{x} - \sqrt{x+h}}{h(\sqrt{x+h} \cdot \sqrt{x})}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \frac{(\sqrt{x} - \sqrt{x+h})(\sqrt{x} + \sqrt{x+h})}{h(\sqrt{x}\sqrt{x+h})(\sqrt{x} + \sqrt{x+h})}$$

$$=\underset{h\to0}{\text{Lt}}\ \frac{x-x-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$

$$=\underset{h\to0}{\text{Lt}}\ \frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$

$$=\underset{h\to0}{\text{Lt}}\ \frac{1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}=-\frac{1}{2x\sqrt{x}}.$$

Ex. 4. Find the value of $f'(x)$ for given value of x when

$f(x)=\sec 2x\ \left(x=\frac{\pi}{8}\right)$ [Joint Entrance 1987]

$$f'(x)=\underset{h\to0}{\text{Lt}}\ \frac{f\left(\frac{\pi}{8}+h\right)-f\left(\frac{\pi}{8}\right)}{h}$$

$$=\underset{h\to0}{\text{Lt}}\ \frac{\sec 2\left(\frac{\pi}{8}+h\right)-\sec 2.\frac{\pi}{8}}{h}$$

$$=\underset{h\to0}{\text{Lt}}\ \frac{\dfrac{1}{\cos\left(\frac{\pi}{4}+2h\right)}-\dfrac{1}{\cos\frac{\pi}{4}}}{h}=\underset{h\to0}{\text{Lt}}\ \frac{\cos\frac{\pi}{4}-\cos\left(\frac{\pi}{4}+2h\right)}{h\cos\left(\frac{\pi}{4}+2h\right)\cos\frac{\pi}{4}}$$

$$=\underset{h\to0}{\text{Lt}}\ \frac{2\sin\left(\frac{\pi}{4}+h\right)\sin h}{h\cos\left(\frac{\pi}{4}+2h\right)\cos\frac{\pi}{4}}=\frac{2}{\cos\frac{\pi}{4}}\ \underset{h\to0}{\text{Lt}}\ \frac{\sin\left(\frac{\pi}{4}+h\right)}{\cos\left(\frac{\pi}{4}+2h\right)}\cdot\underset{h\to0}{\text{Lt}}\ \frac{\sin h}{h}$$

$$=\frac{2}{\frac{1}{\sqrt{2}}}\ .\tan\frac{\pi}{4}.1=2\sqrt{2}.1\ 1=2\sqrt{2}$$

Ex. 5. Find from the first principle the derivatives of the following functions :

(i) $f(x)=\sin^2 x$ (ii) $f(x)=\sin x^2$ [H. S. '84]

(iii) $f(x)=e^{\sqrt{x}}$ [Joint Entrance' 79 ; '84]

(iv) $f(x)=\sec 3x$ [H. S. '85] (v) $f(x)=\sin 4x$ [H. S. '88]

(vi) $f(x)=e^{-2x}$; at $x=0$ (H. S. 1987)

(i) $$f'(x)=\underset{h\to0}{Lt}\ \frac{f(x+h)-f(x)}{h}=\underset{h\to0}{Lt}\ \frac{\sin^2(x+h)-\sin^2x}{h}$$

$$=\underset{h\to0}{Lt}\ \frac{\sin(x+h+x)\sin(x+h-x)}{h}$$

$$=\underset{h\to0}{Lt}\ \frac{\sin(2x+h)\sin h}{h}=\underset{h\to0}{Lt}\ \sin(2x+h)\,\underset{h\to0}{Lt}.\frac{\sin h}{h}$$

$$=\sin 2x.1=\sin 2x.$$

(ii) $$f'(x)=\underset{h\to0}{Lt}\ \frac{f(x+h)-f(x)}{h}=\underset{h\to0}{Lt}\ \frac{\sin(x+h)^2-\sin x^2}{h}$$

$$=\underset{h\to0}{Lt}\ \frac{2\cos\left\{\frac{(x+h)^2+x^2}{2}\right\}\sin\left\{\frac{(x+h)^2-x^2}{2}\right\}}{h}$$

$$=\underset{h\to0}{Lt}\ \frac{2\cos\frac{2x^2+2hx+h^2}{2}\sin\frac{2hx+h^2}{2}}{h}$$

$$=2\underset{h\to0}{Lt}\ \cos\left(\frac{2x^2+2hx+h^2}{2}\right)\underset{h\to0}{Lt}\ \left\{\frac{\sin h\left(\frac{2x+h}{2}\right)}{h\left(\frac{2x+h}{2}\right)}.\left(\frac{2x+h}{2}\right)\right\}$$

$$=2\cos x^2\,\underset{h\to0}{Lt}\ \frac{\sin h\left(\frac{2x+h}{2}\right)}{h\left(\frac{2x+h}{2}\right)}\quad\underset{h\to0}{Lt}\left(\frac{2x+h}{2}\right)$$

$$=2\cos x^2.\ 1.\ x=2x\cos x^2$$

$$\left[\text{when } h\to0,\text{ then } h\left(\frac{2x+h}{2}\right)\to0,\right]$$

(iii) $$f'(x)=\frac{f(x+h)-f(x)}{h}$$

$$=\underset{h\to0}{Lt}\ \frac{e^{\sqrt{x+h}}-e^{\sqrt{x}}}{h}$$

$$=\underset{h\to0}{Lt}\ \left\{\frac{e^{\sqrt{x+h}}-e^{\sqrt{x}}}{\sqrt{x+h}-\sqrt{x}}.\frac{\sqrt{x+h}-\sqrt{x}}{h}\right\}$$

$$[\because\ h\to0,\ \sqrt{x+h}-\sqrt{x}\neq0\ ]$$

$$= \underset{h\to 0}{Lt}\ \frac{e^{\sqrt{x+h}} - e^{\sqrt{x}}}{\sqrt{x+h}-\sqrt{x}}\ \underset{h\to 0}{Lt}\ \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Now let $\sqrt{x+h}-\sqrt{x}=k$ and when $h\to 0$, then $k\to 0$.

$$\therefore\ \underset{h\to 0}{Lt}\ \frac{e^{\sqrt{x+h}} - e^{\sqrt{x}}}{\sqrt{x+h}-\sqrt{x}} = \underset{k\to 0}{Lt}\ \frac{e^{\sqrt{x}+k} - e^{\sqrt{x}}}{k}$$

$$= \underset{k\to 0}{Lt}\ \frac{e^{\sqrt{x}}(e^{k}-1)}{k} = e^{\sqrt{x}}\ \underset{k\to 0}{Lt}\ \frac{e^{k}-1}{k} = e^{\sqrt{x}}.1 = e^{\sqrt{x}}$$

Also $\underset{h\to 0}{Lt}\ \dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \dfrac{1}{2\sqrt{x}}$

$$\therefore\ f'(x) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}.$$

(iv) $f'(x) = \underset{h\to 0}{Lt}\ \dfrac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\ \dfrac{\sec 3(x+h) - \sec 3x}{h}$

$$= \underset{h\to 0}{Lt}\ \frac{\dfrac{1}{\cos 3(x+h)} - \dfrac{1}{\cos 3x}}{h} = \underset{h\to 0}{Lt}\ \frac{\cos 3x - \cos 3(x+h)}{h \cos 3(x+h) \cos 3x}$$

$$= \underset{h\to 0}{Lt}\ \frac{2 \sin\dfrac{3h}{2} \sin\left(\dfrac{6x+3h}{2}\right)}{h \cos 3(x+h) \cos 3x}$$

$$= \underset{h\to 0}{Lt}\left\{\frac{\sin\dfrac{3h}{2}}{\dfrac{3h}{2}}.\ 3\right\}\ \underset{h\to 0}{Lt}\left\{\frac{\sin\left(\dfrac{6x+3h}{2}\right)}{\cos 3(x+h) \cos 3x}\right\}$$

$$= 3\left\{\underset{h\to 0}{Lt}\ \frac{\sin\dfrac{3h}{2}}{\dfrac{3h}{2}}\right\} \frac{\sin 3x}{\cos 3x.\cos 3x} = 3.1.\ \tan 3x \sec 3x.$$

$= 3 \tan 3x \sec 3x.$

(v) $f'(x) = \underset{h\to 0}{Lt}\ \dfrac{\sin 4(x+h) - \sin 4x}{h}$

$$= \underset{h\to 0}{Lt}\ \frac{2 \cos (4x+2h) \sin 2h}{h}.$$

$$=2\underset{h\to0}{Lt}.\cos(4x+2h)\underset{h\to0}{Lt}\left(\frac{\sin 2h}{2h}.2\right)$$

$$=4.\cos 4x.\underset{h\to0}{Lt}\frac{\sin 2h}{2h}=4\cos 4x.$$

(vi) $$f'(0)=\underset{h\to0}{Lt}\frac{f(h)-f(0)}{h}=\underset{h\to0}{Lt}\frac{e^{-2h}-1}{h}$$

$$=\underset{h\to0}{Lt}\left\{\frac{e^{-2h}-1}{-2h}(-2)\right\}=-2\underset{h\to0}{Lt}\frac{e^{-2h}-1}{-2h}.$$

$$=-2.1=-2.$$

Ex. 6 Find from the definition, the derivatives of the following functions.

(i) $f(x)=\tan^{-1}x$ [Joint Entrance '85] (ii) $f(x)=\sin^{-1}x$

(iii) $f(x)=\log(\cos x)$ [Joint Entrance '88] (iv) $f(x)=x^x$.

(i) Let $y=\tan^{-1}x$ $\therefore$ $\tan y=x.$

As $x\to x+h$, let $y\to y+k.$

$\therefore$ $y+k=\tan^{-1}(x+h)$ or, $x+h=\tan(y+k)$

$h=\tan(y+k)-\tan y$

and when $h\to0$, then $k\to0.$

$$\therefore\ \frac{dy}{dx}=\underset{h\to0}{Lt}\frac{\tan^{-1}(x+h)-\tan^{-1}x}{h}=\underset{h\to0}{Lt}\frac{y+k-y}{h}$$

$$=\underset{h\to0}{Lt}\frac{k}{h}=\underset{k\to0}{Lt}\frac{k}{\tan(y+k)-\tan y}$$

$$=\underset{k\to0}{Lt}\frac{k}{\frac{\sin(y+k)}{\cos(y+k)}-\frac{\sin y}{\cos y}}=\underset{k\to0}{Lt}\frac{k\cos(y+k)\cos y}{\sin(y+k)\cos y-\cos(y+k)\sin y}$$

$$=\underset{k\to0}{Lt}\frac{k\cos(y+k)\cos y}{\sin k}=\frac{\underset{k\to0}{Lt}\cos(y+k)\cos y}{\underset{k\to0}{Lt}\frac{\sin k}{k}}$$

$$=\frac{\cos^2 y}{1}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}$$

(ii) Let $y=\sin^{-1}x$ and $y+k=\sin^{-1}(x+h)$

$\therefore$ $k=\sin^{-1}(x+h)-\sin^{-1}x$ and $\sin y=x$, $\sin(y+k)=x+h$

$\therefore$ If $h\to 0$, $k\to 0$.

Now $\dfrac{dy}{dx}=\underset{h\to 0}{Lt}\ \dfrac{\sin^{-1}(x+h)-\sin^{-1}x}{h}$

$$=\underset{h\to 0}{Lt}\ \frac{k}{x+h-x}=\underset{k\to 0}{Lt}\ \frac{k}{\sin(y+k)-\sin y}$$

$$=\underset{k\to 0}{Lt}\ \frac{1}{\dfrac{\sin(y+k)-\sin y}{k}}=\frac{1}{\underset{k\to 0}{Lt}\ \dfrac{2\cos\left(y+\frac{k}{2}\right)\sin\frac{k}{2}}{k}}$$

$$=\frac{1}{\underset{k\to 0}{Lt}\ \cos\left(y+\frac{k}{2}\right)\ \underset{h\to 0}{Lt}\ \dfrac{\sin\frac{k}{2}}{\frac{k}{2}}}=\frac{1}{\cos y\ .\ 1}=\frac{1}{\cos y}$$

$$=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$$

(iii) Let $y=(\cos x)$ and $y+k=\cos(x+h)$

$\therefore$ $k=\cos(x+h)-\cos x$

$\therefore$ When $h\to 0$, then $k\to 0$.

Now $\dfrac{d}{dx}\{\log(\cos x)\}=\underset{h\to 0}{Lt}\ \dfrac{\log\cos(x+h)-\log\cos x}{h}$

$$=\underset{h\to 0}{Lt}\ \frac{\log(y+k)-\log y}{h}=\underset{h\to 0}{Lt}\ \frac{\log\dfrac{y+k}{y}}{h}$$

$$=\underset{h\to 0}{Lt}\ \left\{\frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}}\cdot\frac{1}{y}\cdot\frac{k}{h}\right\}$$

$$=\frac{1}{y}\ \underset{k\to 0}{Lt}\ \frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}}\cdot\underset{h\to 0}{Lt}\ \frac{\cos(x+h)-\cos x}{h}$$

$$=\frac{1}{\cos x}\cdot 1.\underset{h\to 0}{Lt}\ \frac{2\sin\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h}$$

$$=-\frac{1}{\cos x}\ \underset{h\to 0}{Lt}\ \sin\left(x+\frac{h}{2}\right)\underset{h\to 0}{Lt}\ \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$=-\frac{1}{\cos x}\cdot\sin x.\ 1=-\tan x.$$

(iv) $f'(x)=\underset{h\to 0}{Lt}\ \dfrac{f(x+h)-f(x)}{h}$

$$\underset{h\to 0}{Lt}\ \frac{(x+h)^{x+h}-x^{x}}{h}=\underset{h\to 0}{Lt}\ \frac{e^{\log(x+h)^{(x+h)}}-e^{\log x^{x}}}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{e^{(x+h)\log(x+h)}-e^{x\log x}}{h}$$

$$=\underset{h\to 0}{Lt}\left\{\left(e^{x\log x}\right)\frac{e^{(x+h)\log(x+h)-x\log x}-1}{h}\right\}$$

$$=e^{x\log x}\cdot\underset{h\to 0}{Lt}\left\{\frac{e^{(x+h)\log(x+h)-x\log x}-1}{(x+h)\log(x+h)-x\log x}\times\frac{(x+h)\log(x+h)-x\log x}{h}\right\}$$

$$=e^{\log x^{x}}\ \underset{h\to 0}{Lt}\left\{\frac{e^{k}-1}{k}\cdot\frac{(x+h)\log(x+h)-x\log x}{h}\right\}$$

where $k=(x+h)\log(x+h)-x\log x$

$\therefore$ when $h\to 0$, then $k\to 0$

[Here we have assumed $\log x$ as a continuous function]

$$=x^{x}\cdot\underset{k\to 0}{Lt}\ \frac{e\ -1}{k}\ \underset{h\to 0}{Lt}\left[x\left\{\frac{\log(x+h)-\log x}{h}\right\}+\log(x+h)\right]$$

$$=x^{x}\ .\ 1\left[x\underset{h\to 0}{Lt}\ \frac{\log\left(\frac{x+h}{h}\right)}{h}+\underset{h\to 0}{Lt}\ \log(x+h)\right]$$

$$=x^x\left\{x\underset{h\to 0}{Lt}\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}\cdot x}+\log x\right\}$$

$\left[\because\right.$ log x is a continuous function, $\underset{h\to 0}{Lt}\log(x+h)=\log x\left.\right]$

$$=x^x\left\{x.\frac{1}{x}.1+\log x\right]=x^x(1+\log x)$$

Ex. 7. If $f(x)=x+\frac{1}{x}$, where $x\neq 0$, then prove from the first principle that $f'(1)=0$ [Joint Entrance 1983]

$$f'(1)=\underset{h\to 0}{Lt}\frac{f(1+h)-f(1)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{(1+h)+\frac{1}{1+h}-2}{h}$$

$$=\underset{h\to 0}{Lt}\frac{(1+h)^2+1-2(1+h)}{(1+h)h}$$

$$=\underset{h\to 0}{Lt}\frac{1+2h+h^2+1-2-2h}{(1+h)h}$$

$$=\underset{h\to 0}{Lt}\frac{h^2}{(1+h)h}=\frac{\underset{h\to 0}{Lt}\,h}{\underset{h\to 0}{Lt}(1+h)}=\frac{0}{1}=0$$

Ex. 8. If $f(x)$ is differentiable at $x=h$, find the value of $\underset{x\to h}{Lt}\frac{(x+h)f(x)-2h\,f(h)}{x-h}$ [H. S. 1986]

$$\underset{x\to h}{Lt}\frac{(x+h)f(x)-2h\,f(h)}{x-h}$$

$$=\underset{x\to h}{Lt}\frac{x\,f(x)+h\,f(x)-hf(h)-hf(h)}{x-h}$$

$$=\underset{x\to h}{Lt}\frac{x\,f(x)-x\,f(h)+x\,f(h)-h\,f(h)+h\,f(x)-h\,f(h)}{x-h}$$

$$=\underset{x\to h}{Lt}\,x\left\{\frac{f(x)-f(h)}{x-h}\right\}+\underset{x\to h}{Lt}\frac{(x-h)\,f(h)}{x-h}+\underset{x\to h}{Lt}\,h\left\{\frac{f(x)-f(h)}{x-h}\right\}$$

$$=\underset{x\to h}{Lt}\, x \underset{x\to h}{Lt} \frac{f(x)-f(h)}{x-h}+\underset{x\to h}{Lt} f(h)+h\underset{x\to h}{Lt} \frac{f(x)-f(h)}{x-h}$$

$$=hf'(h)+f(h)+hf'(h)=f(h)+2h\,f'(h)$$

Ex. 9. Find $\frac{dy}{dx}$ using formulas if.

(i) $y=x^{10}$ (ii) $y=\sqrt[5]{x}$ (iii) $y=\frac{1}{x}$ (iv) $y=\frac{1}{x^2}$

(v) $y=\frac{1}{\sqrt[3]{x^5}}$ (vi) $y=\sqrt[5]{x^3}$.

(i) $\frac{dy}{dx}=\frac{d}{dx}(x^{10})=10x^{10-1}=10x^9$.

(ii) $\frac{dy}{dx}=\frac{d}{dx}(\sqrt[5]{x})=\frac{d}{dx}(x^{\frac{1}{5}})=\frac{1}{5}x^{\frac{1}{5}-1}=\frac{1}{5}x^{-\frac{4}{5}}$

$=\frac{1}{5x^{\frac{4}{5}}}=\frac{1}{5\sqrt[5]{x^4}}$.

(iii) $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{d}{dx}(x^{-1})=-1.x^{-1-1}=-x^{-2}=-\frac{1}{x^2}$.

(iv) $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{x^2}\right)=\frac{d}{dx}(x^{-2})=-2.x^{-2-1}$

$=-2x^{-3}=-\frac{2}{x^3}$.

(v) $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x^5}}\right)=\frac{d}{dx}\left(x^{-\frac{5}{3}}\right)=-\frac{5}{3}x^{-\frac{5}{3}-1}$

$=-\frac{5}{3}x^{-\frac{8}{3}}=-\frac{5}{3x^{\frac{8}{3}}}=-\frac{5}{3.\sqrt[3]{x^8}}$

(vi) $\frac{dy}{dx}=\frac{d}{dx}(\sqrt[5]{x^3})=\frac{d}{dx}\left(x^{\frac{3}{5}}\right)=\frac{3}{5}x^{\frac{3}{5}-1}$

$=\frac{3}{5}x^{-\frac{2}{5}}=\frac{3}{5x^{\frac{2}{5}}}=\frac{3}{5\sqrt[5]{x^2}}$.

Ex. 10. If (i) $y=\sin(-x)$ (ii) $y=\cos^2\frac{x}{2}$, find $\frac{dy}{dx}$.

(i) $\frac{dy}{dx}=\frac{d}{dx}\{\sin(-x)\}=\frac{d}{dx}\{-\sin x\}=-\frac{d}{dx}(\sin x)=-\cos x$.

(ii) $\frac{dy}{dx}=\frac{d}{dx}\left(\cos^2\frac{x}{2}\right)=\frac{d}{dx}\{\frac{1}{2}(1+\cos x)\}$

$=\frac{1}{2}\left\{\frac{d}{dx}(1)+\frac{d}{dx}(\cos x)\right\}=\frac{1}{2}(0-\sin x)=-\frac{1}{2}\sin x$

Ex. 11. (i) $y=\log_2 x$; find $\frac{dy}{dx}$ [Joint Entrance '84]

(ii) $y=\log_{10}x^n$; find $\frac{dy}{dx}$

(i) $y=\log_2 x=\log_e x \log_2 e$

$\therefore \frac{dy}{dx}=\frac{d}{dx}(\log_e x \log_2 e)=\log_2 e \frac{d}{dx}(\log_e x)$

$=\log_2 e.\frac{1}{x}$

(ii) $y=\log_{10}x^n=n\log_{10}x=n\log_e x \log_{10}e$

$\therefore \frac{dy}{dx}=\frac{d}{dx}(n\log_e x \log_{10}e)=n\log_{10}e\frac{d}{dx}(\log_e x)$

$=n\log_{10}e\frac{1}{x}.$

Ex. 12. Determine the co-ordinates of the points at which $\frac{dy}{dx}=0$ for the function $y=x+\frac{1}{x}$. Also write the geometrical significance.

[H. S. 1985]

$y=x+\frac{1}{x}$ $\therefore$ $\frac{dy}{dx}=1-\frac{1}{x^2}$

$\therefore$ if $\frac{dy}{dx}=0$, then $1-\frac{1}{x^2}=0$ or, $x^2-1=0$

or, $x=\pm 1$, when $x=\pm 1$, then $y=\pm 2$.

At these two points the tangents to the graph of the function will be parallel to the x-axis and the function will have maximum or minimum values. [See Chapters One and Three of the portion 'Application of Calculus.']

Ex. 13. Find $\dfrac{dy}{dx}$ when

(i) $y=x^5+\dfrac{6}{x}$ (ii) $y=x^2+2e^x+3\sin x-4\cos x.$

(iii) $y=e^x\sec x$ (H. S. 1978) (iv) $y=\dfrac{\sin x}{\log x}$ (H. S. 1978)

(v) $y=\dfrac{x+2}{(x-1)(x+5)}$ (H. S. 1978)

(i) $\dfrac{dy}{dx}=\dfrac{d}{dx}\left(x^5+\dfrac{6}{x}\right)=\dfrac{d}{dx}(x^5)+6\dfrac{d}{dc}\left(\dfrac{1}{x}\right)$

$=5x^4+6.\left(-\dfrac{1}{x^2}\right)=5x^4-\dfrac{6}{x^2}.$

(ii) $\dfrac{dy}{dx}=\dfrac{d}{dx}(x^2+2e^x+3\sin x-4\cos x)$

$=\dfrac{d}{dx}(x^2)+\dfrac{d}{dx}(2e^x)+\dfrac{d}{dx}(3\sin x)-\dfrac{d}{dx}(4\cos x)$

$=2x+2.\dfrac{d}{dx}(e^x)+3\dfrac{d}{dx}(\sin x)-4\dfrac{d}{dx}(\cos x)$

$=2x+2e^x+3\cos x-4(-\sin x)$

$=2x+2e^x+3\cos x+4\sin x$

(iii) $\dfrac{dy}{dx}=\dfrac{d}{dx}(e^x\sec x)=e^x\dfrac{d}{dx}(\sec x)+\sec x\dfrac{d}{dx}(e^x)$

$=e^x\sec x\tan x+\sec x\, e^x=e^x\sec x(\tan x+1)$

(iv) $\dfrac{dy}{dx}=\dfrac{d}{dx}\left(\dfrac{\sin x}{\log x}\right)=\dfrac{\log x\dfrac{d}{dx}(\sin x)-\sin x\dfrac{d}{dx}(\log x)}{(\log x)^2}$

$=\dfrac{\log x.\cos x-\sin x.\dfrac{1}{x}}{(\log x)^2}=\dfrac{x\log x.\cos x-\sin x}{x(\log x)^2}$

(v) $\dfrac{dy}{dx}=\dfrac{d}{dx}\left\{\dfrac{x+2}{(x-1)(x+5)}\right\}=\dfrac{d}{dx}\left\{\dfrac{x+2}{x^2+4x-5}\right\}$

$=\dfrac{(x^2+4x-5)\dfrac{d}{dx}(x+2)-(x+2)\dfrac{d}{dx}(x^2+4x-5)}{(x^2+4x-5)^2}$

$$=\frac{(x^2+4x-5).1-(x+2)(2x+4)}{(x^2+4x-5)^2}$$

$$=\frac{(x^2+4x-5)-(2x^2+8x+8)}{(x^2+4x-5)^2}=-\frac{x^2+4x+13}{(x^2+4x-5)^2}$$

Ex. 14. If $f(x)=\frac{6-4x}{1+2x+2x^2}$, determine $f'(x)$. Also find those values of x for which $f'(x)=0$ [H. S. 1984]

$$f'(x)=\frac{d}{dx}\left(\frac{6-4x}{1+2x+2x^2}\right)$$

$$=\frac{(1+2x+2x^2)\frac{d}{dx}(6-4x)-(6-4x)\frac{d}{dx}(1+2x+2x^2)}{(1+2x+2x^2)^2}$$

$$=\frac{(1+2x+2x^2)(-4)-(6-4x)(2+4x)}{(1+2x+2x^2)^2}$$

$$=\frac{-16-24x+8x^2}{(1+2x+2x^2)^2}=\frac{8(x^2-3x-2)}{(1+2x+2x^2)^2}$$

If $f'(x)=0$, $\frac{8(x^2-3x-2)}{(1+2x+2x^2)^2}=0$

or, $x^2-3x-2=0$ $\therefore$ $x=\frac{3\pm\sqrt{9+8}}{2}=\frac{3\pm\sqrt{17}}{2}$

Ex. 15. Expressing $\tan x$ as $\frac{\sin x}{\cos x}$, show that the derivative of $\tan x$ is $\sec^2 x$.

$$\frac{d}{dx}(\tan x)=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{\cos x\frac{d}{dx}(\sin x)-\sin x\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$=\frac{\cos x.\cos x-\sin x(-\sin x)}{\cos^2 x}$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x.$$

Ex. 16. If $f(x)=\frac{x-1}{2x^2-7x+5}$ when $x\neq 1$

$=-\frac{1}{3}$ when $x=1$,

find the derivative of $f(x)$ at $=x=1$ [I. I. I. 1979] when $x\neq 1$, $f(x)=\dfrac{x-1}{2x^2-7x+5}=\dfrac{x-1}{(x-1)(2x-5)}=\dfrac{1}{2x-5}$

$$\therefore\quad f'(1)=\underset{h\to 0}{Lt}\ \frac{f(1+h)-f(1)}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{\dfrac{1}{2(1+h)-5}-\left(-\dfrac{1}{3}\right)}{h}=\underset{h\to 0}{Lt}\ \frac{\dfrac{1}{2h-3}+\dfrac{1}{3}}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{2h}{h(2h-3)3}=\underset{h\to 0}{Lt}\ \frac{2}{(2h-3).3.}=-\frac{2}{9}.$$

Ex. 17. A function $f(x)$ is defined as follows :

$f(x)=\dfrac{x^2}{a}-a$ when $0\leqslant x\leqslant a$

$=a-\dfrac{x^2}{a}$ when $x\geqslant a$

Test the continuity and differentiability of the function at $x=a$

$$\underset{x\to a-}{Lt}\ f(x)=\underset{x\to a-}{Lt}\ \left(\frac{x^2}{a}-a\right)=\frac{a^2}{a}-a=0$$

$$\underset{x\to a+}{Lt}\ f(x)=\underset{x\to a+}{Lt}\ \left(a-\frac{x^2}{a}\right)=a-\frac{a^2}{a}=0.$$

$$\therefore\quad \underset{x\to a-}{Lt}\ f(x)=\underset{x\to a+}{Lt}\ f(x)=0$$

$$\therefore\quad \underset{x\to a}{Lt}\ f(x)=0=f(a)$$

$\therefore$ $f(x)$ is continuous at $x=a$.

Again, $f'(a-)=\underset{h\to 0-}{Lt}\ \dfrac{f(a+h)-f(a)}{h}$

$$=\underset{h\to 0-}{Lt}\ \frac{\dfrac{(a+h)^2}{a}-a-0}{h}=\underset{h\to 0-}{Lt}\ \frac{a^2+2ha+h^2-a^2}{ah}$$

$$=\underset{h\to 0-}{Lt}\ \frac{h(2a+h)}{ah}=\underset{h\to 0}{Lt}\ \frac{(2a+h)}{a}=2.$$

$$'(a+)=\underset{h\to 0+}{Lt}\ \frac{f(a+h)-f(a)}{h}=\underset{h\to 0+}{Lt}\ \frac{a-\dfrac{(a+h)^2}{a}-0}{h}$$

$$=\underset{h\to 0+}{Lt}\ \frac{a^2-(a^2+2ah+h^2)}{ah}=\underset{h\to 0+}{Lt}\ \frac{-h(2a+h)}{ah}$$

$$=\underset{h\to 0+}{Lt}\ -\left(\frac{2a+h}{a}\right)=-2.$$

$\therefore\ f'(a-)\neq f'(a+)$

$\therefore\ f'(a)$ does not exist i.e., the function is not differentiable at $x=a$,

Ex. 18. Let $f(x+y)=f(x).f(y)$, for all x and suppose $f(5)=2$ and $f'(0)=3$, find $f'(5)$ [I. I. T. 1981]

$f(x+y)=f(x).\ f(y)$.

Putting $x=5$, $y=0$, $f(5+0)=f(5).\ f(0)$

or, $f(5)=f(5).\ f(0)$ $\therefore$ $f(0)=1$ [$\because$ $f(5)=2\neq 0$]

Now $f'(5)=\underset{h\to 0}{Lt}\ \dfrac{f(5+h)-f(5)}{h}$

$$=\underset{h\to 0}{Lt}\ \frac{f(5).\ f(h)-f(5)}{h}=\underset{h\to 0}{Lt}\ \frac{2f(h)-2}{h}$$

$$=2\underset{h\to 0}{Lt}\ \frac{f(h)-1}{h}=2\underset{h\to 0}{Lt}\ \frac{f(h)-f(0)}{h}$$

$=2\times f'(0)=2\times 3=6.$

Ex. 19. If $f(a)=2$, $f'(a)=1$, $g(a)=-1$, $g'(a)=2$, find the value of $\underset{x\to a}{Lt}\ \dfrac{g(x)f(a)-g(a)f(x)}{x-a}$ [C. F. I. I. T. 1983]

$$\underset{x\to a}{Lt}\ \frac{g(x)\ f(a)-g(a)f(x)}{x-a}$$

$$=\underset{x\to a}{Lt}\ \frac{g(x)f(a)-g(a)f(a)+g(a)f(a)-g(a)f(x)}{x-a}$$

$$=\underset{x\to a}{Lt}\ \frac{\{g(x)-g(a)\}f(a)-g(a)\{f(x)-f(a)\}}{x-a}$$

$$=\underset{x\to a}{Lt}\ \left\{\frac{g(x)-g(a)}{x-a}\ f(a)-g(a)\ \frac{f(x)-f(a)}{x-a}\right\}$$

$$=f(a)\underset{x\to a}{Lt}\ \frac{g(x)-g(a)}{x-a}-g(a)\underset{x\to a}{Lt}\ \frac{f(x)-f(a)}{x-a}$$

$=f(a)g'(a)-g(a)f'(a)=2.2-(-1).1=4+1=5.$

Ex. 20. Show that the function $f(x)=1+|\sin x|$ is not differentiable at $x=0$ and $x=\pi$ [C. F. I. I. T. 1986]

$f(x)=1+|\sin x|=1+\sin x$ if $\sin x\geqslant 0$

$\qquad\qquad\qquad\quad =1-\sin x$ if $\sin x\leqslant 0$

So, when $-\pi<x<0$, $\sin x<0$, $f(x)=1-\sin x$

when $0<x<\pi$, $\sin x>0$, $f(x)=1+\sin x$

when $\pi<x<2\pi$, $\sin x<0$, $f(x)=1-\sin x$

$f(0)=1+|\sin 0|=1$; $f(\pi)=1+(\sin\pi)=1$

$$\therefore\quad f'(0-)=\underset{h\to0-}{Lt}\ \frac{f(0+h)-f(0)}{h}=\underset{h\to0-}{Lt}\ \frac{f(h)-f(0)}{h}$$

$$=\underset{h\to0-}{Lt}\ \frac{1-\sin h-1}{h}=\underset{h\to0-}{Lt}\ \frac{-\sin h}{h}=-1$$

$$f'(0+)=\underset{h\to0+}{Lt}\ \frac{f(0+h)-f(0)}{h}=\underset{h\to0+}{Lt}\ \frac{f(h)-f(0)}{h}$$

$$=\underset{h\to0+}{Lt}\ \frac{1+\sin h-1}{h}=\underset{h\to0+}{Lt}\ \frac{\sin h}{h}=-1$$

$\therefore\quad f'(0-)\neq f'(0+).\qquad \therefore\ f'(0)$ does not exist.

$$f'(\pi-)=\underset{h\to0-}{Lt}\ \frac{f(\pi+h)-f(\pi)}{h}=\underset{h\to0-}{Lt}\ \frac{1+\sin(\pi+h)-1}{h}$$

$$=\underset{h\to0-}{Lt}\ \frac{-\sin h}{h}=-1.$$

$$f'(\pi+)=\underset{h\to0+}{Lt}\ \frac{f(\pi+h)-f(\pi)}{h}=\underset{h\to0+}{Lt}\ \frac{1-\sin(\pi+h)}{h}$$

$$=\underset{h\to0+}{Lt}\ \frac{\sin h}{h}=1$$

$\therefore\quad f'(\pi-)\neq f'(\pi+)\qquad \therefore\ f'(\pi)$ does not exist.

Ex. 21. Show that the derivative of every even function is an odd function.

Let $f(x)$ be an even function. $\quad\therefore\ f(-x)=f(x)$

$$\text{Now, } f'(-x)=\underset{h\to0}{Lt}\ \frac{f(-x+h)-f(-x)}{h}$$

$$=\underset{h\to0}{Lt}\ \frac{f(x-h)-f(x)}{h}$$

[$\because$ $f(x)=f(-x)$, So $f(-x+h)=f(x-h)$]

$$=-\underset{h\to 0}{Lt}\ \frac{f(x-h)-f(x)}{-h}$$

$$=-f'(x)$$

$$\therefore\quad f'(-x)=-f'(x)$$

$\therefore$ $f'(x)$ is an odd function.

Exercise 5A

1. Determine the derivatives of the following functions with respect to x from the definition.

(i) $y=x^4$ (ii) $y=-3x$ (Tripura 1982)

(iii) $y=x^{2\cdot 5}$ (iv) $y=ax^2+bx+c$ (v) $y=\frac{1}{x^2}$

2. Find the differential coefficients of the following functions with respect to x at the specified points

(i) $f(x)=\frac{1}{\sqrt{x}}$ (at $x=4$)

(ii) $f(x)=\frac{1}{x^3}$ (at $x=2$)

(iii) $f(x)=\sqrt[3]{x}$ (at $x=8$)

(iv) $f(x)=x^3+x$ (at $x=2$)

3. Find from the first principle the derivatives of the following functions with respect to x.

(i) $y=\sin 5x$ (ii) $y=\cos^2 x$ (iii) $y=\sin^2 2x$

(iv) $y=\tan\frac{x}{2}$ (v) $y=a\sin\left(\frac{x}{a}\right)$ (vi) $y=\sqrt{\tan x}$.

4. Find from the first principle the derivatives of the following functions with respect to x.

(i) $f(x)=e^{2x}$ (ii) $f(x)=e^{mx}$ [Tripura 1983]

(iii) $f(x)=e^{\sin x}$.

5. Find the differential coefficients *ab initio*.

(i) $\log_{10} x$ (ii) $\log\sin\left(\frac{x}{a}\right)$ (iii) $\log\sin x$.

6. Find from the definition the differential coefficients with respect to x.

(i) $\cos x^2$ (ii) $\sin(\log x)$ (iii) $e^{\cos x}$ (at $x=0$)
(iv) $\sin x^3$ (v) $\operatorname{cosec} 3x$

7. Find from the first principle the derivatives of the following functions with respect to x.

(i) $x^{\cos x}$ (ii) $\cos^{-1} x$ (iii) $x \cos x$ (iv) $x^3+\frac{1}{x^3}$.

8. Write down the differential coefficients of :

(i) 56 (ii) x (iii) x^7 (iv) x^{-5} (v) $5x^6$ (vi) $\frac{x^6}{6}$ (vii) $\frac{x^n}{n}$

(viii) $\sqrt[3]{x}$ (ix) $\frac{1}{\sqrt[3]{x}}$ (x) (ax^m) (xi) $\sqrt[6]{x^{13}}$ (xii) $\frac{1}{\sqrt[6]{x^{13}}}$.

9. Write down the differential coefficients with respect to x of

(i) $-5 \log x$ (ii) $\log e^{2x}$ (iii) $e^{\log 2x}$

10. Find the derivatives with respect to x (Ex. 10—Ex. 13) :
(i) $ax^2+2bx+c-5$ (ii) $(\alpha x)^n+\beta^m$ (iii) $4x^2+5\cos x$
(iv) $x+2e^x+3\log x+4\sin x-5\cos x+6\tan x-7\cot x$
$+8\sec x-9\operatorname{cosec} x$
(v) $(a+bx)^3$ (vi) $(a-bx)^2$ (vii) $\frac{(1+x)^3}{x}$ (viii) $\frac{(1-x)^3}{x^2}$

(ix) $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}$ (x) $\sqrt{x}+2x\sqrt{x}+3x^3/x$.

11. (i) $\log_2 x$ (ii) $\log x^n$ (iii) $\log x+\log x^2+\log x^3+\cdots+\log x^{10}$
(iv) $\log_a x+\log_a x^2+\log_a x^3$.

12. (i) $x^5 e^x$ (ii) $e^x \sin x$ (iii) $3^x \sin x$ (iv) $5^x x^5$
(v) $(x^2+1)\cos x$ (vi) $x^2 \log x$ (vii) $x^2 \log x^3$
(viii) $(x+2)e^x$ (ix) $(x^2+2)\log x$ (x) $e^x \log x$.

13. (i) $e^x \cos x+\sin x\, e^x$ (ii) $x^3 e^x+\cot x$
(iii) $e^x x \cot x \log x^x$ (iv) $x \log x \cot x$ (v) $(x-1)(x-2)(x-3)$

14. Express $\sin 2x$ as $2 \sin x \cos x$ and find the derivative of $\sin 2x$.

15. Expressing $\tan x$ as $\sin x \sec x$; show that $\frac{d}{dx}(\tan x)=\sec^2 x$

16. Find the derivatives with respect to x

(i) $\frac{x^n}{\log x}$ (ii) $\frac{\sin x}{x}$ (iii) $\frac{x}{\sin x}$ (iv) $\frac{\cos x}{\log x}$

(v) $\frac{3x^2+2}{\sin x+\cos x}$ (vi) $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$ (vii) $\frac{\cot x}{x+e^x}$ (viii) $\frac{\sin x}{2^x}$

17. (i) Express $\tan x$ as $\frac{1}{\cot x}$ (ii) $\sec x$ as $\frac{1}{\cos x}$

(iii) $\operatorname{cosec} x$ as $\frac{1}{\sin x}$ and show that

$$\frac{d}{dx}(\tan x)=\sec^2 x,\ \frac{d}{dx}(\sec x)=\sec x \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x \cot x.$$

18. Find $\frac{dy}{dx}$(Ex. 18—22)

(i) $y=\frac{e^x}{x}+x^2+2x$ (ii) $y=\frac{x}{\sin x}+3e^x+x\tan x$

(iii) $y=5x^2+7x\cos x+\frac{8\sin x}{x^2}$.

19. (i) $y=\frac{1-\cos x}{1+\cos x}$ (ii) $y=\frac{1+\sqrt{x}}{1-\sqrt{x}}$

(iii) $y=\frac{\cos x-\cos 2x}{1-\cos x}$ [H. S. 1981]

20. (i) $y=\frac{x^4+x^2+1}{x^2-x+1}$ (ii) $y=\frac{x^5-1}{x-1}$.

21. (i) $y=\frac{x^2-4}{x^2(x+4)}$ [H. S. 1980]

(ii) $y=\frac{x\sin x+\cos x}{x\cos x-\sin x}$ [H. S. 1982] (iii) $\frac{\sin x-x\cos x}{x\sin x+\cos x}$

(iv) $y=\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}$

22. (i) $y=\left(\dfrac{x^b}{x^c}\right)^{b+c}\left(\dfrac{x^c}{x^a}\right)^{c+a}\left(\dfrac{x^a}{x^b}\right)^{a+b}$

(ii) $y=\dfrac{1}{1+x^{a-b}+x^{a-c}}+\dfrac{1}{1+x^{b-c}+x^{b-a}}+\dfrac{1}{1+x^{c-a}+x^{c-b}}$

23. Express cot x as $\dfrac{\cos x}{\sin x}$ and hence show that

$$\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x.$$

24. $y=\dfrac{x-3}{x+3}$. Show that $2x\dfrac{dy}{dx}=1-y^2$

25. v and r denote the volume and radius of a sphere respectively; show that $\dfrac{dv}{dr}$ represents the area of the curved surface of the sphere.

26. $f(x)=\sec x+\tan x$; show that $f'(x)=f(x)\sec x$.

27. A function $f(x)$ is as follows :

$$\begin{aligned} f(x)&=x \text{ when } 0<x<1\\ &=2-x,\ 1\leqslant x\leqslant 2\\ &=3x-x^2,\ x>2 \end{aligned}$$

Show that the function is discontinuous at $x=2$. Also verify that the function is not differentiable at $x=2$.

28. $$\begin{aligned} \phi(x)&=\tfrac{1}{2}(b^2-a^2),\ 0\leqslant x\leqslant a\\ &=\tfrac{1}{2}b^2-\tfrac{1}{6}x^2-\tfrac{1}{3}\left(\frac{a^3}{x}\right),\ a\leqslant x\leqslant b\\ &=\tfrac{1}{3}\frac{(b^3-a^3)}{x},\ x\geqslant b. \end{aligned}$$

Show that $\phi'(x)$ is continuous for every positive value of x.

[C. U.]

29. $$\begin{aligned} f(x)&=3+2x,\ -\tfrac{3}{2}<x\leqslant 0\\ &=3-2x,\ 0<x<\tfrac{3}{2}. \end{aligned}$$

Show that $f(x)$ is continuous at $x=0$ but $f'(0)$ does not exist.

[C. U.]

In § 5·5 we have discussed the formula of differentiation of a function of a function. Here let $y = \phi(z)$,

$z = g(u)$, $u = h(v)$, $v = \psi(w)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{dz} \dfrac{dz}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dw} \cdot \dfrac{dw}{dx}$.

The formula is explained with the help of an example.

Let $y = \sin\{\log \cos(e^{x})\}$.

Here $y = \sin z$, where $z = \log \cos(e^{x^2})$.

$z = \log u$, where $u = \cos(e^{x^2})$

$u = \cos v$, where $v = e^{x^2}$

$v = e^{w}$ where $w = x^2$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

$$= \cos z \frac{1}{u} (-\sin v) e^{w} . 2x$$

$$= -\cos \{ \log \cos (e^{x^2}) \} \frac{1}{\cos (ex^2)} \sin(e^{x^2}) e^{x^2} . 2x$$

$$= -2x \cos \{ \log (\cos e^{x^2}) \} . \frac{1}{\cos(e^{x^2})} \sin(e^{x^2}) e^{x^2}$$

§ 5·8. Derivatives of Implicit Functions.

In chapter two we have discussed what is meant by an Implicit Function. If y be an implicit function of x, then to find $\dfrac{dy}{dx}$ i.e., derivative of y with respect to x, one has to differentiate both sides of the implicit relation between x and y. After differentiation, solving the new relation, $\dfrac{dy}{dx}$ is obtained. The following example will explain the method.

Examples

(i) If $x^2 + y^2 = a^2$, find $\dfrac{dy}{dx}$.

Differentiating both sides of the equation
$x^2+y^2=a^2$ with respect to x we get

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(a^2).$$

[Here y is a function of x]

or $$2x+\frac{d}{dy}(y^2)\frac{dy}{dx}=0$$

or, $$2x+2y\frac{dy}{dx}=0 \quad \therefore \quad 2y\frac{dy}{dx}=-2x.$$

or, $$\frac{dy}{dx}=-\frac{x}{y}.$$

(ii) $xy=\sin(x+y)$. Find $\frac{dy}{dx}$

Differentiating both sides of the given equation with respect to x we get

$$\frac{d}{dx}(xy)=\frac{d}{dx}\{\sin(x+y)\}$$

or, $$y\frac{d}{dx}(x)+x\frac{d}{dx}(y)=\cos(x+y)\ \frac{d}{dx}(x+y)$$

or, $$y.\ 1+x\frac{dy}{dx}=\cos(x+y)\left(1+\frac{dy}{dx}\right)$$

or, $$x\frac{dy}{dx}-\cos(x+y)\frac{dy}{dx}=\cos(x+y)-y.$$

or, $$\frac{dy}{dx}\{x-\cos(x+y)\}=\cos(x+y)-y.$$

or, $$\frac{dy}{dx}=\frac{\cos(x+y)-y}{x-\cos(x+y)}.$$

Note. Try to understand the differentiations done above.

In case of $\frac{d}{dx}(y^2)$, y^2 is at the first instance a function of y and then y is a function of x.

So y^2 is a function of a function of x.

So, $$\frac{d}{dx}(y^2)=\frac{d}{dy}(y^2)\frac{dy}{dx}=2y\frac{dy}{dx}.$$

Similarly $\frac{d}{dx}(y^3)=\frac{d}{dy}(y^3)\,\frac{dy}{dx}=3y^2\frac{dy}{dx}$.

$\frac{d}{dx}(\sin y)=\frac{d}{dy}(\sin y)\,\frac{dy}{dx}=\cos y\frac{dy}{dx}$.

$\frac{d}{dx}(e^y)=\frac{d}{dy}(e^y)\,\frac{dy}{dx}=e^y\frac{dy}{dx}$.

$\frac{d}{dx}(\log y)=\frac{d}{dy}(\log y)\frac{dy}{dx}=\frac{1}{y}\frac{dy}{dx}$.

Again in case of $\frac{d}{dx}\{\sin(x+y)\}$,

$\sin(x+y)$ is a function of $(x+y)$ and $(x+y)$ is a function of x.

So $\frac{d}{dx}\{\sin(x+y)\}$

$=\frac{d}{dx}(\sin z)$ [where $z=x+y$]

$=\frac{d}{dz}(\sin z)\,\frac{dz}{dx}$

$=\cos z\frac{d}{dx}(x+y)=\cos(x+y)\left\{1+\frac{dy}{dx}\right\}$.

§ 5·9. Relation between the derivatives of mutually inverse functions.

In chapter two we have said that if solving x from the relation $y=f(x)$, we can get a relation of the form $x=\phi(y)$, then the two functions $f(x)$, ane $\phi(y)$ are said to be inverse functions of each other.

Let $y=x^3+1$ $\therefore$ $x^3=y-1$ or, $x=\sqrt[3]{y-1}$.

If we sey, $f(x)=x^3+1$ and $\phi(y)=\sqrt[3]{y-1}$,

then $f(x)$ and $\phi(y)$ are inverse functions of each other.

Let $y=f(x)$ and $x=\phi(y)$ are two mutually inverse functions.

From the formula of differentiation of function of a function we get.

$$\frac{d}{dx}\{\phi(y)\}=\frac{d}{dy}\{\phi(y)\}\frac{dy}{dx}.$$

or $$\frac{d}{dx}\{x\}=\frac{d}{dy}(x)\frac{dy}{dx}\ [\because \phi(y)=x]$$

or $$1=\frac{dx}{dy}\cdot\frac{dy}{dx}.$$

Corollary $$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

§ 5·10. Derivatives of Inverse circular functions.

(i) Derivative of $\sin^{-1}x$.

Let $y=\sin^{-1}x \quad \therefore \quad \sin y=x$.

Differentiating both sides with respect to x we get

$$\cos y\frac{dy}{dx}=1 \quad \text{or,} \quad \frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}$$

or, $$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}.$$

(ii) Derivative of $\cos^{-1}x$.

Let $y=\cos^{-1}x \quad \therefore \quad \cos y=x$.

Differentiating both sides with respect to x we get

$$-\sin y\frac{dy}{dx}=1 \text{ or, } \frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-\cos^2 y}}$$

or, $$\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}.$$

(iii) Derivative of $\tan^{-1}x$.

Let $y=\tan^{-1}x \quad \therefore \quad \tan y=x$.

Differentiating both sides with respect to x we get

$$\sec^2 y\frac{dy}{dx}=1 \quad \text{or,} \quad \frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}$$

$$\therefore \quad \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}.$$

(iv) Derivative of $\cot^{-1}x$

Let $y=\cot^{-1}x$ $\quad\therefore\quad \cot y=x$

Differentiating both sides with respect to x we get

$$-\operatorname{cosec}^2 y\frac{dy}{dx}=1.$$

$$\text{or,}\quad \frac{dy}{dx}=-\frac{1}{\operatorname{cosec}^2 y}=-\frac{1}{1+\cot^2 y}=-\frac{1}{1+x^2}$$

$$\therefore\quad \frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$$

(v) Derivative of $\sec^{-1}x$.

Let $y=\sec^{-1}x$ $\quad\therefore\quad \sec y=x.$

Differentiating both sides with respect to x we get

$$\sec y\tan y\frac{dy}{dx}=1\quad\therefore\quad\frac{dy}{dx}=\frac{1}{\sec y\tan y}=\frac{1}{\sec y\sqrt{\sec^2 y-1}}$$

$$\text{or}\quad \frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$$

(vi) Derivative of $\operatorname{cosec}^{-1}x$.

Let $y=\operatorname{cosec}^{-1}x$ $\quad\therefore\quad \operatorname{cosec} y=x.$

Differentiating both sides with respect to x we get

$$-\operatorname{cosec} y\cot y\frac{dy}{dx}=1\qquad\therefore\quad\frac{dy}{dx}=\frac{1}{-\operatorname{cosec} y\cot y}$$

$$=-\frac{1}{\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y-1}}$$

$$\text{So,}\quad \frac{d}{dx}(\operatorname{cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}$$

Note. As $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

$$\therefore\quad \frac{d}{dx}\left(\cos^{-1}x\right)=\frac{d}{dx}\left(\frac{\pi}{2}-\sin^{-1}x\right)$$

$$=\frac{d}{dx}\left(\frac{\pi}{2}\right)-\frac{d}{dx}\left(\sin^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}}$$

Similarly $\frac{d}{dx}\left(\cot^{-1}x\right)$ and $\frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right)$ could be determined

§ 5·11. Derivatives of parametric functions.

Let $x=f(t)$ and $y=\phi(t)$ where t is a parameter.

$$\text{Then } \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\phi'(t)}{f'(t)}.$$

Example. Let $x=a\cos t,\ y=b\sin t.$

$$\therefore\quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{d}{dt}(b\sin t)}{\frac{d}{dx}(a\cos t)}=\frac{b\cos t}{-a\sin t}=-\frac{b}{a}\tan t$$

§ 5·12. Logarithmic Differentiation.

In the determination of $\frac{dy}{dx}$ from a relation connecting x and y sometimes it is found convenient or sometimes it is found essential to differentiate with respect to x the relation obtained after taking logarithm of both sides. Let the index of a variable is also a variable, i.e., y is of the form $\{f(x)\}^{\phi(x)}$; in these cases logarithmic differentiation is essential (if, of course, the differentiation is not done from the first principle).

Examples.

(i) Let $y=x^{\sin x}$.

Taking logarithm on both sides we get

$$\log y=\log x^{\sin x}=\sin x\log x.$$

Differentiating both sides with respect to x we get

$$\frac{1}{y}\frac{dy}{dx}=\log x\cos x+\sin x.\frac{1}{x}.$$

$$\therefore\quad \frac{dy}{dx}=y\left(\log x\cos x+\frac{\sin x}{x}\right)$$

$$=x^{\sin x}\left(\log\cos x+\frac{\sin x}{x}\right)$$

(ii) Find $\frac{dy}{dx}$ if $y=(x-a)(x-b)(x-c)(x-d)(x-e)(x-f)(x-g)$

$$y=(x-a)(x-b)(x-c)(x-d)(x-e)(x-f)$$

or, $\log y=\log\{(x-a)(x-b)(x-c)(x-d)(x-e)(x-f)$

$$=\log(x-a)+\log(x-b)+\log(x-c)+\log(x-d)$$
$$+\log(x-e)+\log(x-f)$$

Differentiating both sides with respect to x we get,

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\frac{1}{x-d}+\frac{1}{x-e}+\frac{1}{x-f}$$

$$\therefore\quad \frac{dy}{dx}=y\left\{\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\frac{1}{x-d}+\frac{1}{x-e}+\frac{1}{x-f}\right\}$$
$$=(x-a)(x-b)(x-c)(x-d)(x-e)(x-f)\times$$
$$\left\{\frac{1}{(x-a)}+\frac{1}{(x-b)}+\frac{1}{(x-c)}+\frac{1}{(x-d)}+\frac{1}{(x-e)}+\frac{1}{(x-f)}\right\}$$

§ 5·13. Differentiation with respect to a function.

$f(x)$ and $g(x)$ are two functions of x.

If one has to differentiate $f(x)$ with respect to $g(x)$, then let $y=f(x)$ and $z=g(x)$.

$\therefore$ we are to find $\frac{dy}{dz}$

$$\text{Now}\quad \frac{dy}{dx}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{f'(x)}{g'(x)}.$$

Example. If $y=\sin x$ and $z=x^3$, derivative of $\sin x$ with respect to x^3.

$$=\frac{dy}{dz}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x^3)}=\frac{\cos x}{3x^2}.$$

§ 5·14. Differentiation of an infinite series or an infinite product.

In this elementary discourse we shall assume that every infinite series or infinite product asked to differentiate, is convergent and satisfies all requisite conditions to be differentiable. The methods have been shown in Examples 5B.

Examples 5B

Example 1. Find $\frac{dy}{dx}$ when,

(i) $y=(7x+11)^{101}$ (ii) $y=\sin 5x$. (iii) $y=\tan 4x$.

(iv) $y=e^{2x}$ (v) $y=\cos(2-3x)$ (vi) $y=e^{3x+4}$

(vii) $y=\log(2x+3)$ (viii) $y=\sqrt{x^2+5}$ (ix) $y=\frac{1}{\sqrt{x^2+a^2}}$

(x) $y=\sqrt[5]{x^2+x+1}$.

(i) $y=(7x+11)^{101}=z^{101}$ where $7x+11=z$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(z^{101})\ \frac{d}{dx}(7x+11)$$

$$=101\,z^{100}\left\{\frac{d}{dx}(7x)+\frac{d}{dx}(11)\right\}$$

$$=101\,(7x+11)^{100}.\ 7=707\,(7x+11)^{100}$$

(ii) $y=\sin 5x=\sin z$ where $z=5x$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(\sin z)\ \frac{d}{dx}(5x)=\cos z.\ 5$$

$$=5\cos 5x.$$

(iii) $y=\tan 4x=\tan z$ where $z=4x$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(\tan z)\ \frac{d}{dx}(4x)=\sec^2 z.4$$

$$=4\sec^2 4x$$

(iv) $y=e^{2x}=e^{z}$ where $2x=z$.

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}{}^{z}(e)\ \frac{d}{dx}(2x)=e^{z}.\ 2=2e^{2x}.$$

(v) $y=\cos(2-3x)=\cos z$ where $z=2-3x$.

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(\cos z)\ \frac{d}{dx}(2-3x)$$

$$=-\sin z\,(-3)=3\sin(2-3x).$$

(vi) $y=e^{3x+4}=e^{z}$ where $z=3x+4$.

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}\,(e^{z})\quad\frac{d}{dx}\,(3x+4)=e^{z}\;.\;3.$$

$$=3\,e^{3x+4}.$$

(vii) $y=\log\,(2x+3)=\log\,z$ where $z=2x+3$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}\,(\log\,z)\quad\frac{d}{dx}\,(2x+3)$$

$$=\frac{1}{z}\cdot 2=\frac{1}{2x+3}\cdot 2=\frac{2}{2x+3}.$$

(viii) $y=\sqrt{x^{2}+5}=\sqrt{z}$ where $z=x^{2}+5$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}\,(\sqrt{z})\quad\frac{d}{dx}\,(x^{2}+5)$$

$$=\frac{d}{dz}\left(z^{\frac{1}{2}}\right)\left\{\frac{d}{dx}\,(x^{2})+\frac{d}{dx}\,(5)\right\}$$

$$=\tfrac{1}{2}\,z^{-\frac{1}{2}}\,(2x)=x\,\frac{1}{z^{\frac{1}{2}}}=\frac{x}{\sqrt{x^{2}+5}}$$

(ix) $y=\dfrac{1}{\sqrt{x^{2}+a^{2}}}=\dfrac{1}{\sqrt{z}}$ where $z=x^{2}+a^{2}$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}\left(\frac{1}{\sqrt{z}}\right).\quad\frac{d}{dx}\,(x^{2}+a^{2})$$

$$=\frac{d}{dz}\left(z^{-\frac{1}{2}}\right).\;2x=-\tfrac{1}{2}.\;z^{-\frac{3}{2}}.\;2x$$

$$=-x\,\frac{1}{z^{\frac{3}{2}}}=-\frac{x}{(x^{2}+a^{2})^{\frac{3}{2}}}.$$

(x) $y=\sqrt[5]{x^{2}+x+1}=(x^{2}+x+1)^{\frac{1}{5}}=z^{\frac{1}{5}}$ where $z=x^{2}+x+1$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}\left(z^{\frac{1}{5}}\right)\;\frac{d}{dx}\,(x^{2}+x+1)$$

$$=\tfrac{1}{5}\,z^{-\frac{4}{5}}.\;(2x+1)=\frac{2x+1}{5.\;z^{\frac{4}{5}}}=\frac{2x+1}{5(x^{2}+x+1)^{\frac{4}{5}}}$$

Ex. 2. Find $\frac{dy}{dx}$ when.

(i) $y=\sin x^2$ (ii) $y=e^{\sin x}$ (iii) $y=\cos(e^x)$

(iv) $y=\cos(\log x)$ (v) $y=\log(\sin x)$ (vi) $y=\log(e^x+3)$

(vii) $y=\tan\{\phi(x)\}$ (viii) $y=\phi(\sin x)$ (ix) $y=e^{f(x)}$.

(x) $y=\log\{f(x)\}$.

(i) $y=\sin x^2=\sin z$ where $z=x^2$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{d}{dx}(\sin z).\ \frac{d}{dx}(x^2)=\cos z.\ 2x$$

$=2x\cos x^2$.

(ii) $y=e^{\sin x}=e^z$ where $z=\sin x$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(e^z)\ \frac{d}{dx}(\sin x)=e^z.\ \cos x=e^{\sin x}.\ \cos x$$

(iii) $y=\cos(e^x)=\cos z$ where $z=e^x$.

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(\cos z).\ \frac{d}{dx}(e^x)=-\sin z\ e^x$$

$=-e^x\sin(e^x)$.

(iv) $y=\cos(\log x)=\cos z$, where $z=\log x$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(\cos z).\ \frac{d}{dx}(\log x)=-\sin z\ \frac{1}{x}$$

$=-\sin(\log x).\ \frac{1}{x}$

(v) $y=\log(\sin x)=\log z$ where $z=\sin x$.

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(\log z)\ \frac{d}{dx}(\sin x)=\frac{1}{z}\cos x=\frac{\cos x}{\sin x}=\cot x$$

(vi) $y=\log(e^x+3)=\log z$ where $z=e^x+3$.

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(\log z).\ \frac{d}{dx}(e^x+3)=\frac{1}{z}.\ e^x=\frac{e^x}{e^x+3}.$$

(vii) $y=\tan\phi(x)=\tan z$ where $\phi(x)=z$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(\tan z)\ \frac{d}{dx}\{\phi(x)\}=\sec^2 z\ \phi'(x)$$

$$=\sec^2\{\phi(x)\}\ \phi'(x)$$

(viii) $y=\phi(\sin x)=\phi(z)$ where $z=\sin x$.

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}\{\phi(z)\}\ \frac{d}{dx}(\sin x)=\phi'(z)\cos x$$

$$=\phi'(\sin x)\cos x.$$

(ix) $y=e^{f(x)}=e^{z}$ where $f(x)=z$.

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(e^{z}).\ \frac{d}{dx}\{f(x)\}=e^{z}f'(x)=e^{f(x)}f'(x)$$

(x) $y=\log\{f(x)\}=\log z$, where $z=f(x)$.

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(\log z).\ \frac{d}{dx}\{f(x)\}=\frac{1}{z}f'(x)=\frac{f'(x)}{f(x)}$$

Ex. 3. Find $\frac{dy}{dx}$ when

(i) $y=\sin^3 x$ (ii) $y=\sqrt{\cos x}$ (iii) $y=\tan^{\frac{4}{3}} x$

(iv) $y=\sec^5 x$ (v) $y=e^{e^{x}}$ (vi) $y=e^{x^4+x^2+1}$

(vii) $y=\cos^2 x$ (viii) $\log\ \tan\frac{x}{2}$ [H. S. 1983]

(i) $y=\sin^3 x=(\sin x)^3=z^3$ where $z=\sin x$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(z^3).\ \frac{d}{dx}(\sin x)=3z^2.\cos x$$

$=3\sin^2 x\cos x.$

(ii) $y=\sqrt{\cos x}=\sqrt{z}$ where $z=\cos x$.

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(\sqrt{z})\ \frac{d}{dx}(\cos x)=\frac{d}{dz}\left(z^{\frac{1}{2}}\right)(-\sin x)$$

$$=\tfrac{1}{2}z^{-\frac{1}{2}}(-\sin x)=-\frac{\sin x}{2\sqrt{z}}=-\frac{\sin x}{2\sqrt{\cos x}}.$$

(iii) $y=\tan^{\frac{4}{3}} x=(\tan x)^{\frac{4}{3}}=z^{\frac{4}{3}}$ where $\tan x=z$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}\left(z^{\frac{4}{3}}\right).\ \frac{d}{dx}(\tan x)=\tfrac{4}{3}z^{\frac{1}{3}}.\sec^2 x$$

$$=\tfrac{4}{3}(\tan x)^{\frac{1}{3}}.\sec^2 x=\tfrac{4}{3}\sqrt[3]{\tan x}\sec^2 x.$$

(iv) $y=\sec^5 x=(\sec x)^5=z^5$ where $z=\sec x$

$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(z^5)\ \frac{d}{dx}(\sec x)=5z^4\sec x\tan x.$

$=5(\sec x)^4.\sec x\tan x=5\sec^5 x\tan x.$

(v) $y=e^{e^x}=e^z$ where $z=e^x$.

$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(e^z).\ \frac{d}{dx}(e^x)=e^z.\,e^x$

$=e^{e^x}.\,e^x.$

(vi) $y=e^{x^4+x^2+1}=e^z$ where $z=x^4+x^2+1$

$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(e^z)\ \frac{d}{dx}(x^4+x^2+1)=e^z(4x^3+2x)$

$=(4x^3+2x)\ e^{x^4+x^2+1}$

(vii) $y=\cos^2 x=(\cos x)^2=z^2$ where $z=\cos x$.

$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}(z^2).\ \frac{d}{dx}(\cos x)=2z(-\sin x)$

$=2\cos x(-\sin x)=-2\sin x\cos x=-\sin 2x.$

Ex. 4 Find $\frac{dy}{dx}$ when

(i) $y=e^{\sin x^2}$ (ii) $y=\cos(e^{\tan x})$ (iii) $y=\{\log(\sin x^2)\}^n$
(iv) $y=\log\sec(ax+b)^3$ (v) $\log\log\log x^2$ (vi) $\sqrt{\sin(e^x)}$
(vii) $y=\log\sin(\cos x)$ (viii) $y=\log_e\tan\frac{x}{2}$ [H. S. 1988]

(i) $y=e^{\sin x^2}=e^z$ where $z=\sin x^2=\sin u$ where $u=x^2$

$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{du}\cdot\frac{du}{dx}=\frac{d}{dz}(e^z)\frac{d}{du}(\sin u).\frac{d}{dx}(x^2)$

$=e^z.\cos u\ 2x=2x\,e^{\sin x^2}\ \cos x^2$

(ii) $y=\cos(e^{\tan x})=\cos z$; where $z=e^{\tan x}=e^u$ where $u=\tan x$.

$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{du}\cdot\frac{du}{dx}=\frac{d}{dz}(\cos z)\frac{d}{du}(e^u)\frac{d}{dx}(\tan x)$

$=-\sin z.\,e^u\sec^2 x=-\sin(e^{\tan x}),\ e^{\tan x}\ \sec^2 x$

(iii) $y=\{\log\sin(x^2)\}^n=z^n$ where $z=\log\sin(x^2)$
$=\log u$, where $u=\sin(x^2)=\sin(v)$ where $v=x^2$.

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}=\frac{d}{dz}(z^n)\frac{d}{du}(\log u)\frac{d}{dv}(\sin v)\frac{d}{dx}(x^2)$$

$$=nz^{n-1}\frac{1}{u}\cdot\cos v.\,2x$$

$$=2n\{\log\sin(x^2)\}^{n-1}\frac{1}{\sin x^2}\cos(x^2).x$$

$$=2nx\{\log\sin(x^2)\}^{n-1}\cot(x^2).$$

(iv) $y=\log\sec(ax+b)^3=\log z$ where $z=\sec(ax+b)^3$
$=\sec u$; where $u=(ax+b)^3=v^3$; where $v=ax+b$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{du}\frac{du}{dv}\frac{dv}{dx}=\frac{d}{dz}(\log z)\frac{d}{du}(\sec u)\frac{d}{dv}(v^3)\frac{d}{dx}(ax+b)$$

$$=\frac{1}{z}\cdot\sec u\tan u.\,3v^2.a$$

$$=\frac{3a}{\sec\{(ax+b)^3\}}\sec\{(ax+b)^3\}\tan\{(ax+b)^3\}(ax+b)^2$$
$$=3a(ax+b)^2\tan(ax+b)^3$$

(v) Let $\log\log x^2=z$; $\log x^2=u,\ v=x^2$
$\therefore\ y=\log z,\ z=\log u,\ u=\log v.$

$$\text{Now}\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

$$=\frac{d}{dz}(\log z)\frac{d}{du}(\log u)\frac{d}{dv}(\log v)\frac{d}{dv}(x^2)$$

$$=\frac{1}{z}\cdot\frac{1}{u}\cdot\frac{1}{v}\cdot 2x$$

$$=\frac{2x}{\log\log x^2}\cdot\frac{1}{\log x^2}\cdot\frac{1}{x^2}$$

$$=\frac{2x}{\log\log x^2}\,\frac{1}{2\log x}\cdot\frac{1}{x^2}$$

$$=\frac{1}{\log\log x^2}\cdot\frac{1}{\log x}\cdot\frac{1}{x}$$

(vi) $y=\sqrt{\sin(e^x)}$

Let $z=\sin(e^x)$, $u=e^x$.

∴ $y=\sqrt{z}$, $z=\sin u$.

Now, $y=\frac{dy}{ds}\cdot\frac{dz}{du}\cdot\frac{du}{dx}=\frac{d}{dz}(\sqrt{z}).\frac{d}{du}(\sin u)\frac{d}{dx}(e^x)$

$=\frac{1}{2\sqrt{z}}\cdot\cos u.e^x=\frac{1}{2\sqrt{\sin(e^x)}}\cdot\cos(e^x).e^x.$

(vii) $y=\log\sin(\cos x)=\log z$ where $z=\sin(\cos x)$

$=\sin u$ where $u=\cos x$

∴ $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{du}\cdot\frac{du}{dx}=\frac{1}{z}\cos u(-\sin x)$

$=-\frac{1}{\sin(\cos x)}.\cos(\cos x)\sin x=-\sin x\cot(\cos x).$

(viii) $y=\log_e\tan\frac{x}{2}=\log_e z$ where $z=\tan\frac{x}{2}$

$=\tan u$; where $u=\frac{x}{2}$.

∴ $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{du}\frac{du}{dx}=\frac{d}{dz}(\log_e z)\frac{d}{du}(\tan u)\frac{d}{dx}\left(\frac{x}{2}\right)$

$=\frac{1}{z}.\sec^2 u\ \tfrac{1}{2}=\frac{1}{2\tan\frac{x}{2}}\sec^2\left(\frac{x}{2}\right).$

$$=\frac{1}{\frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}}}\cdot\frac{1}{\cos^2\frac{x}{2}}=\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{1}{\sin x}=\operatorname{cosec} x.$$

Ex. 5. Find $\frac{dy}{dx}$ where

(i) $y=\log(\sec x+\tan x)$

(ii) $y=\log_e(x+\sqrt{x^2+a^2})$ [H. S. 1982 ; '85]

(iii) $y=\log_e\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ [H. S. 1986] or, $3(y^2-ax)\dfrac{dy}{dx}=3(ay-x^2)$

$$\therefore \quad \frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}.$$

(iii) Differentiating both sides of the given equation with respect to x we get

$$2x+2y\frac{dy}{dx}=0 \quad \therefore \quad \frac{dy}{dx}=-\frac{x}{y}.$$

(iv) $$\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}}=\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

$$=\frac{1+\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1-\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}=\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}=\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}$$

$$=\frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cos\frac{x}{2}\sin\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}=\frac{1+\sin x}{\cos x}$$

$=\sec x+\tan x.$

$\therefore$ $e^y=\sec x+\tan x.$

or, $\dfrac{d}{dx}(e^y)=\dfrac{d}{dx}(\sec x+\tan x)$

or, $e^y\dfrac{dy}{dx}=\sec x\tan x+\sec^2 x$

or, $\dfrac{dy}{dx}=\dfrac{\sec x(\sec x+\tan x)}{e^y}=\dfrac{\sec x(\sec x+\tan x)}{\sec x+\tan x}$

$=\sec x.$

(v) $$e^y-\frac{a+b\tan x}{a-b\tan x}=0$$

or, $$e^y=\frac{a+b\tan x}{a-b\tan x}=\frac{a+b\dfrac{\sin x}{\cos x}}{a-b\dfrac{\sin x}{\cos x}}$$

$$=\frac{a\cos x+b\sin x}{a\cos x-b\sin x}$$

$$\therefore\quad y=\log_e\left(\frac{a\cos x+b\sin x}{a\cos x-b\sin x}\right)$$

$$=\log(a\cos x+b\sin x)-\log(a\cos x-b\sin x)$$

$$\therefore\quad \frac{dy}{dx}=\frac{d}{dx}\{\log(a\cos x+b\sin x)\}-\frac{d}{dx}\{\log(a\cos x-b\sin x)\}$$

$$=\frac{1}{a\cos x+b\sin x}\;\frac{d}{dx}(a\cos x+b\sin x)-$$

$$\frac{1}{a\cos x-b\sin x}\;\frac{d}{dx}(a\cos x-b\sin x)$$

$$=\frac{-a\sin x+b\cos x}{a\cos x+b\sin x}-\frac{1}{a\cos x-b\sin x}(-a\sin x-b\cos x)$$

$$=-\frac{a\sin x-b\cos x}{a\cos x+b\sin x}+\frac{a\sin x+b\cos x}{a\cos x-b\sin x}$$

$$=\frac{\begin{array}{c}-a^2\cos x\sin x+ab\cos^2x+ab\sin^2x-b^2\cos x\sin x\\+a^2\sin x\cos x+ab\cos^2x+ab\sin^2x+b^2\cos x\sin x\end{array}}{(a\cos x+b\sin x)(a\cos x-b\sin x)}$$

$$=\frac{2ab(\cos^2x+\sin^2x)}{a^2\cos^2x-b^2\sin^2x}=\frac{2ab}{a^2\cos^2x-b^2\sin^2x}.$$

Ex. 7. Find $\frac{dy}{dx}$ if

(i) $x+y=\sin(xy)$ (ii) $\log(xy)=x+y$

(iii) $e^{x+y}=xy$ (iv) $xy=\cos(x+y)$

(v) $e^{xy}-4xy=4$ [H. S. 1981]

(i) $x+y=\sin(xy)$

or, $\frac{d}{dx}(x+y)=\frac{d}{dx}\{\sin(xy)\}$

or, $1+\frac{dy}{dx}=\frac{d}{dx}\sin z$ where $xy=z$

$$=\frac{d}{dz}(\sin z)\frac{dz}{dx}$$

$$=\cos z\frac{d}{dx}(xy)$$

$$=\cos(xy)\left(y+x\frac{dy}{dx}\right)$$

or, $\frac{dy}{dx}-x\cos(xy)\frac{dy}{dx}=y\cos(xy)-1$

or, $\frac{dy}{dx}\{1-x\cos(xy)\}=y\cos(xy)-1$

or, $\frac{dy}{dx}=\frac{y\cos(xy)-1}{1-x\cos(xy)}$

(ii) $\log(xy)=x+y$ or, $\log x+\log y=x+y$

differentiating both sides with respect to x we get

$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1+\frac{dy}{dx}$ or, $\frac{dy}{dx}\left(\frac{1}{y}-1\right)=1-\frac{1}{x}$

or, $\frac{dy}{dx}\left(\frac{1-y}{y}\right)=\frac{x-1}{x}$ $\therefore$ $\frac{dy}{dx}=\frac{y}{x}\left(\frac{x-1}{1-y}\right)$

(iii) $e^{x+y}=xy$

or, $\frac{d}{dx}(e^{x+y})=\frac{d}{dx}(xy)$ or, $e^{x+y}\frac{d}{dx}(x+y)=y+x\frac{dy}{dx}$.

or, $e^{x+y}\left(1+\frac{dy}{dx}\right)=y+x\frac{dy}{dx}$

or, $\frac{dy}{dx}(e^{x+y}-x)=y-e^{x+y}$

$\therefore$ $\frac{dy}{dx}=\frac{y-e^{x+y}}{e^{x+y}-x}$

(iv) $xy=\cos(x+y)$

or, $\frac{d}{dx}(xy)=\frac{d}{dx}\{\cos(x+y)\}$

or, $y+x\frac{dy}{dx}=-\sin(x+y)\frac{d}{dx}(x+y)$

$$=-\sin(x+y)\left\{1+\frac{dy}{dx}\right\}$$

or, $\frac{dy}{dx}\{x+\sin(x+y)\}=-\{(y+\sin(x+y)\}$

$$\frac{dy}{dx}=-\frac{y+\sin(x+y)}{x+\sin(x+y)}$$

(v) $e^{xy}-4xy=4$

differentiating both sides with respect to x we get

$$\frac{d}{dx}(e^{xy})-\frac{d}{dx}(4xy)=\frac{d}{dx}(4)$$

or, $e^{xy}\frac{d}{dx}(xy)-4\frac{d}{dx}(xy)=0$

or, $e^{xy}\left(x\frac{dy}{dx}+y\right)-4\left(x\frac{dy}{dx}+y\right)=0$

or, $\frac{dy}{dx}(xe^{xy}-4x)=4y-ye^{xy}$

or, $\frac{dy}{dx}x(e^{xy}-4)=y(4-e^{xy})$

or, $\frac{dy}{dx}=\frac{y(4-e^{xy})}{-x(4-e^{xy})}=-\frac{y}{x}.$

Ex. 8. Find $\frac{dy}{dx}$ when,

(i) $y=\frac{1}{\sqrt{x+a}-\sqrt{x+b}}$ (ii) $y=\log(\sqrt{x+a}+\sqrt{x+b})$

(iii) $\log_{10}(\sqrt{x+a}+\sqrt{x+b})$ (iv) $\log_{10}\sin x.$

(i) $y=\frac{1}{\sqrt{x+a}-\sqrt{x+b}}=\frac{\sqrt{x+a}+\sqrt{x+b}}{(\sqrt{x+a}-\sqrt{x+b})(\sqrt{x+a}+\sqrt{x+b})}$

$=\frac{\sqrt{x+a}+\sqrt{x+b}}{(x+a)-(x+b)}=\frac{\sqrt{x+a}+\sqrt{x+b}}{a-b}$

$\therefore$ $\frac{dy}{dx}=\frac{1}{a-b}\frac{d}{dx}(\sqrt{x+a}+\sqrt{x+b})$

$=\frac{1}{a-b}\left\{\frac{1}{2}\frac{1}{\sqrt{x+a}}+\frac{1}{2\sqrt{x+b}}\right\}$

$=\frac{1}{2(a-b)}\left\{\frac{1}{\sqrt{x+a}}+\frac{1}{\sqrt{x+b}}\right\}$

(ii) $y=\log(\sqrt{x+a}+\sqrt{x+b})=\log z$

[where $z=\sqrt{x+a}+\sqrt{x+b}$]

$$\therefore \quad \frac{dy}{dx}=\frac{d}{dx}(\log z)=\frac{d}{dz}(\log z)\frac{dz}{dx}=\frac{1}{z}\frac{d}{dx}(\sqrt{x+a}+\sqrt{x+b})$$

$$=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\left(\frac{1}{2\sqrt{x+a}}+\frac{1}{2\sqrt{x+b}}\right)$$

(iii) $y=\log_{10}(\sqrt{x+a}+\sqrt{x+b})=\log_e(\sqrt{x+a}+\sqrt{x+b})\log^e_{10}$

$$\therefore \quad \frac{dy}{dx}=\log^e_{10}\quad \frac{d}{dx}\log_e\left(\sqrt{x+a}+\sqrt{x+b}\right)\Big\}$$

[$\because$ $\log^e_{10}$ is constant]

$$=\log^e_{10}\quad \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\,\frac{1}{2}\left(\frac{1}{\sqrt{x+a}}+\frac{1}{\sqrt{x+b}}\right)$$

[form (ii) above]

(iv) $\log_{10}\sin x=\log_e\sin x\log^e_{10}$

$$\therefore \quad \frac{dy}{dx}=\frac{d}{dx}\left(\log_e{}^{\sin x}\ \log^e_{10}\ \right)$$

$$=\log^e_{10}\quad \frac{d}{dx}(\log\sin x)=\log^e_{10}\ \frac{d}{dx}(\log z)$$

[$z=\sin x$ (say)]

$$=\log^e_{10}\quad \frac{d}{dx}\log(z)\frac{dz}{dx}=\log^e_{10}\quad \frac{1}{z}\cdot\frac{d}{dx}(\sin x)$$

$$=\log^e{}_{10}\quad \frac{1}{\sin x}\cos x=\log^e_{10}\quad \cot x.$$

Ex. 9. Find $\frac{dy}{dx}$ when

(i) $y=\sin^{-1}\left(\frac{x}{a}\right)$ (ii) $y=\tan^{-1}\sqrt{x}$ (iii) $y=\tan^{-1}(\sec x)$

(i) $y=\sin^{-1}\left(\frac{x}{a}\right)=\sin^{-1}z$ where $z=\frac{x}{a}$

$$\therefore \quad \frac{dy}{dx}=\frac{d}{dx}(\sin^{-1}z)=\frac{d}{dz}\left(\sin^{-1}z\right)\frac{dz}{dx}$$

$$=\frac{1}{\sqrt{1-z^2}}\frac{d}{dx}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\frac{1}{a}=\frac{1}{\sqrt{a^2-x^2}}$$

(ii) $y=\tan^{-1}\sqrt{x}=\tan^{-1}z$ where $z=\sqrt{x}$ (say)

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(\tan^{-1}z)=\frac{d}{dz}(\tan^{-1}z)\frac{dz}{dx}$$

$$=\frac{1}{1+z^2}\frac{d}{dx}(\sqrt{x})=\frac{1}{1+x}\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}(1+x)}$$

(iii) $y=\tan^{-1}(\sec x)=\tan^{-1}z$; $z=\sec x$ (say)

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(\tan^{-1}z)=\frac{d}{dz}(\tan^{-1}z)\frac{dz}{dx}=\frac{1}{1+z^2}\frac{d}{dx}(\sec x)$$

$$=\frac{1}{1+z^2}\sec x\tan x=\frac{1}{1+\sec^2 x}\sec x\tan x.$$

$$=\frac{1}{1+\frac{1}{\cos^2 x}}\cdot\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}=\frac{\sin x}{1+\cos^2 x}.$$

Ex. 10. Find $\frac{dy}{dx}$ when

(i) $y=\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}$ [H. S. 1979]

(ii) $y=\sin^{-1}\frac{x}{1+x}$ [H. S. 1980]

(iii) $y=\tan^{-1}\frac{\cos x}{1+\sin x}$ [H. S. 1981 ; Joint Entrance 1979

(iv) $y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ [H. S. 1983]

(v) $y=\cos^{-1}\frac{1-x}{1+x}$ [H. S. 1984]

(vi) $y=\tan^{-1}\sqrt{\frac{1+\cos 2x}{1-\cos 2x}}$ [H. S. 1986]

(i) let $x=\tan\theta$ $\therefore\ \theta=\tan^{-1}x$

$$\therefore\ y=\frac{1}{2}\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\frac{1}{2}\sin^{-1}\sin 2\theta=\frac{1}{2}2\theta$$

$$=\theta=\tan^{-1}x$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{1+x^2}$$

(ii) $y=\sin^{-1}\frac{x}{1+x}=\sin^{-1}z\left[\text{where } z=\frac{x}{1+x}\right]$

$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(\sin^{-1}z)=\frac{d}{dz}\sin^{-1}z)\frac{dz}{dx}$

$=\frac{1}{\sqrt{1-z^2}}\frac{d}{dx}\left(\frac{x}{1+x}\right)$

$=\frac{1}{\sqrt{1-\left(\frac{x}{1+x}\right)^2}}\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x)^2}$

$=\frac{1+x}{\sqrt{(1+x)^2-x^2}}\cdot\frac{(1+x).1-x.1}{(1+x)^2}=\frac{1+x}{\sqrt{2x+1}}\cdot\frac{1}{(1+x)^2}$

$=\frac{1}{\sqrt{2x+1}}\cdot\frac{1}{1+x}$.

(iii) $y=\tan^{-1}\frac{\cos x}{1+\sin x}=\tan^{-1}\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cos\frac{x}{2}\sin\frac{x}{2}}$

$=\tan^{-1}\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}=\tan^{-1}\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}$

$=\tan^{-1}\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}=\tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{\pi}{4}-\frac{x}{2}$

$\therefore\ \frac{dy}{dx}=-\frac{1}{2}$.

(iv) $y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}=\tan^{-1}\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}=\tan^{-1}\tan\frac{x}{2}=\frac{x}{2}$

$\therefore\ \frac{dy}{dx}=\frac{1}{2}$.

(v) $y=\cos^{-1}\frac{1-x}{1+x}=\cos^{-1}\frac{1-\tan^2\theta}{1+\tan\theta}=\cos^{-1}\cos 2\theta=2\theta$.

[where $x=\tan^2\theta$ or, $\tan\theta=\sqrt{x}$ or, $\theta=\tan^{-1}\sqrt{x}$]

$\therefore\ y=2\tan^{-1}\sqrt{x}\quad\therefore\ \frac{dy}{dx}=2\frac{d}{dx}(\tan^{-1}\sqrt{x})$

$$=2\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{x}(1+x)}$$

(vi) $y=\tan^{-1}\sqrt{\frac{1+\cos 2x}{1-\cos 2x}}=\tan^{-1}\sqrt{\frac{2\cos^2 x}{2\sin^2 x}}=\tan^{-1}\cot x$

$$=\tan^{-1}\tan\left(\frac{\pi}{2}-x\right)=\frac{\pi}{2}-x$$

$$\therefore\ \frac{dy}{dx}=-1.$$

Ex. 11. Find $\frac{dy}{dx}$ when

(i) $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$ [H. S. 1988]

(ii) $y=\sin^{-1}(x^2\sqrt{1-x}-\sqrt{x}\sqrt{1-x^4})$ [Joint Entrance 1982]

(iii) $y=\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$ [Joint Entrance 1983]

(i) Let $\sin^{-1}x=\theta$ $\sin\theta=x$ $\therefore$ $\cos\theta=\sqrt{1-x^2}$

$\therefore$ $\theta=\cos^{-1}\sqrt{1-x^2}$

or, $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}$

$$\therefore\ y=\cos^{-1}\sqrt{1-x^2}+\sin^{-1}\sqrt{1-x^2}=\frac{\pi}{2}$$

$$\therefore\ \frac{dy}{dx}=0.$$

(ii) $y=\sin^{-1}(x^2\sqrt{1-(\sqrt{x})^2}-\sqrt{x}\sqrt{1-(x^2)^2})$

$=\sin^{-1}x^2-\sin^{-1}\sqrt{x}$

$$\therefore\ \frac{dy}{dx}=\frac{1}{\sqrt{1-x^4}}\frac{d}{dx}(x^2)-\frac{1}{\sqrt{1-x}}\frac{d}{dx}(\sqrt{x})$$

$$=\frac{2x}{\sqrt{1-x^4}}-\frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}$$

(iii) $y=\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)=\sin^{-1}z$ where $z=\frac{a+b\cos x}{b+a\cos x}$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}z\right)=\frac{d}{dz}\left(\sin^{-1}z\right)\frac{dz}{dx}$$

$$=\frac{1}{\sqrt{1-z^2}}\ \frac{d}{dx}\left(\frac{a+b\cos x}{b+a\cos x}\right)$$

$$=\frac{1}{\sqrt{1-\left(\dfrac{a+b\cos x}{b+a\cos x}\right)^2}}.$$

$$\frac{(b+a\cos x)\ \dfrac{d}{dx}\ (a+b\cos x)-(a+b\cos x)\ \dfrac{d}{dx}\ (b+a\cos x)}{(b+a\cos x)^2}.$$

$$=\frac{b+a\cos x}{\sqrt{(b+a\cos x)^2-(a+b\cos x)^2}}.$$

$$\frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)\times(-a\sin x)}{(b+a\cos x)^2}$$

$$=\frac{1}{\sqrt{\{b^2+2ab\cos x+a^2\cos^2 x-a^2-2ab\cos x-b^2\cos^2 x\}}}$$

$$\frac{-b^2\sin x-ab\sin x\cos x+a^2\sin x+ab\sin x\cos x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{\{(b^2-a^2)-(b^2-a^2)\cos^2 x\}}}\cdot\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{\{(b^2-a^2)(1-\cos^2 x)\}}}\cdot\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}\sin x}\cdot\frac{(a^2-b^2)\ \sin x}{b+a\cos x}$$

$$=-\frac{\sqrt{b^2-a^2}}{b+a\cos x}$$

Ex. 12. (i) If $y=e^t\cos t,\ x=e^t\sin t$ verify that $\dfrac{dy}{dx}\cdot\dfrac{dx}{dy}=1$

[H. S. 1983]

(i) $y=e^t\cos t\ ;\ x=e^t\sin t$

$$\frac{dy}{dt}=e^t\ \frac{d}{dt}\ (\cos t)+\cos t\,\frac{d}{dt}(e^t)$$

$$=e^t(-\sin t)+\cos t\ e^t=e^t(\cos t-\sin t)$$

$$\frac{dx}{dt}=e^t\ \frac{d}{dt}\ (\sin t)+\sin t\ \frac{d}{dt}\ (e^t)$$

$$=e^t\ \cos t+\sin t\ e^t=e^t\ (\cos t+\sin t)$$

$$\therefore\quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{e^t(\cos t-\sin t)}{e^t(\cos t+\sin t)}=\frac{\cos t-\sin t}{\cos t+\sin t}$$

$$\therefore\quad \frac{dx}{dy}=\frac{\frac{dx}{dt}}{\frac{dy}{dt}}=\frac{e^t(\cos t+\sin t)}{e^t(\cos t-\sin t)}=\frac{\cos t+\sin t}{\cos t-\sin t}$$

$$\therefore\quad \frac{dy}{dx}\cdot\frac{dx}{dy}=\frac{\cos t-\sin t}{\cos t+\sin t}\cdot\frac{\cos t+\sin t}{\cos t-\sin t}=1.$$

(ii) Differentiate $y=x^3+1$ w.r, to x at $x=3$. Also differentiating the inverse function $3\sqrt{y-1}$

with respect to y at $y=28$ verify the formula $\frac{dy}{dx}\times\frac{dx}{dy}=1$.

(ii) $y=x^3+1 \quad \therefore \quad \frac{dy}{dx}=3x^2 \quad \therefore \quad \left[\frac{dy}{dx}\right]_{x=3}=3.3^2=27$

$$x=\sqrt[3]{y-1}=(y-1)^{\frac{1}{3}}$$

$$\therefore\quad \frac{dx}{dy}=\frac{1}{3}(y-1)^{-\frac{2}{3}} \quad \therefore \quad \left[\frac{dx}{dy}\right]_{x=28}=\frac{1}{3}(28-1)^{-\frac{2}{3}}$$

$$=\frac{1}{3}\cdot\frac{1}{(27)^{\frac{2}{3}}}=\frac{1}{3}\cdot\frac{1}{9}=\frac{1}{27}$$

$$\therefore\quad \frac{dy}{dx}\cdot\frac{dx}{dy}=27\cdot\frac{1}{27}=1$$

Ex. 13. (a) Differentiate the functions with respect to x

(i) $2x\tan^{-1}x-\log_e(1+x^2)$

(ii) $\sin^{-1}\sqrt{x-1}$

State the restrictions on the value of x for the second of these functions.

(b) Using differential calculus state the number of real roots of the equation $\sin^{-1}x=2x$.

[Joint Entrance 1978]

(a) (i) $\frac{d}{dx}\left\{2x\tan^{-1}x-\log_e(1+x^2)\right\}$

$$=\frac{d}{dx}\left\{2x\tan^{-1}x\right\}-\frac{d}{dx}\left\{\log_e(1+x^2)\right\}$$

$$=2x\frac{d}{dx}\left(\tan^{-1}x\right)+\tan^{-1}x\frac{d}{dx}(2x)-\frac{d}{dz}(\log z)\frac{dz}{dx}$$

[where $z=1+x^2$]

$$=(2x)\frac{1}{1+x^2}+\tan^{-1}x.2-\frac{1}{z}.2x$$

$$=\frac{2x}{1+x^2}+2\tan^{-1}x-\frac{2x}{1+x^2}=2\tan^{-1}x$$

(ii) If $x>2$, $\sqrt{x-1}>1$ $\therefore$ $\sin^{-1}\sqrt{x+1}$ is undefined

Again if $x<1$. $\sqrt{x-1}$ is imaginary $\therefore 1<x<2$.

Now, $y=\sin^{-1}\sqrt{x-1}=\sin^{-1}z$; where $\sqrt{x-1}=z$ (say)

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{d}{dz}\left(\sin^{-1}z\right)\frac{d}{dx}\left(\sqrt{x-1}\right)$$

$$=\frac{1}{\sqrt{1-z^2}}\frac{1}{2\sqrt{x-1}}=\frac{1}{\sqrt{1-(x-1)}}\cdot\frac{1}{2\sqrt{x-1}}=\frac{1}{2\sqrt{2-x}\sqrt{x-1}}$$

(*b*) $\sin^{-1}x=2x$ or, $\frac{d}{dx}(\sin^{-1}x)=\frac{d}{dx}(2x)$

or, $\frac{1}{\sqrt{1-x^2}}=2$ or, $\frac{1}{1-x^2}=4$ (squaring both sides)

or, $4-4x^2=1$ or, $4x^2=3$, or, $x^2=\frac{3}{4}$ or, $x=\pm\frac{\sqrt{3}}{2}$

Ex. 14. (a) Find $\frac{dy}{dx}$, when

(i) $x=a\cos\frac{\theta}{2}, y=b\sin\frac{\theta}{2}$ [H. S. 1989]

(ii) $x=a\cos\theta, y=b\sin\theta$ [H. S. 1980]

(iii) $x=a\sec^2\theta, y=a\tan^3\theta$; when $\theta=\frac{\pi}{4}$ [H. S. 1981]

(*b*) (i) $y=\tan^{-1}\frac{2t}{1-t^2}$ and $x=\sin^{-1}\frac{2t}{1+t^2}$, show that $\frac{dy}{dx}=1$.

[H. S. 1982]

(ii) If $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$, express $\frac{dy}{dx}$ in simplest form and show that $\frac{dy}{dx}=-1$ when $\theta=\frac{\pi}{2}$ [H. S.]

(*a*) (i) $x=a\cos\frac{\theta}{2},\ y=b\sin\frac{\theta}{2}$

$$\frac{dx}{d\theta}=\frac{d}{d\theta}\left(a\cos\frac{\theta}{2}\right)=-a.\frac{1}{2}.\sin\frac{\theta}{2}$$

$$\frac{dy}{d\theta}=\frac{d}{d\theta}\left(b\sin\frac{\theta}{2}\right)=b.\frac{1}{2}\cos\frac{\theta}{2}$$

$$\therefore\ \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{b.\frac{1}{2}\cos\frac{\theta}{2}}{-a.\frac{1}{2}\sin\frac{\theta}{2}}=-\frac{b}{a}\cot\frac{\theta}{2}$$

(ii) $\frac{dx}{d\theta}=\frac{d}{d\theta}(a\cos\theta)=-a\sin\theta$

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(b\sin\theta)=b\cos\theta$$

$$\therefore\ \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{b\cos\theta}{-a\sin\theta}=-\frac{b}{a}\cot\theta.$$

(iii) $x=a\sec^2\theta\quad\therefore\quad\frac{dx}{d\theta}=\frac{d}{d\theta}(a\sec^2\theta)$

$$=2a\sec\theta\sec\theta\tan\theta=2a\sec^2\theta\tan\theta$$

$y=a\tan^3\theta.\quad\therefore\quad\frac{dy}{d\theta}=\frac{d}{d\theta}(a\tan^3\theta)=3a\tan^2\theta\sec^2\theta.$

$$\therefore\ \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{3a\tan^2\theta\sec^2\theta}{2a\sec^2\theta\tan\theta}=\frac{3}{2}\tan\theta$$

$$\therefore\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}}=\frac{3}{2}\tan\frac{\pi}{4}=\frac{3}{2}.1=\frac{3}{2}.$$

(*b*) (i) Let $t=\tan\theta$.

$$\therefore\quad y=\tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}=\tan^{-1}\tan 2\theta=2\theta.\qquad\therefore\quad \frac{dy}{d\theta}=2.$$

$$x=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}\sin 2\theta=2\theta\quad\therefore\quad\frac{dx}{d\theta}=2.$$

$$\therefore\quad\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{2}{2}=1.$$

(ii) $x=a(\theta+\sin\theta)$; $\quad\frac{dx}{d\theta}=a(1+\cos\theta)$

$y=a(1+\cos\theta)$; $\quad\frac{dy}{d\theta}=-a\sin\theta.$

$$\therefore\quad\frac{dy}{d\theta}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a(-\sin\theta)}{a(1+\cos\theta)}=\frac{-\sin\theta}{1+\cos\theta}$$

$$=\frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=-\tan\frac{\theta}{2}$$

$$\therefore\quad\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{2}}=-\tan\frac{\pi}{4}=-1.$$

Ex. 15. Find $\frac{dy}{dx}$.

(i) $x=\frac{3at}{1+t^2}$, $y=\frac{3at^2}{1+t^2}$ (ii) $x=a\cos^3\theta$, $y=b\sin^3\theta$

(i) $$\frac{dx}{dt}=\frac{(1+t^2)\frac{dx}{dt}(3at)-3at\frac{d}{dt}(1+t^2)}{(1+t^2)^2}=\frac{(1+t^2)3a-3at.\,2t}{(1+t^2)^2}$$

$$=\frac{3a(1+t^2-2t^2)}{(1+t^2)^2}=\frac{3a(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{(1+t^2)\frac{d}{dt}(3at^2)-3at^2\frac{d}{dt}(1+t^2)}{(1+t^2)^2}=\frac{(1+t^2)6at-3at^2.\,2t}{(1+t^2)^2}$$

$$=\frac{6at\,(1+t^2-t^2)}{(1+t^2)^2}=\frac{6at}{(1+t^2)^2}$$

$$\therefore\quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{6at}{(1+t^2)^2}}{\frac{3a(1-t^2)}{(1+t^2)^2}}=\frac{2t}{1-t^2}$$

(ii) $x=a\cos^3\theta\quad\therefore\quad \frac{dx}{d\theta}=\frac{d}{d\theta}(a\cos^3\theta)=3a\cos^2\theta(-\sin\theta)$

$y=b\sin^3\theta\quad\therefore\quad \frac{dy}{d\theta}=\frac{d}{d\theta}(b\sin^3\theta)=3b\sin^2\theta.\cos\theta$

$$\therefore\quad \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{3b\sin^2\theta\cos\theta}{3a\cos^2\theta\,(-\sin\theta)}=-\frac{b}{a}\tan\theta.$$

Ex. 16. (i) If $y=e^{\sin^{-1}x}$ and $z=e^{-\cos^{-1}x}$, then show that the value of $\frac{dy}{dz}$ does not depend on x [Joint Entrance 1987]

(ii) If $y=\cos^{-1}(8x^4-8x^2+1)$, find $\frac{dy}{dx}$.

(i) $y=e^{\sin^{-1}x}=e^u$; [$\sin^{-1}x=u$(say)]

$$\therefore\quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=e^u\cdot\frac{1}{\sqrt{1-x^2}}=e^{\sin^{-1}x}\,\frac{1}{\sqrt{1-x^2}}$$

$z=e^{-\cos^{-1}x}=e^v$; where $v=-\cos^{-1}x$

$$\therefore\quad \frac{dz}{dx}=\frac{dz}{dv}\cdot\frac{dv}{dx}=e^v\frac{d}{dx}(-\cos^{-1}x)=e^{-\cos^{-1}x}\,\frac{1}{\sqrt{1-x^2}}$$

$$\therefore\quad \frac{dy}{dz}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{e^{\sin^{-1}x}\frac{1}{\sqrt{1-x^2}}}{e^{\cos^{-1}x}\frac{1}{\sqrt{1-x^2}}}=\frac{e^{\sin^{-1}x}}{e^{-\cos^{-1}x}}=e^{\sin^{-1}x+\cos^{-1}x}$$

$$=e^{\frac{\pi}{2}}$$

which is independent of x. So, the value of, $\frac{dy}{dz}$ does not depend on x.

(ii) $y=\cos^{-1}(8x^4-8x^2+1)=\cos^{-1}\{2(2x^2-1)^2-1\}$

$=\cos^{-1}\{2(2\cos^2\theta-1)^2-1\}$ where $x=\cos\theta$.

$=\cos^{-1}(2\cos^2 2\theta-1)=\cos^{-1}(\cos 4\theta)=4\theta$

$=4\cos^{-1}x$

$$\therefore\quad \frac{dy}{dx}=\frac{d}{dx}\left(4\cos^{-1}x\right)=4\frac{d}{dx}\left(\cos^{-1}x\right)=4\frac{-1}{\sqrt{1-x^2}}=-\frac{4}{\sqrt{1-x^2}}$$

Ex. 17. Find $\frac{dy}{dx}$ if.

(i) $y=x^x$ (ii) $y=x^{x^x}$ (iii) $x^y=y^x$ [H. S. 1985]

(iv) $x^3y^4=(x+y)^7$

(i) $y=x^x$ or, $\log y=\log(x^x)=x\log x$.

Differentiating both sides with respect to x we get,

$$\frac{1}{y}\frac{dy}{dx}=x\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(x)$$

$$=x.\frac{1}{x}+\log x.1=1+\log x$$

$$\therefore\quad \frac{dy}{dx}=y(1+\log x)=x^x(1+\log x)=x^x(\log e+\log x)$$

$=x^x(\log ex)$

(ii) $y=x^{x^x}$ or, $\log y=\log(x^{x^x})=x^x\log(x)$

Differentiating both sides with respect to x we get

$$\frac{1}{y}\frac{dy}{dx}=x^x\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(x^x)$$

$$=x^x\cdot\frac{1}{x}+\log x.x^x(\log x+1)\quad[\text{ See (i) }]$$

$=x^x\left\{\frac{1}{x}+\log x(1+\log x)\right\}$

$$\frac{dy}{dx}=yx^x\left\{\frac{1}{x}+\log x(1+\log x)\right\}$$

$$=x^{x^x}x^x\left\{\frac{1}{x}+\log x(1+\log x)\right\}$$

(iii) $x^y=y^x$ or, $\log(x^y)=\log(y^x)$

or, $y\log x=x\log y$.

Differentiating both sides with respect to x we get

$$y\frac{d}{dx}(\log x)+\frac{dy}{dx}\log x=x\frac{d}{dx}(\log y)+\log y\frac{d}{dx}(x)$$

or, $y.\frac{1}{x}+\frac{dy}{dx}\log x=x\frac{1}{y}\frac{dy}{dx}+\log y.1$

or, $\frac{dy}{dx}\left(\log x-\frac{x}{y}\right)=\log y-\frac{y}{x}$

or, $\frac{dy}{dx}\left(\frac{y\log x-x}{y}\right)=\frac{x\log y-y}{x}$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{x\log y-y}{x}}{\frac{y\log x-x}{y}}=\frac{y}{x}\,\frac{(x\log y-y)}{(y\log x-x)}$$

(iv): $x^3y^4=(x+y)^7$ [Joint Entrance 1982]

or, $\log(x^3y^4)=\log(x+y)^7$

or, $3\log x+4\log y=7\log(x+y)$

Differentiating both sides with respect to x we get,

$$\frac{3}{x}+\frac{4}{y}\frac{dy}{dx}=\frac{7}{x+y}\frac{d}{dx}\left(x+y\right)=\frac{7}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

or, $\frac{dy}{dx}\left(\frac{4}{y}-\frac{7}{x+y}\right)=\frac{7}{x+y}-\frac{3}{x}$

or, $\frac{dy}{dx}\left(\frac{4x+4y-7y}{y(x+y)}\right)=\frac{7x-3x-3y}{x(x+y)}$

or, $\frac{dy}{dx}\left(\frac{4x-3y}{y(x+y)}\right)=\frac{4x-3y}{x(x+y)}$

or, $\frac{dy}{dx}=\frac{y}{x}$.

Ex. 18. Find $\frac{dy}{dx}$ when

(i) $y=x^{\sin\frac{x}{2}}$ [H. S. 1979]

(ii) $y=\sqrt{\frac{x^2+1}{x^2-1}}$ [H. S. 1980]

(iii) $y=(\sin x)^{\log x}$ [H. S. 1980, '83]

(iv) $y=x^{\sin x}$ [H. S. 1984]

(v) $y^2=(\sin x)^{-x}$ [H. S. 1987]

(vi) $y^y=\sin x$ [H. S. 1987]

(viii) $y=(1+x)^x$ [Joint Entrance 1989]

(viii) $y=\sin(\log\sqrt{x})+x^{\cos x}$ [Joint Entrance 1981]

(ix) $x^y y^x=e^{xy}-3x$ [Joint Entrance 1983]

(x) $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}$ [Joint Entrance 1980]

(xi) $x^{\sin y}+y^{\cos x}=1$ [Joint Entrance 1984]

(xii) $y=e^{\cos^{-1}x}+x^{\sqrt{x}}$ [Joint Entrance 1986]

(i) $y=x^{\sin x}$ or, $\log y=\log x^{\sin\frac{x}{2}}=\sin\frac{x}{2}\log x$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}\left(\log y\right)=\frac{d}{dx}\left(\sin\frac{x}{2}\log x\right)$$

$$\text{or, }\frac{1}{y}\frac{dy}{dx}=\sin\frac{x}{2}\frac{d}{dx}\left(\log x\right)+\log x\frac{d}{dx}\left(\sin\frac{x}{2}\right)$$

$$=\sin\frac{x}{2}\cdot\frac{1}{x}+\log x\cos\frac{x}{2}\cdot\frac{1}{2}$$

$$=\frac{1}{x}\sin\frac{x}{2}+\frac{1}{2}\log x\cos\frac{x}{2}$$

$$\therefore\ \frac{dy}{dx}=y\left(\frac{1}{x}\sin\frac{x}{2}+\frac{1}{2}\log x\cos\frac{x}{2}\right)$$

$$=x^{\sin\frac{x}{2}}\left(\frac{1}{x}\sin\frac{x}{2}+\frac{1}{2}\log x\cos\frac{x}{2}\right)$$

(ii) $\sqrt{\frac{x^2+1}{x^2-1}}$ or, $\log y=\log\left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}}=\frac{1}{2}\log\left(x^2+1\right)$

$$-\tfrac{1}{2}\log(x^2-1)$$

differentiating both sides with respect to x we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\cdot\frac{d}{dx}\left\{\log(x^2+1)\right\}-\frac{1}{2}\frac{d}{dx}\left\{\log(x^2-1)\right\}$$

$$=\frac{1}{2}\frac{1}{x^2+1}\frac{d}{dx}\left(x^2+1\right)-\frac{1}{2}\frac{1}{x^2-1}\frac{d}{dx}\left(x^2-1\right)$$

$$=\frac{1}{2}\frac{1}{x^2+1}(2x)-\frac{1}{2}\frac{1}{x^2-1}\cdot 2x$$

$$=\frac{x}{x^2+1}-\frac{x}{x^2-1}=\frac{x^3-x-x^3-x}{(x^2+1)(x^2-1)}=-\frac{2x}{x^4-1}$$

$$\therefore\ \frac{dy}{dx}=-\frac{2xy}{x^4-1}$$

(iii) $y=(\sin x)^{\log x}$ or, $\log y=\log\{(\sin x)^{\log x}\}$
$=\log x\log(\sin x)$

differentiating both sides with respect to x we get

$$\frac{1}{y}\frac{dy}{dx}=\log x\ \frac{d}{dx}\left(\log\sin x\right)+\log\sin x\ \frac{d}{dx}(\log x)$$

$$=\log x\cdot\frac{1}{\sin x}\frac{d}{dx}(\sin x)+\log\sin x\cdot\frac{1}{x}$$

$$=\log x\cdot\frac{1}{\sin x}\cdot\cos x+\frac{\log\sin x}{x}$$

$$\therefore\ \frac{dy}{dx}=y\ \left\{(\log x)\cot x+\frac{\log(\sin x)}{x}\right\}$$

$$=(\sin x)^{\log x}\left\{\cot x\log x+\frac{\log(\sin x)}{x}\right\}$$

(iv) $y=x^{\sin x}$ or, $\log y=\log(x^{\sin x})=\sin x\ \log x$

differentiating both sides with respect to x we get,

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left(\sin x\log x\right)=\sin x\ \frac{d}{dx}\left(\log x\right)+\log x\ \frac{d}{dx}\left(\sin x\right)$$

$$=\sin x\cdot\frac{1}{x}+\log x\cdot\cos x$$

$$\therefore\ \frac{dy}{dx}=y\left(\frac{\sin x}{x}+\cos x\ \log x\right)=x^{\sin x}\left(\frac{\sin x}{x}+\cos x\ \log x\right)$$

(v) $y^2=(\sin x)^{-x}$ or, $\log y^2=\log(\sin x)^{-x}=-x\log(\sin x)$

differentiating both sides with respect to x we get,

$$\frac{d}{dx}(2\log y)=-\left\{x\frac{d}{dx}(\log\sin x)+\log\sin x\ \frac{d}{dx}(x)\right\}$$

or, $$2\frac{1}{y}\frac{dy}{dx}=-\left\{x\frac{1}{\sin x}\cos x+\log\sin x.1\right\}$$

or, $$\frac{dy}{dx}=-\frac{y}{2}\left\{x\cot x+\log\ \sin x\right\}$$

(vi) $y^y=\sin x$ or, $\log(y^y)=\log\sin x$ or, $y\log y=\log\sin x$ differentiating both sides with respect to x we get,

$$y\frac{d}{dx}(\log y)+\log y\ \frac{dy}{dx}=\frac{d}{dx}(\log\sin x)$$

or, $$y\frac{1}{y}\frac{dy}{dx}+\log y\ \frac{dy}{dx}=\frac{1}{\sin x}\cos x$$

or, $$\frac{dy}{dx}\left(1+\log y\right)=\cot x \qquad \therefore\ \frac{dy}{dx}=\frac{\cot x}{1+\log y}$$

(vii) $y=(1+x)^x$ or, $\log y=\log(1+x)^x=x\log(1+x)$ differentiating both sides with respect to x we get,

$$\frac{d}{dx}(\log y)=\frac{d}{dx}\{x\log(1+x)\}$$

or, $$\frac{1}{y}\frac{dy}{dx}=\log(1+x)\ .\ \frac{d}{dx}(x)+x\frac{d}{dx}\log(1+x)$$

$$=\log(1+x).1+x.\frac{1}{1+x}$$

$$\therefore\ \frac{dy}{dx}=y\left\{\log(1+x)+\frac{x}{1+x}\right\}=(1+x)^x\left\{\log(1+x)+\frac{x}{1+x}\right\}$$

(viii) $y=\sin(\log\sqrt{x})+x^{\cos x}$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}\left\{\sin(\log\sqrt{x})+x^{\cos x}\right\}$$

$$=\frac{d}{dx}\{\sin(\log\sqrt{x})\}+\frac{d}{dx}\left(x^{\cos x}\right)$$

Let $u=\sin(\log\sqrt{x})=\sin(\frac{1}{2}\log x)$

$$\therefore\ \frac{du}{dx}=\frac{d}{dx}\{\sin(\log\sqrt{x})\}=\frac{d}{dx}\sin\left(\frac{1}{2}\log x\right)$$

$$=\frac{d}{dx}\sin z \text{ where } \frac{1}{2}\log x=z$$

$$=\frac{d}{dz}(\sin z)\frac{dz}{dx}=\cos z.\frac{d}{dx}\left(\frac{1}{2}\log x\right)=\cos\left(\frac{1}{2}\log x\right)\cdot\frac{1}{2x}$$

$v=x^{\cos x}$ (say)

$\log v=\log\left(x^{\cos x}\right)=\cos x\log x.$

differentiating both sides with respect to x we get

$$\frac{d}{dx}(\log v)=\frac{d}{dx}(\cos x\log x)$$

$$\text{or,}\quad \frac{1}{v}\frac{dv}{dx}=\cos x\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(\cos x)$$

$$=\frac{\cos x}{x}-(\log x)\sin x$$

$$\therefore\quad \frac{dv}{dx}=v\left\{\frac{\cos x}{x}-\sin x\log x\right\}$$

$$\text{or,}\quad \frac{d}{dx}\left(x^{\cos x}\right)=x^{\cos x}\left\{\frac{\cos x}{x}-\sin x\log x\right\}$$

$$\therefore\quad \frac{dy}{dx}=\cos\left(\frac{1}{2}\log x\right)\frac{1}{2x}+x^{\cos x}\left\{\frac{\cos x}{x}-\sin x\log x\right\}.$$

(ix) $x^y y^x=e^{xy}-3x$ or, $\frac{d}{dx}(x^y y^x)=\frac{d}{dx}(e^{xy}-3x)$

For $\frac{d}{dx}(x^y y^x)$, let $u=x^y y^x$

$\therefore$ $\log u=y\log x+x\log y$

differentiating both sides with respect to x we get

$$\frac{d}{dx}(\log u)=\frac{d}{dx}(y\log x)+\frac{d}{dx}(x\log y)$$

$$\text{or,}\quad \frac{1}{u}\frac{du}{dx}=y\frac{d}{dx}(\log x)+\log x\frac{dy}{dx}+x\frac{d}{dx}(\log y)+\log y\frac{d}{dx}(x)$$

$$=\frac{y}{x}+\log x\frac{dy}{dx}+x\frac{1}{y}\frac{dy}{dx}+\log y.1$$

$$\therefore \quad \frac{du}{dx}=u\left(\frac{y}{x}+\log x\,\frac{dy}{dx}+\frac{x}{y}\,\frac{dy}{dx}+\log y\right)$$

$$=x^{y}y^{x}\left(\frac{y}{x}+\log x\frac{dy}{dx}+\frac{x}{y}\,\frac{dy}{dx}+\log y\right)$$

$$\frac{d}{dx}(e^{xy}-3x)=\frac{d}{dx}(e^{xy})-\frac{d}{dx}\,(3x)$$

$$=e^{xy}\frac{d}{dx}\,(xy)-3.$$

$$=e^{xy}\left(x\frac{dy}{dx}+y\right)-3$$

$$\therefore \quad x^{y}y^{x}\left(\frac{y}{x}+\log x\,\frac{dy}{dx}+\frac{x}{y}\,\frac{dy}{dx}+\log y\right)=e^{xy}\left(x\frac{dy}{dx}+y\right)-3$$

or, $$\frac{dy}{dx}\left\{x^{y}y^{x}\left(\log x+\frac{x}{y}\right)-xe^{xy}\right\}$$

$$=ye^{xy}-3-x^{y}y^{x}\left(\frac{y}{x}+\log y\right)$$

$$\therefore \quad \frac{dy}{dx}=\frac{ye^{xy}-3-x^{y}y^{x}\left(\frac{y}{x}+\log y\right)}{x^{y}y^{x}\left(\log x+\frac{x}{y}\right)-xe^{xy}}$$

(x) $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}=u+v$

where $u=(\tan x)^{\cot x}$ and $v=(\cot x)^{\tan x}$

$$\therefore \quad \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}.$$

Now, $u=(\tan x)^{\cot x}$ or, $\log u=\cot x(\log \tan x)$

differentiating both sides, with respect to x we get

$$\frac{d}{dx}\,(\log u)=\frac{d}{dx}\,\{\cot x\log(\tan x)\}$$

or, $$\frac{1}{u}\frac{du}{dx}=\log(\tan x)\,\frac{d}{dx}\,(\cot x)+\cot x\,\frac{d}{dx}\,(\log\tan x)$$

$$=-\log(\tan x)\,\text{cosec}^{2}x+\cot x\,\frac{1}{\tan x}\,\sec^{2}x$$

$$\therefore \quad \frac{du}{dx}=u\,\{\cot^2 x \sec^2 x-\log(\tan x)\operatorname{cosec}^2 x\}$$

$$=u\{\operatorname{cosec}^2 x-\log(\tan x)\operatorname{cosec}^2 x\}$$

$$=(\tan x)^{\cot x}\operatorname{cosec}^2 x(1-\log\tan x)$$

Again, $v=(\cot x)^{\tan x}$ or, $\log v=\log(\cot x)^{\tan x}$
$=\tan x\log(\cot x)$

differentiating both sides with respect to x we get

$$\frac{d}{dx}(\log v)=\frac{d}{dx}\{\tan x\log(\cot x)\}$$

or, $$\frac{1}{v}\frac{dv}{dx}=\tan x\frac{d}{dx}(\log\cot x)+\log\cot x\frac{d}{dx}(\tan x)$$

$$=\tan x\frac{1}{\cot x}(-\operatorname{cosec}^2 x)+(\log\cot x)\sec^2 x$$

$$=-\sec^2 x+\log(\cot x)\sec^2 x.$$

$$\therefore \quad \frac{dv}{dx}=v[\sec^2 x\log(\cot x)-1]$$

$$=(\cot x)^{\tan x}\sec^2 x\,\{\log(\cot x)-1\}$$

$$\therefore \quad \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$=(\tan x)^{\cot x}\operatorname{cosec}^2 x\,(1-\log\tan x)$$

$$+(\cot x)^{\tan x}\sec^2 x\,(\log\cot x-1).$$

(xi) Let $x^{\sin y}=u$ and $y^{\cos x}=v$.

$\therefore \quad u+v=1$ or, $\frac{d}{dx}\left(x^{\sin y}+y^{\cos x}\right)=\frac{d}{dx}(1)$

or, $\frac{d}{dx}(u+v)=0$ or, $\frac{du}{dx}+\frac{dv}{dx}=0$ ... (i)

Now, $u=x^{\sin y}$ or, $\log u=\log(x^{\sin y})=\sin y\log x$

differentiating both sides with respect to x we get

$$\frac{d}{dx}(\log u)=\frac{d}{dx}(\sin y\log x)$$

or, $\frac{1}{u}\frac{du}{dx}=\sin y\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(\sin y)$

$$=\frac{\sin y}{x}+\log x\cos y\frac{dy}{dx}$$

$\therefore\ \frac{du}{dx}=u\left(\frac{\sin y}{x}+\log x\cos y\frac{dy}{dx}\right)$

$$=x^{\sin y}\left(\frac{\sin y}{x}+\log x\cos y\frac{dy}{dx}\right)$$

Again, $v=y^{\cos x}$ or, $\log v=\log y^{\cos x}=\cos x\log y$ differentiating both sides with respect to x we get,

$\frac{d}{dx}(\log v)=\frac{d}{dx}(\cos x\log y)$

or, $\frac{1}{v}\frac{dv}{dx}=\cos x\frac{d}{dx}(\log y)+\log y\frac{d}{dx}(\cos x)$

$$=\cos x\frac{1}{y}\frac{dy}{dx}+\log y(-\sin x)$$

or, $\frac{dv}{dx}=v\left\{\frac{\cos x}{y}\frac{dy}{dx}-\log y\sin x\right\}$

or, $\frac{dv}{dx}=y^{\cos x}\left\{\frac{\cos x}{y}\frac{dy}{dx}-\log y\sin x\right\}$

$\therefore$ from equation-(i) we get

$x^{\sin y}\left(\frac{\sin y}{x}+\log x\cos x\frac{dy}{dx}\right)$

$+y^{\cos x}\left(\frac{\cos x}{y}\frac{dy}{dx}-\log y\sin x\right)=0$

or, $\frac{dy}{dx}\left[x^{\sin y}\log x\cos x+y^{\cos x}\frac{\cos x}{y}\right]$

$=y^{\cos x}\log y\sin x-x^{\sin y}\frac{\sin y}{x}$

$\therefore\ \frac{dy}{dx}=\frac{y^{\cos x}\log y\sin x-x^{\sin y}\frac{\sin y}{x}}{x^{\sin y}\log x\cos x+y^{\cos x}\frac{\cos x}{y}}$

(xii) $y=e^{\cos^{-1}x}+x^{\sqrt{x}}=u+v$

where, $u=e^{\cos^{-1}x}$, $v=x^{\sqrt{x}}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$$

Now, $u=e^{\cos^{-1}x}$

$$\therefore \frac{du}{dx}=\frac{d}{dx}\left(e^{\cos^{-1}x}\right)=e^{\cos^{-1}x}\frac{d}{dx}\left(\cos^{-1}x\right)$$

$$=-e^{\cos^{-1}x}\frac{1}{\sqrt{1-x^2}}$$

$v=x^{\sqrt{x}}$ or, $\log v=\log x^{\sqrt{x}}=\sqrt{x}\log x$

differentiating both sides with respect to x we get

$$\frac{d}{dx}(\log v)=\frac{d}{dx}(\sqrt{x}\log x)$$

$$\text{or,}\quad \frac{1}{v}\frac{dv}{dx}=\log x\frac{d}{dx}(\sqrt{x})+\sqrt{x}\frac{d}{dx}(\log x)$$

$$=\log x\frac{1}{2\sqrt{x}}+\sqrt{x}.\frac{1}{x}$$

$$=\frac{1}{\sqrt{x}}(\tfrac{1}{2}\log x+1)$$

$$\therefore \frac{dv}{dx}=v\,\frac{1}{\sqrt{x}}(\tfrac{1}{2}\log x+1)$$

$$=x^{\sqrt{x}}.\frac{1}{\sqrt{x}}(\log\sqrt{x}+1)$$

$$\therefore \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}=-e^{\cos^{-1}x}\frac{1}{\sqrt{1-x^2}}+x^{\sqrt{x}}.\frac{1}{\sqrt{x}}(\log\sqrt{x}+1)$$

Ex. 19. (i) Find the derivative of x^5 with respect to x^2

(ii) Find the derivative of $\sin x$ with respect to $\cos x$.

(iii) If $y=\tan^{-1}\frac{2t}{1-t^2}$ and $x=\sin^{-1}.\frac{2t}{1+t^2}$ show that

$\frac{dy}{dx}=1$ [H. S. 1982]

(iv) Differentiate $\sin^{-1}\dfrac{2x}{1+x^2}$ with respect to $\tan^{-1}\dfrac{2x}{1-x^2}$

[Joint Entrance 1982]

(v) Show that the derivative of $\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$

with respect to $\sqrt{1-x^4}$ is $\dfrac{\sqrt{1-x^4}-1}{x^6}$ [Joint Entrance 1984]

(vi) Find the differential coefficient of $x^{\sin^{-1}x}$ with respect to $\sin^{-1}x$

[Joint Entrance 1987]

(vii) Differentiate $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$

(i) Let $y=x^5$ and $z=x^2$

So, we are to find $\dfrac{dy}{dz}$

Now, $$\frac{dy}{dz}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{\frac{d}{dx}(x^5)}{\frac{d}{dx}(x^2)}=\frac{5x^4}{2x}=\frac{5}{2}x^3$$

(ii) Let $y=\sin x$ and $z=\cos x$.

So we are to find $\dfrac{dy}{dz}$

Now, $$\frac{dy}{dz}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(\cos x)}=\frac{\cos x}{-\sin x}=-\cot x$$

(iii) Let, $t=\tan\theta$

$$\therefore\ y=\tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}=\tan^{-1}\tan 2\theta=2\theta$$

$$x=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}\sin 2\theta=2\theta$$

$$\therefore\ \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\frac{d}{d\theta}(2\theta)}{\frac{d}{d\theta}(2\theta)}=\frac{2}{2}=1.$$

(iv) Here $y=\sin^{-1}\dfrac{2x}{1+x^2}=2\theta$ and $z=\tan^{-1}\dfrac{2x}{1-x^2}=2\theta$

[Putting $x=\tan\theta$ as in (iii) above]

So we are to find $\dfrac{dy}{dz}$

Now, $$\frac{dy}{dz}=\frac{\dfrac{dy}{d\theta}}{\dfrac{dz}{d\theta}}=\frac{\dfrac{d}{d\theta}(2\theta)}{\dfrac{d}{d\theta}(2\theta)}=\frac{2}{2}=1$$

(v) Let $x^2=\cos\theta$

$$\therefore\quad y=\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}$$

$$=\frac{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}=\frac{\sqrt{2}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)}{\sqrt{2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}$$

$$=\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}=\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

Again, $z=\sqrt{1-x^4}=\sqrt{1-\cos^2\theta}=\sin\theta$

$\therefore$ Required differential coefficient $$=\frac{dy}{dz}=\frac{\dfrac{dy}{d\theta}}{\dfrac{dz}{d\theta}}=\frac{\dfrac{d}{d\theta}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}{\dfrac{d}{d\theta}(\sin\theta)}$$

$$=\frac{\sec^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left(-\frac{1}{2}\right)}{\cos\theta}=-\frac{1}{2}\,\frac{\sec^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\cos\theta}$$

$$=-\frac{1}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\theta}=-\frac{1}{\left\{1+\cos 2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}x^2}$$

$$=-\frac{1}{1+\cos\left(\frac{\pi}{2}-\theta\right)x^2}=-\frac{1}{(1+\sin\theta)\,x^2}$$

$$=-\frac{1}{\{1+\sqrt{1-\cos^2\theta}\}\,x^2}=-\frac{1}{\{1+\sqrt{1-x^4}\}\,x^2}$$

$$=-\frac{\sqrt{1-x^4}-1}{\{\sqrt{1-x^4}+1\}\{\sqrt{1-x^4}-1\}\,x^2}=-\frac{\sqrt{1-x^4}-1)}{(1-x^4-1)x^2}$$

$$=\frac{\sqrt{1-x^4}-1}{x^6}$$

(vi) Let $y=x^{\sin^{-1}x}$ and $z=\sin^{-1}x$

Now $\log y=\log\left(x^{\sin^{-1}x}\right)=\sin^{-1}x\log x$

differentiating both sides with respect to x we get

$$\frac{d}{dx}(\log y)=\frac{d}{dx}(\sin^{-1}x\,.\log x)$$

or, $\frac{1}{y}\frac{dy}{dx}=\sin^{-1}x\,\frac{d}{dx}(\log x)+\log x\,\frac{d}{dx}(\sin^{-1}x)$

$$=\frac{\sin^{-1}x}{x}+\frac{\log x}{\sqrt{1-x^2}}$$

$$\therefore\ \frac{dy}{dx}=y\left\{\frac{\sin^{-1}x}{x}+\frac{\log x}{\sqrt{1-x^2}}\right\}=x^{\sin^{-1}x}\left\{\frac{\sin^{-1}x}{x}+\frac{\log x}{\sqrt{1-x^2}}\right\}$$

$$\frac{dz}{dx}=\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$$

$$\therefore\ \frac{dy}{dz}=\frac{\dfrac{dy}{dx}}{\dfrac{dz}{dx}}=\frac{x^{\sin^{-1}x}\left\{\dfrac{\sin^{-1}x}{x}+\dfrac{\log x}{\sqrt{1-x^2}}\right\}}{\dfrac{1}{\sqrt{1-x^2}}}$$

$$=x^{\sin^{-1}x}\left\{\frac{\sqrt{1-x^2}}{x}\sin^{-1}x+\log x.\right\}$$

(vii) Let $y=\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ and $z=\tan^{-1}x$

or, $x=\tan z$

$$\therefore\ y=\tan^{-1}\frac{\sqrt{1+\tan^2 z}-1}{\tan z}=\tan^{-1}\frac{\sec z-1}{\dfrac{\sin z}{\cos z}}=\tan^{-1}\frac{\dfrac{1}{\cos z}-1}{\dfrac{\sin z}{\cos z}}$$

$$=\tan^{-1}\frac{1-\cos z}{\sin z}=\tan^{-1}\frac{2\sin^2\frac{z}{2}}{2\sin\frac{z}{2}\cos\frac{z}{2}}=\tan^{-1}\tan\frac{z}{2}=\frac{z}{2}$$

Now derivative of $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ with respect to z

$$=\frac{dy}{dz}=\frac{d}{dz}\left(\frac{z}{2}\right)=\frac{1}{2}.$$

Ex. 20. (i) If $x^y=e^{x-y}$, prove that $\dfrac{dy}{dx}=\dfrac{\log x}{(\log ex)^2}$.

[Joint Entrance 1983]

(ii) If $\sin y=x\sin(a+y)$, prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$

[Here $a\neq n\pi$, $n=0, 1, 2, \cdots$] [H. S. 1982]

(iii) Given that $\cos y=x\cos(a+y)$, prove that

$\dfrac{dy}{dx}=\dfrac{\cos^2(a+y)}{\sin a}$ where $a(\neq 0)$ is a constant.

(iv) If $\sqrt{1-x^4}+\sqrt{1-y^4}=k(x^2-y^2)$ prove that

$$\frac{dy}{dx}=\frac{x\sqrt{1-y^4}}{y\sqrt{1-x^4}}$$ [[Joint Entrance 1986]

(v) If $\sqrt{1-x^2}+\sqrt{1-y^2}=k(x-y)$, prove that

$$\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

(vi) If $x(1+y)^{\frac{1}{2}}+y(1+x)^{\frac{1}{2}}=0$, prove that

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

(i) $x^y=e^{x-y}$

$\therefore\ \log x^y=\log e^{x-y}$ or, $y\log x=x-y\cdots\ \cdots$(i)

differentiating both sides with respect to x we get

$$\frac{d}{dx}(y\log x)=\frac{d}{dx}(x-y)$$

or, $y\dfrac{d}{dx}(\log x)+\log x\ \dfrac{dy}{dx}=\dfrac{d}{dx}(x)-\dfrac{dy}{dx}$

or, $\dfrac{y}{x}+\log x\dfrac{dy}{dx}=1-\dfrac{dy}{dx}$

or, $\dfrac{dy}{dx}(1+\log x)=1-\dfrac{y}{x}=\dfrac{x-y}{x}$

or, $\frac{dy}{dx}(\log ex)=\frac{y}{x}\log x \quad [\because \log e=1]$

now from (i) we get,

$y(1+\log x)=x,\ \frac{y}{x}=\frac{1}{1+\log x}=\frac{1}{\log ex}$

$\therefore \frac{dy}{dx}(\log ex)=\frac{\log x}{\log ex} \quad \therefore \frac{dy}{dx}=\frac{\log x}{(\log ex)^2}$

(ii) $\sin y=x\sin(a+y)$

differentiating both sides with respect to x we get,

$$\cos y\frac{dy}{dx}=x\frac{d}{dx}\{\sin(a+y)\}+\sin(a+y)\frac{d}{dx}(x)$$

$$=x\cos(a+y)\frac{dy}{dx}+\sin(a+y)$$

or, $\frac{dy}{dx}\{\cos y-x\cos(a+y)\}=\sin(a+y)$

or, $\frac{dy}{dx}\left\{\cos y-\frac{\sin y}{\sin(a+y)}\cos(a+y)\right\}=\sin(a+y)$

$[\because \sin y=x\sin(a+y)]$

or. $\frac{dy}{dx}\left\{\frac{\sin(a+y)\cos y-\cos(a+y)\sin y)}{\sin(a+y)}\right\}=\sin(a+y)$

or, $\frac{dy}{dx}\sin(a+y-y)=\sin^2(a+y)$

or, $\frac{dy}{dx}\sin a=\sin^2(a+y)$ or, $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$

(iii) $\cos y=x\cos(a+y)$

differentiating both sides with respect to x we get,

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}\{x\cos(a+y)\}$$

or, $-\sin y\frac{dy}{dx}=x\frac{d}{dx}\left\{\cos(a+y)\right\}+\cos(a+y)\frac{d}{dx}(x)$

$$=-x\sin(a+y)\frac{dy}{dx}+\cos(a+y)$$

or, $\frac{dy}{dx}\left\{x \sin(a+y)-\sin y\right\}=\cos(a+y)$

or, $\frac{dy}{dx}\left\{\frac{\cos y}{\cos(a+y)} \sin(a+y)-\sin y\right\}=\cos(a+y)$

$[\because \quad \cos y=x \cos(a+y)]$

or, $\frac{dy}{dx}\left\{\frac{\sin(a+y)\cos y-\cos(a+y)\sin y}{\cos(a+y)}\right\}=\cos(a+y)$

or, $\frac{dy}{dx} \sin(a+y-y)=\cos^2(a+y)$

or, $\frac{dy}{dx} \sin a=\cos^2(a+y)$ or, $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

(iv) $\sqrt{1-x^4}+\sqrt{1-y^4}=k(x^2-y^2)\cdots\cdots$(i)

or, $\frac{d}{dx}\left\{\sqrt{1-x^4}+\sqrt{1-y^4}\right\}=\frac{d}{dx}\left\{k(x^2-y^2)\right\}$

or, $\frac{d}{dx}(\sqrt{1-x^4})+\frac{d}{dx}(\sqrt{1-y^4})=k\left\{\frac{d}{dx}(x^2)-\frac{d}{dx}(y^2)\right\}$

or, $\frac{1}{2\sqrt{1-x^4}}\frac{d}{dx}(1-x^4)+\frac{1}{2\sqrt{1-y^4}}\frac{d}{dx}(1-y^4)$

$=k\left\{2x-2y\frac{dy}{dx}\right\}$

or, $\frac{1}{2\sqrt{1-x^4}}(-4x^3)+\frac{1}{2\sqrt{1-y^4}}\left(-4y^3\frac{dy}{dx}\right)$

$=2k\left(x-y\frac{dy}{dx}\right)$

or, $\frac{-x^3}{\sqrt{1-x^4}}-\frac{y^3}{\sqrt{1-y^4}}\frac{dy}{dx}=k\left(x-y\frac{dy}{dx}\right)$

or, $\frac{dy}{dx}\left\{ky-\frac{y^3}{\sqrt{1-y^4}}\right\}=kx+\frac{x^3}{\sqrt{1-x^4}}$

or, $\frac{dy}{dx}=\frac{kx\sqrt{1-x^4}+x^3}{\sqrt{1-x^4}}\times\frac{\sqrt{1-y^4}}{ky\sqrt{1-y^4}-y^3}$

$=\frac{x\sqrt{1-y^4}}{y\sqrt{1-x^4}}\frac{k\sqrt{1-x^4}+x^2}{k\sqrt{1-y^4}-y^2}$

Now $y^4 - x^4 = (1 - x^4) - (1 - y^4)$ ... (ii)

∴ From (ii) ÷ (i)

$$\frac{y^4 - x^4}{k(x^2 - y^2)} = \frac{(1 - x^4) - (1 - y^4)}{\sqrt{1 - x^4} + \sqrt{1 - y^4}}$$

or, $-(y^2 + x^2) = k\{\sqrt{1 - x^4} - \sqrt{1 - y^4}\}$

or, $k\sqrt{1 - x^4} + x^2 = k\sqrt{1 - y^4} - y^2$

∴ $\dfrac{k\sqrt{1 - x^4} + x^2}{k\sqrt{1 - y^4} - y^2} = 1$

∴ $\dfrac{dy}{dx} = \dfrac{x\sqrt{1 - y^4}}{y\sqrt{1 - x^4}}$

(v) $\sqrt{1 - x^2} + \sqrt{1 - y^2} = k(x - y)$ ... (i)

differentiating both sides with respect to x we get

$$\frac{d}{dx}\sqrt{1 - x^2} + \frac{d}{dx}\sqrt{1 - y^2} = \frac{d}{dx}\{k(x - y)\}$$

or, $\dfrac{1}{2\sqrt{1 - x^2}}(-2x) + \dfrac{1}{2\sqrt{1 - y^2}}\left(-2y\dfrac{dy}{dx}\right) = k - k\dfrac{dy}{dx}$

∴ $\dfrac{dy}{dx}\left\{k - \dfrac{y}{\sqrt{1 - y^2}}\right\} = k + \dfrac{x}{\sqrt{1 - x^2}}$

or, $\dfrac{dy}{dx}\left\{\dfrac{k\sqrt{1 - y^2} - y}{\sqrt{1 - y^2}}\right\} = \dfrac{k\sqrt{1 - x^2} + x}{\sqrt{1 - x^2}}$

or, $\dfrac{dy}{dx} = \dfrac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}} \cdot \dfrac{k\sqrt{1 - x^2} + x}{k\sqrt{1 - y^2} - y}$.

Now, $(1 - x^2) - (1 - y^2) = y^2 - x^2$ ... (ii)

From (ii) ÷ (i)

$$\sqrt{1 - x^2} - \sqrt{1 - y^2} = -\left(\frac{x + y}{k}\right)$$

or, $k\sqrt{1 - x^2} - k\sqrt{1 - y^2} = -x - y$

or, $k\sqrt{1 - x^2} + x = k\sqrt{1 - y^2} - y$

or, $\dfrac{k\sqrt{1 - x^2} + x}{k\sqrt{1 - y^2} - y} = 1$ ∴ $\dfrac{dy}{dx} = \dfrac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}}$

(vi) $x(1+y)^{\frac{1}{2}}+y(1+x)^{\frac{1}{2}}=0.$

or, $x(1+y)^{\frac{1}{2}}=-y(1+x)^{\frac{1}{2}}$

or, $x^2(1+y)=y^2(1+x)$ [squaring]

or, $y^2(1+x)-y.x^2-x^2=0$

$$\therefore\ y=\frac{x^2\pm\sqrt{x^4+4x^2(1+x)}}{2(1+x)}=\frac{x^2\pm\sqrt{x^2(x^2+4x+4)}}{2(1+x)}$$

$$=\frac{x^2\pm x(x+2)}{2(1+x)}=\frac{2x^2+2x}{2(1+x)}\quad\text{or,}\quad -\frac{2x}{2(1+x)}$$

$$=x\qquad\text{or,}\quad -\frac{x}{1+x}\quad\text{But } y\neq x$$

$$\therefore\ y=-\frac{x}{1+x}\quad\therefore\ \frac{dy}{dx}=-\frac{(1+x)\frac{d}{dx}(x)-x\cdot\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$=-\frac{(1+x)-x}{(1+x)^2}=-\frac{1}{(1+x)^2}$$

Ex. 21. (i) If $f(x)=\left(\frac{a+x}{b+x}\right)^x+\cos x$, find the value of $f'(0)$.

(ii) If $f(x)=\left(\frac{a+x}{b+x}\right)^{a+b+2x}$, show that

$f'(0)=\left(2\log\frac{a}{b}+\frac{b^2-a^2}{ab}\right)\left(\frac{a}{b}\right)^{a+b}$ [C. U.]

(iii) If $y=e^{x\sin x^2}+(\tan x)^x$, find $\frac{dy}{dx}$ [I. I. T. 1981]

(i) $f(x)=\left(\frac{a+x}{b+x}\right)^x+\cos x$

$$\therefore\ f'(x)=\frac{d}{dx}\left\{\left(\frac{a+x}{b+x}\right)^x\right\}+\frac{d}{dx}(\cos x)$$

Now let, $u=\left(\frac{a+x}{b+x}\right)^x$

$$\therefore\ \log u=\log\left(\frac{a+x}{b+x}\right)^x=x\log(a+x)-x\log(b+x)$$

differentiating both sides with respect to x we get

$$\frac{d}{dx}(\log u)=\frac{d}{dx}\{x\log(a+x)-x\log(b+x)\}$$

or, $$\frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}\log(a+x)+\log(a+x)\frac{d}{dx}(x)$$

$$-x\frac{d}{dx}\log(b+x)-\log(b+x)\frac{d}{dx}(x)$$

$$=\frac{x}{a+x}+\log(a+x)-\frac{x}{b+x}-\log(b+x)$$

$$=\log\frac{a+x}{b+x}-x\left\{\frac{b-a}{(a+x)(b+x)}\right\}$$

$$\therefore\quad \frac{du}{dx}=u\left\{\log\frac{a+x}{b+x}-x\left\{\frac{b-a}{(a+x)(b+x)}\right\}\right.$$

$$\therefore\quad f'(x)=\left(\frac{a+x}{b+x}\right)^x\left[\log\frac{a+x}{b+x}-x\left\{\frac{b-a}{(a+x)(b+x)}\right]-\sin x\right.$$

$$\therefore\quad f'(0)=\left(\frac{a}{b}\right)^0\left(\log\frac{a}{b}\right)-\sin 0=\log\frac{a}{b}$$

(ii) Let $y=f(x)=\left(\dfrac{a+x}{b+x}\right)^{a+b+2x}$

$$\therefore\quad \log y=\log\left\{\left(\frac{a+x}{b+x}\right)^{a+b+2x}\right\}$$

$$=(a+b+2x)\log\left(\frac{a+x}{b+x}\right)=(a+b+2x)\{\log(a+x)-\log(b+x)\}$$

differentiating both sides with respect to x we get,

$$\frac{d}{dx}(\log y)=(a+b+2x)\left\{\frac{d}{dx}\log(a+x)-\frac{d}{dx}\log(b+x)\right\}$$

$$+\{\log(a+x)-\log(b+x)\}\frac{d}{dx}(a+b+2x)$$

or, $$\frac{1}{y}\frac{dy}{dx}=(a+b+2x)\left\{\frac{1}{a+x}-\frac{1}{b+x}\right\}+\log\frac{a+x}{b+x}\cdot 2$$

$$\therefore\quad \frac{dy}{dx}=y\left[(a+b+2x)\left\{\frac{b-a}{(a+x)(b+x)}\right\}+2\log\frac{a+x}{b+x}\right]$$

or, $$f'(x)=\left(\frac{a+x}{b+x}\right)a+b+2x\left[(a+b+2x)\left\{\frac{(b-a)}{(a+x)(b+x)}\right.\right.$$

$$\left.\left.+2\log\frac{a+x}{b+x}\right\}\right]$$

$$\therefore \quad f'(0)=\left(\frac{a}{b}\right)^{a+b}\left[(a+b)\left(\frac{b-a}{ab}\right)+2\log\frac{a}{b}\right]$$

$$=\left(2\log\frac{a}{b}+\frac{b^2-a^2}{ab}\right)\left(\frac{a}{b}\right)^{a+b}.$$

(iii) $y=e^{x\sin x^3}+(\tan x)^x$

$$\therefore \quad \frac{dy}{dx}=\frac{d}{dx}\left(e^{x\sin x^3}\right)+\frac{d}{dx}(\tan x)^x$$

Now $\dfrac{d}{dx}\left(e^{x\sin x^3}\right)=e^{x\sin x^3}\dfrac{d}{dx}(x\sin x^3)$

$$=e^{x\sin x^3}\left\{x\frac{d}{dx}(\sin x^3)+\sin x^3\frac{d}{dx}(x)\right\}$$

$$=e^{x\sin x^3}\{x(\cos x^3).3x^2+\sin x^3\}$$

Again let,

$u=(\tan x)^x \quad \therefore \quad \log u=\log(\tan x)^x=x\log(\tan x)$

differentiating both sides with respect to x we get

$$\frac{d}{dx}(\log u)=\frac{d}{dx}\{x\log(\tan x)\}$$

or, $\dfrac{1}{u}\dfrac{du}{dx}=x\dfrac{d}{dx}\{\log(\tan x)\}+\log\tan x\dfrac{d}{dx}(x)$

$$=x\frac{1}{\tan x}\frac{d}{dx}(\tan x)+\log\tan x.$$

$$=x\frac{\sec^2 x}{\tan x}+\log\tan x$$

$$\therefore \quad \frac{du}{dx}=u\left\{x\frac{\sec^2 x}{\tan x}+\log\tan x\right\}$$

or, $\dfrac{d}{dx}\{(\tan x)^x\}=(\tan x)^x\left\{x\dfrac{\sec^2 x}{\tan x}+\log\tan x\right\}$

$$\therefore \quad \frac{dy}{dx}=e^{x\sin x^3}\left\{3x^3(\cos x^3)+\sin x^3\right\}$$

$$+(\tan x)^x\left\{x\frac{\sec^2 x}{\tan x}+\log\tan x\right\}.$$

Ex. 22. (i) Find the derivative of $\log_{\cos x}^{\sin x}$ with respect to x

(ii) If $\log_e (x+y)=x^2+y^2$ find $\frac{dy}{dx}$ when $x=1, y=1$

[H. S. 1981]

(iii) If $\tan^2 y=\frac{1+\cos 2x}{1-\cos 2x}$, find $\frac{dy}{dx}$ [H. S. 1985]

(iv) $y=x^3\sqrt{\frac{x^2+4}{x^2+3}}$; find $\frac{dy}{dx}$. [C. U.]

(v) $y=x^2\sqrt{\frac{x^2-x+1}{x^2+x+1}}\cos x$; find $\frac{dy}{dx}$.

(i) $\log_{\cos x}^{\sin x}=\log_e^{\sin x}\times\log^e\cos x=\frac{\log_e^{\sin x}}{\log_e^{\cos x}}$

$$\therefore\ \frac{d}{dx}\left(\log_{\cos x}^{\sin x}\right)=\frac{\log\cos x\frac{d}{dx}(\log\sin x)-\log\sin x\frac{d}{dx}(\log\cos x)}{(\log\cos x)^2}$$

$$=\frac{(\log\cos x)\frac{1}{\sin x}\frac{d}{dx}(\sin x)-(\log\sin x)\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)}{(\log\cos x)^2}$$

$$=\frac{(\log\cos x)\frac{1}{\sin x}\cdot\cos x+(\log\sin x)\frac{1}{\cos x}\cdot\sin x}{(\log\cos x)^2}$$

$$=\frac{\cot x\log(\cos x)+\tan x\log(\sin x)}{(\log\cos x)^2}$$

(ii) $\log_e (x+y)=x^2+y^2$.

or, $\frac{d}{dx}\{\log_e (x+y)\}=\frac{d}{dx}(x^2+y^2)$

or, $\frac{1}{x+y}\frac{d}{dx}(x+y)=2x+2y\frac{dy}{dx}$.

or, $\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)=2x+2y\frac{dy}{dx}$

or, $\frac{dy}{dx}\left(\frac{1}{x+y}-2y\right)=2x-\frac{1}{x+y}$

or, $\frac{dy}{dx}\left(\frac{1-2xy-2y^2}{x+y}\right)=\frac{2x^2+2xy-1}{x+y}$

$\therefore\ \frac{dy}{dx}=\frac{2x^2+2xy-1}{1-2xy-2y^2}.$

$\therefore$ when $x=1$, $y=1$,

$\frac{dy}{dx}=\frac{2+2-1}{1-2-2}=-\frac{3}{3}=-1.$

(iii) $\tan^2 y=\frac{1+\cos 2x}{1-\cos 2x}=\frac{2\cos^2 x}{2\sin^2 x}=\cot^2 x=\tan^2\left(\frac{\pi}{2}-x\right)$

$\therefore\ \tan y=\pm\tan\left(\frac{\pi}{2}-x\right)$ or, $y=\pm\left(\frac{\pi}{2}-x\right)$

$\therefore\ \frac{dy}{dx}=\pm(-1)=\mp 1.$

(iv) $y=x^3\sqrt{\frac{x^2+4}{x^2+3}}$

$\therefore\ \log y=3\log x+\frac{1}{2}\log(x^2+4)-\frac{1}{2}\log(x^2+3)$

$\therefore\ \frac{d}{dx}(\log y)=\frac{d}{dx}\{3\log x+\frac{1}{2}\log(x^2+4)-\frac{1}{2}\log(x^2+3)\}$

or, $\frac{1}{y}\frac{dy}{dx}=3\frac{d}{dx}(\log x)+\frac{1}{2}\frac{d}{dx}\{\log(x^2+4)\}-\frac{1}{2}\frac{d}{dx}\{\log(x^2+3)\},$

$$=3.\frac{1}{x}+\frac{1}{2}\cdot\frac{1}{x^2+4}\frac{d}{dx}(x^2+4)-\frac{1}{2}.\frac{1}{x^2+3}\frac{d}{dx}(x^2+3)$$

$$=\frac{3}{x}+\frac{1}{2}\frac{1}{x^2+4}.(2x)-\frac{1}{2}.\frac{1}{x^2+3}(2x)$$

$$=\frac{3}{x}+\frac{x}{x^2+4}-\frac{x}{x^2+3}$$

$\therefore\ \frac{dy}{dx}=y\left\{\frac{3}{x}+\frac{x}{x^2+4}-\frac{x}{x^2+3}\right\}$

$$=x^3\sqrt{\frac{x^2+4}{x^2+3}}\left\{\frac{3}{x}+\frac{x}{x^2+4}-\frac{x}{x^2+3}\right\}$$

(v) $y=x^2\sqrt{\dfrac{x^2-x+1}{x^2+x+1}}\cos x$

or, $\log y=\log x^2+\frac{1}{2}\log(x^2-x+1)-\frac{1}{2}\log(x^2+x+1)$
$+\log\cos x$

$\therefore\ \dfrac{d}{dx}(\log y)=\dfrac{d}{dx}(2\log x)+\dfrac{1}{2}\dfrac{d}{dx}\{\log(x^2-x+1\}$

$$-\frac{1}{2}\frac{d}{dx}\{\log(x^2+x+1)\}+\frac{d}{dx}(\log\cos x)$$

or, $\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{2}{x}+\dfrac{1}{2}\dfrac{1}{x^2-x+1}\dfrac{d}{dx}(x^2-x+1)$

$-\dfrac{1}{2}\dfrac{1}{x^2+x+1}\dfrac{d}{dx}(x^2+x+1)+\dfrac{1}{\cos x}\dfrac{d}{dx}(\cos x)$

$=\dfrac{2}{x}+\dfrac{1}{2(x^2-x+1)}(2x-1)-\dfrac{1}{2(x^2+x+1)}(2x+1)+\dfrac{1}{\cos x}(-\sin x$

$\therefore\ \dfrac{dy}{dx}=y\left\{\dfrac{2}{x}+\dfrac{2x-1}{2(x^2-x+1)}-\dfrac{2x+1}{2(x^2+x+1)}-\tan x\right\}$

$=x^2\sqrt{\dfrac{x^2-x+1}{x^2+x+1}}\cos x\left\{\dfrac{2}{x}+\dfrac{2x-1}{2(x^2-x+1)}-\dfrac{2x+1}{2(x^2+x+1)}-\tan x\right\}$

Ex. 23. (i) Find the derivative of $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$ at $x=\frac{1}{2}$

[I. I. T. 1986]

(ii) Find the derivative with respect to x of the function $\left(\log\dfrac{\sin x}{\cos x}\right)\left(\log\dfrac{\cos x}{\sin x}\right)^{-1}+\sin^{-1}\left(\dfrac{2x}{1-x^2}\right)$ at $x=\dfrac{\pi}{4}$

[I. I. T. 1984]

(iii) If $y=f\left(\dfrac{2x-1}{x^2+1}\right)$ and $f'(x)=\sin x^2$, then find $\dfrac{dy}{dx}$

[c. f. I. I. T. 1982]

(iv) Let $y=e^x\sin x^3+(\tan x)^x$.

Find $\dfrac{dy}{dx}$

[I. I. T. 1981]

(i) Let $y=\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$, $z=\sqrt{1-x^2}$, $x=\cos\theta$.

$$\therefore \quad y=\sec^{-1}\frac{1}{2\cos^2\theta-1}=\sec^{-1}\frac{1}{\cos 2\theta}=\sec^{-1}\sec 2\theta=2\theta$$

$$z=\sqrt{1-x^2}=\sqrt{1-\cos^2\theta}=\sin\theta$$

$$\therefore \quad \frac{dy}{dz}=\frac{\frac{dy}{d\theta}}{\frac{dz}{d\theta}}=\frac{\frac{d}{d\theta}(2\theta)}{\frac{d}{d\theta}(\sin\theta)}=\frac{2}{\cos\theta}=\frac{2}{x}$$

$$\therefore \quad \frac{dy}{dx}=\frac{2}{\frac{1}{2}}=4 \text{ at } x=\tfrac{1}{2}$$

[Here derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}=\frac{dy}{dz}$]

(ii) $\left(\log\frac{\sin x}{\cos x}\right)\left(\log\frac{\cos x}{\sin x}\right)^{-1}+\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$=\left(\log\frac{\sin x}{\cos x}\right)\left(\frac{1}{\log\frac{\sin x}{\cos x}}\right)^{-1}+\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \; [\tan\theta=x(\text{say})]$$

$$=\left(\log\frac{\sin x}{\cos x}\right)^2+\sin^{-1}\sin 2\theta=\left(\log\frac{\sin x}{\cos x}\right)^2+2\theta.$$

$$=\left(\log\frac{\sin x}{\cos x}\right)^2+2\tan^{-1}x.$$

$\therefore$ derivative of the given function.

$$=2\left(\log\frac{\sin x}{\cos x}\right)\frac{d}{dx}\left(\frac{\log\sin x}{\log\cos x}\right)+\frac{2}{1+x^2}$$

$$=2\left(\log\frac{\sin x}{\cos x}\right)\frac{\cot x\log(\cos x)+\tan x\log(\sin x)}{(\log\cos x)^2}+\frac{2}{1+x^2}$$

$\therefore$ derivative of the given function at $x=\frac{\pi}{4}$

$$=2\left(\log\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}\right)\frac{\cot\frac{\pi}{4}\log\left(\cos\frac{\pi}{4}\right)+\tan\frac{\pi}{4}\log\left(\sin\frac{\pi}{4}\right)}{\left(\log\cos\frac{\pi}{4}\right)^2}$$

$$+\frac{2}{1+\left(\frac{\pi}{4}\right)^2}$$

$$=2\left(\log\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)\left\{\frac{1\log\left(\frac{1}{\sqrt{2}}\right)+1\log\left(\frac{1}{\sqrt{2}}\right)}{\log\left(\frac{1}{\sqrt{2}}\right)^2}\right\}+\frac{2}{1+\frac{\pi^2}{16}}$$

$$=2.1\left(\frac{\log\frac{1}{\sqrt{2}}(1+1)}{2\log\left(\frac{1}{\sqrt{2}}\right)}\right)+\frac{2}{1+\frac{\pi^2}{16}}$$

$$=2+\frac{2}{1+\frac{\pi^2}{16}}=2+\frac{32}{16+\pi^2}$$

(iii) $y=f\left(\frac{2x-1}{x^2+1}\right)=f(z)\left[z=\frac{2x-1}{x^2+1}\text{ (say)}\right]$

$$\therefore\quad \frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{d}{dz}\left\{f(z)\right\}\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$$

$$=f'(z)\,\frac{(x^2+1)\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$=f'(z)\frac{(x^2+1).2-(2x-1)2x}{(x^2+1)^2}$$

$$=(\sin z^2)\,\frac{2x^2+2-4x^2+2x}{(x^2+1)^2}\qquad\left[\because\ f'(x)=\sin(x^2)\right]$$

$$=\sin\left\{\left(\frac{2x-1}{x^2+1}\right)^2\right\}\times\frac{2(1+x-x^2)}{(x^2+1)^2}$$

Ex. 24. Find $\frac{dy}{dx}$ when

(i) $y=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)$

(ii) $y=-\sqrt{(8+x-x^2)}+\frac{1}{2}\sin^{-1}\frac{2x-1}{\sqrt{33}}$

(iii) $y=\sin^{-1}(3x-4x^3)$

(iv) $x=\cot^{-1}\frac{1+t}{1-t}$; $y=\cot^{-1}\sqrt{(1+t^2)}-t$

(i) $\frac{dy}{dx}=\frac{d}{dx}\left\{\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right\}$

$$=\tfrac{1}{2}\left\{x\frac{d}{dx}\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\frac{d}{dx}(x)\right\}+\tfrac{1}{2}a^2\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)$$

$$=\tfrac{1}{2}\left\{x\,\frac{1}{2\sqrt{a^2-x^2}}\,\frac{d}{dx}\,(a^2-x^2)+\sqrt{a^2-x^2}.\,1\right\}$$

$$+\tfrac{1}{2}\,a^2\,\frac{1}{a}\,\frac{1}{\sqrt{1-\dfrac{x^2}{a^2}}}$$

$$=\tfrac{1}{2}\left\{\frac{x}{2\sqrt{a^2-x^2}}\,(-2x)+\sqrt{a^2-x^2}\right\}+\frac{a}{2}\,\frac{a}{\sqrt{a^2-x^2}}$$

$$=\tfrac{1}{2}\,\frac{x}{2\sqrt{(a^2-x^2)}}\,(-2x)+\tfrac{1}{2}\sqrt{a^2-x^2}+\frac{a^2}{2\sqrt{a^2-x^2}}$$

$$=\tfrac{1}{2}\left\{\frac{-x^2}{\sqrt{a^2-x^2}}+\sqrt{a^2-x^2}+\frac{a^2}{\sqrt{a^2-x^2}}\right\}$$

$$=\tfrac{1}{2}\,\frac{-x^2+a^2-x^2+a^2}{\sqrt{a^2-x^2}}=\tfrac{1}{2}\,\frac{2\,(a^2-x^2)}{\sqrt{a^2-x^2}}=\sqrt{a^2-x^2}$$

(ii) $$\frac{dy}{dx}=\frac{d}{dx}\left\{-\sqrt{(8+x-x^2)}+\tfrac{1}{2}\sin^{-1}\frac{2x-1}{\sqrt{33}}\right\}$$

$$=-\frac{d}{dx}\sqrt{(8+x-x^2)}+\tfrac{1}{2}\,\frac{d}{dx}\sin^{-1}\frac{2x-1}{\sqrt{33}}$$

$$=-\frac{d}{dx}\sqrt{u}+\tfrac{1}{2}\,\frac{d}{dx}\sin^{-1}v\quad\left[u=8+x-x^2,\ v=\frac{2x-1}{\sqrt{33}}\ (\text{ say })\right]$$

$$=-\frac{d}{du}\sqrt{u}\,\frac{du}{dx}+\tfrac{1}{2}\,\frac{d}{dv}\,(\sin^{-1}v)\,\frac{dv}{dx}$$

$$=-\frac{1}{2\sqrt{u}}\,\frac{d}{dx}\,(8+x-x^2)+\frac{1}{2\sqrt{1-v^2}}\,\frac{d}{dx}\left(\frac{2x-1}{\sqrt{33}}\right)$$

$$=-\frac{1}{2\sqrt{8+x-x^2}}\,(1-2x)+\frac{1}{2\sqrt{1-\dfrac{(2x-1)^2}{33}}}\left(\frac{2}{\sqrt{33}}\right)$$

$$=-\frac{1-2x}{2\sqrt{8+x-x^2}}+\frac{1}{\dfrac{2\sqrt{33-4x^2+4x-1}}{\sqrt{33}}}\cdot\frac{2}{\sqrt{33}}$$

$$=\frac{2x-1}{2\sqrt{8+x-x^2}}+\frac{1}{\sqrt{32+4x-4x^2}}$$

$$=\frac{2x-1}{2\sqrt{8+x-x^2}}+\frac{1}{2\sqrt{8+x-x^2}}=\frac{2x-1+1}{2\sqrt{8+x-x^2}}$$

$$=\frac{2x}{2\sqrt{8+x-x^2}}=\frac{x}{\sqrt{8+x-x^2}}$$

(iii) $y=\sin^{-1}(3x-4x^3)=3\sin^{-1}x$

$$\therefore\ \frac{dy}{dx}=\frac{3}{\sqrt{1-x^2}}.$$

(iv) Let $t=\tan\theta$

$$\therefore\ x=\cot^{-1}\frac{1+\tan\theta}{1-\tan\theta}=\cot^{-1}\tan\left(\frac{\pi}{4}+\theta\right)$$

$$=\cot^{-1}\cot\left(\frac{\pi}{4}-\theta\right)=\frac{\pi}{4}-\theta\quad\therefore\ \frac{dx}{d\theta}=-1$$

$$y=\cot^{-1}\left(\sqrt{1+t^2}-t\right)$$
$$=\cot^{-1}\left(\sqrt{1+\tan^2\theta}-\tan\theta\right)$$

$$=\cot^{-1}(\sec\theta-\tan\theta)=\cot^{-1}\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)$$

$$=\cot^{-1}\frac{1-\sin\theta}{\cos\theta}=\cot^{-1}\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$$

$$=\cot^{-1}\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}=\cot^{-1}\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}=\cot^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$=\cot^{-1}\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{\pi}{4}+\frac{\theta}{2}$$

$$\therefore\ \frac{dy}{d\theta}=\tfrac{1}{2}.$$

$$\therefore\ \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\frac{1}{2}}{-1}=-\tfrac{1}{2}.$$

Ex. 25. (i) From the formula

$$1+x+x^2+\cdots+x^n=\frac{1-x^{n+1}}{1-x},$$

find the sum of the series $1+2x+3x^2+\cdots+nx^{n-1}$

(ii) If $(1+x)^n=c_0+c_1x+c_2x^2+\cdots+c_nx^n$
Prove that $c_0+2c_1+3c_2+\cdots+(n+1)c_n=(n+2)2^{n-1}$

(iii) If S_n and r respectively denote the sum of the first n terms of a geometrical progression and its common ratio, then show that

$$(r-1)\frac{dS_n}{dr}=(n-1)S_n-nS_{n-1}$$

(i) $1+x+x^2+\cdots+x^n=\dfrac{1-x^{n+1}}{1-x}$

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}(1+x+x^2+\cdots+x^n)=\frac{d}{dx}\left(\frac{1-x^{n-1}}{1-x}\right)$$

or, $1+2x+3x^2+\cdots+nx^{n-1}$

$$=\frac{(1-x)\dfrac{d}{dx}(1-x^{n+1})-(1-x^{n+1})\dfrac{d}{dx}(1-x)}{(1-x)^2}$$

$$=\frac{-(1-x)(n+1)x^n+(1-x^{n+1})}{(1-x)^2}$$

$$=\frac{(n+1)(-x^n+x^{n+1})+1-x^{n+1}}{(1-x)^2}$$

$$=\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$$

(ii) $(1+x)^n=c_0+c_1x+c_2x^2+\cdots+c_nx^n$

Differentiating both sides with respect to x we get

$$n(1+x)^{n-1}=c_1+2c_2x+3c_3x^2+\cdots+nc_nx^{n-1}$$

Putting $x=1$

$$n.2^{n-1}=c_1+2c_2+3c_3+\cdots+nc_n \quad \cdots \quad (1)$$

Putting $x=1$ in the given relation

$$2^n=c_0+c_1+c_2+\cdots+c_n. \quad \cdots \quad (2)$$

From (i)+(ii) we get,

$$c_0+2c_1+3c_2+\cdots+(n+1)c_n=n.2^{n-1}+2^n=(n+2).2^{n-1}$$

(iii) $S_n=\dfrac{a(r^n-1)}{r-1}$ or, $(r-1)S_n=a(r^n-1)$

Differentiating both sides with respect to r we get

$$(r-1)\frac{dS_n}{dr}+S_n=anr^{n-1} \quad \text{(i)}$$

Again $S_n=S_{n-1}+ar^{n-1}$ $[t_n=ar^{n-1}]$

$\therefore$ $nS_n=nS_{n-1}+nar^{n-1}$ $\therefore$ $nar^{n-1}=nS_n-nS_{n-1}$.

$\therefore$ From (1) we get

$$(r-1)\frac{dS_n}{dr}+S_n=nS_n-nS_{n-1}$$

or, $(r-1)\dfrac{dS_n}{dr}=(n-1)S_n-nS_{n-1}$

Ex. 26. (i) If $y=x^{x^{x\cdots \text{to infinity}}}$,

Prove that $x\dfrac{dy}{dx}=\dfrac{y^2}{1-y\log x}$

(ii) $y=(\sin x)^{(\sin x)^{(\sin x)\cdots \text{to infinity}}}$

prove that $\dfrac{dy}{dx}=\dfrac{y^2\cot x}{1-y\log\sin x}$

(i) $y=x^{x^{x\cdots(\text{to infinity})}}$

$=x^y$

$\therefore$ $\log y=\log x^y=y\log x$

$\therefore$ $\dfrac{d}{dx}(\log y)=\dfrac{d}{dx}(y\log x)$

or, $\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{dy}{dx}\log x+\dfrac{y}{x}$

or, $\dfrac{dy}{dx}\left(\dfrac{1}{y}-\log x\right)=\dfrac{y}{x}$

or, $\dfrac{dy}{dx}\left(\dfrac{1-y\log x}{y}\right)=\dfrac{y}{x}$

or, $\dfrac{dy}{dx}=\dfrac{y^2}{x(1-y\log x)}$

(ii) $y=(\sin x)^{(\sin x)^{(\sin x)\cdots \text{to infinity}}}$

$=(\sin x)^y$

or, $\log y=\log(\sin x)^y=y\log\sin x$

$$\therefore \quad \frac{d}{dx}(\log y)=\frac{d}{dx}\,(y\log\sin x)$$

$$\text{or,}\quad \frac{1}{y}\,\frac{dy}{dx}=y\frac{d}{dx}\,(\log\sin x)+\frac{dy}{dx}\log\sin x$$

$$=y\frac{1}{\sin x}\,\cos x+\frac{dy}{dx}\log\sin x$$

$$\text{or,}\quad \frac{dy}{dx}\left(\frac{1}{y}-\log\sin x\right)=y\cot x$$

$$\text{or,}\quad \frac{dy}{dx}\left(\frac{1-y\log\sin x}{y}\right)=y\cot x$$

$$\text{or,}\quad \frac{dy}{dx}=\frac{y^2\cot x}{1-y\log\sin x}.$$

Exercise 5B

Determine the derivatives of the following functions with respect to x. (Exs. 1—10)

1. (i) $(x+11)^5$ (ii) $(2x-7)^{11}$ (iii) $(3-5x)^9$ (iv) $\sqrt{(ax^2+bx+c)}$

(v) $\dfrac{1}{\sqrt{ax^2+bx+c}}$.

2. (i) $\sin 4x$ (ii) $\cos 5x$ (iii) $\tan 6x$ (iv) $\operatorname{cosec} 7x$ (v) $\log 8x$

3. (i) e^{3x} (ii) $(e^{4x})^2$ (iii) e^{3x+5} (iv) e^{5-2x} (v) e^{ax^2+bx+c}

4. (i) $\sin x^3$ (ii) $\cos x^4$ (iii) $\sin(3x+2)$ (iv) $\tan(5-2x)$

(v) $e^{\cos x}$ (vi) $\sin\{\phi(x)\}$ (vii) $\sec(e^x)$ (viii) $\sec(e^{2x})$.

5. (i) $\sin^6 x$ (ii) $\sqrt{\sin x}$ (iii) $\sqrt{\tan 2x}$ (iv) $\sec^7 x$ (v) $\sec^7 2x$

6. (i) $\log(\cos x)$ (ii) $\log(\tan x)$ (iii) $\log(ax^2+bx+c)$

(iv) $\log(2x+3)$ (v) $\log(3-2x)$ (vi) $\log(x^7+5)$

(vii) $\log(1+\sin x)$ (viii) $\log(\log x)$

7. (i) $e^{\cos x^3}$ (ii) $\{f(x)\}^n$ (iii) $\cos\{e^{\tan^2 2x}\}$ (iv) $\sin(\cos x^3)$

(v) $\cot\left\{e^{\sin^2 \frac{1}{x}}\right\}$ (vi) $(e)^{e^x}$ (vii) $\log\dfrac{e^x}{e^x+1}$.

8. (i) $\dfrac{e^{\sin x}}{\sin x^7}$ (ii) $(\cos\sqrt{x})\log\sin x$ (iii) $\dfrac{\tan x^3}{ax+b}$

(iv) $\log\left\{\dfrac{2x+3}{5x+6}\right\}$.

9. (i) $e^{(1+\log x)}+\tan(\log x)$ (ii) $\cot(e^{x})-5\log(ax^4+b)$

10. (i) $\log(\sec x-\tan x)$ (ii) $\log(\operatorname{cosec} x-\cot x)$
(iii) $3\log_e(x+\sqrt{x^2-a^2})$

11. Find $\dfrac{dy}{dx}$ if (Exs. 11—13)

(i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$ (ii) $y^2=4ax$ (iii) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

(iv) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ (v) $3x^4-x^2y+2y^3=0$ [C. U.]
(vi) $ax^2+2hxy+by^2+2gx+2fy+c=0$,
(vii) $x^4+x^2y^2+y^4=0$ [C. U.]

12. (i) $xy=\sin(x+y)$ (ii) $x+y=\tan(xy)$
(iii) $y=\sin(x+y)$ (iv) $\sin ax=\cos by$

13. (i) $y=e^{ax}\sin bx$ (ii) $y=e^{ax}\cos bx$

14. (a) Determine the inverse function $\phi(y)$ of the function $y=\dfrac{x}{x+4}$. Show that the value of y corresponding to $x=8$ is $\frac{2}{3}$.

and $\left[\dfrac{dy}{dx}\right]_{x=8}\times\left[\dfrac{dx}{dy}\right]_{y=\frac{2}{3}}=1.$

(b) For the function $y=\dfrac{x}{x+4}$, prove that

$x\dfrac{dy}{dx}=y(1-y)$.

15. Find $\dfrac{dy}{dx}$ when,

(i) $y=\sin^{-1}\sqrt{x}$ (ii) $\tan^{-1}\left(\dfrac{x}{a}\right)$ (iii) $\cos^{-1}\sqrt{(2x+3)}$
(iv) $\sec(\tan^{-1}x)$ [C. U.] (v) $\tan^{-1}(\sec x+\tan x)$
(vi) $\operatorname{cosec}^{-1}(\operatorname{cosec} x+\cot x)$

16. Find the derivatives of the following functions with respect to x.

(i) $\cos^{-1}\dfrac{1-x}{1+x}$ (ii) $\sec^{-1}\dfrac{x^2+1}{x^2-1}$ (iii) $\cos^{-1}(4x^3-3x)$

(iv) $\tan^{-1}\dfrac{1}{\sqrt{x^2-1}}$ [C. U.] (v) $\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

(vi) $\tan^{-1}\dfrac{2x}{1-x^2}$ (vii) $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ [C. U.]

(viii) $\tan^{-1}\dfrac{a+bx}{b-ax}$ (ix) $\cot^{-1}(\sqrt{1+x^2}-x)$

(x) $2\tan^{-1}\left(\sqrt{\dfrac{x-a}{b-x}}\right)$.

17. Find the derivatives with respect to x.

(i) $\sin^{-1}\{2ax\sqrt{(1-a^2x^2)}$ (ii) $\tan^{-1}\dfrac{\cos x-\sin x}{\cos x+\sin x}$

(iii) $\sin\left\{2\tan^{-1}\sqrt{\left(\dfrac{1-x}{1+x}\right)}\right\}$ (iv) $\tan^{-1}\left\{\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}\right\}$

(v) $\tan^{-1}\dfrac{x}{\sqrt{1+x^2}}$ (vi) $\tan^{-1}\left(e^{2x+1}\right)$

(vii) $\cos^{-1}\dfrac{x-x^{-1}}{x+x^{-1}}$ (viii) $\sin^{-1}\dfrac{1}{\sqrt{1+x^2}}$

18. Find $\dfrac{dy}{dx}$ if

(i) $y=\tan^{-1}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$ (ii) $y=\sin^{-1}\dfrac{x^2}{\sqrt{x^4+a^4}}$

(iii) $\tan y=\dfrac{\tan x+\sec x-1}{\tan x-\sec x+1}$

19. Find $\dfrac{dy}{dx}$ when

(i) $x=at^2$, $y=2at$ (ii) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$

(iii) $x=at$, $y=\dfrac{a}{t}$ (iv) $x=a\sec\theta$, $y=b\tan\theta$

(v) $x=\sin^2\theta$, $y=\tan\theta$

20. Find $\dfrac{dy}{dx}$ when,

(i) $x=\sec^{-1}\dfrac{1+t^2}{1-t^2}$, $y=\tan^{-1}\dfrac{a-t}{1+at}$

(ii) $x=\cos^{-1}\dfrac{1-t^2}{1+t^2}$, $y=\sin^{-1}\dfrac{2t}{1+t^2}$

(iii) $x=\sin^{-1}(2t\sqrt{1-t^2})$, $y=\cos^{-1}(1-2t^2)$

(iv) $x-\sin^{-1}(3t-4t^3)$, $y=\cos^{-1}(8t^4-8t+1)$

21. Find the derivatives with respect to x of the following functions.

(i) $x^{\log x}$ (ii) $x^{\cos^{-1}x}$ (iii) x^{1+x+x^2} (iv) e^{e^x}

(v) $(\sin x)^{\tan x}$ (vi) $(\sin x)^{\cos x}+(\cos x)^{\sin x}$

(vii) x^{e^x} (viii) a^{x^x} (ix) $(x^x)^x$ (x) $(\log x)^{\cos x}$

22. Find $\dfrac{dy}{dx}$ when

(i) $y=x^y$ (ii) $x^y y^x=1$ (iii) $(x+y)^{m+n}=x^m.y^n$

(iv) $(\cos x)^y=(\sin y)^x$ (v) $x^{\sin y}+y^{\sin x}=1.$

23. If $x=e^{\tan^{-1}x}$, $y=e^{-\cot^{-1}x}$, show that the value of $\dfrac{dy}{dx}$ is independent of x.

24. $$y=1+\frac{a_1}{x-a_1}+\frac{a_2x}{(x-a_1)(x-a_2)}+\frac{a_3x^2}{(x-a_1)(x-a_2)(x-a_3)}$$

Show that,

$$\frac{dy}{dx}=\frac{y}{x}\left\{\frac{a_1}{a_1-x}+\frac{a_2}{a^2-x}+\frac{a_3}{a_3-x}\right\}$$ [Joint Entrance-1987]

25. Find the derivatives of

(i) $\sec x$ with respect to x.

(ii) $\cos^{-1}\dfrac{1-x^2}{1+x}$ with respect to $\tan^{-1}\dfrac{2x}{1-x^2}$.

26. If $y=\tan^{-1}\dfrac{v}{\sqrt{1-v^2}}$, $x=\sec^{-1}\dfrac{1}{2v^2-1}$, show that

$$\frac{dy}{dx}=-\frac{1}{2}.$$

27. If $f(x)=\log\dfrac{\sqrt{(a+bx)}-\sqrt{(a-bx)}}{\sqrt{(a+bx)}+\sqrt{(a-bx)}}$

Show that $\dfrac{1}{f'(x)}$ will be equal to 0 when,

$x=0,\ \pm\left(\dfrac{a}{b}\right)$.

28. Show that,

(i) If $y=x\sqrt{(x^2+a^2)}+a^2\log(x+\sqrt{x^2+a^2})$, then

$$\frac{dy}{dx}=2\sqrt{(x^2+a^2)}$$

(ii) If $y=\log\frac{1+x}{1-x}+\frac{1}{2}\log\frac{1+x+x^2}{1-x+x^2}+\sqrt{3}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}$,

show that $\frac{dy}{dx}=\frac{6}{1-x^6}$.

29. If $\sin x\sin\left(\frac{\pi}{n}+x\right)\sin\left(\frac{2\pi}{n}+x\right)\ldots\ldots\sin\left(\frac{n-1}{n}\pi+x\right)$

$=\frac{\sin nx}{2n-1}$, show that

$$\cot x+\cot\left(\frac{\pi}{n}+x\right)+\cot\left(\frac{2\pi}{n}+x\right)+\cdots\cdots$$

$$+\cot\left(\frac{n-1}{n}\pi+x\right)=n\cot nx$$

[C. U.]

30. (i) Show that if $y=\sqrt{x+\sqrt{x+\sqrt{x+}}}\ldots\ldots$, then

$$\frac{dy}{dx}=\frac{1}{2y-1}$$

(ii) Show that it $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x}+\ldots\ldots}}$, then

$$\frac{dy}{dx}=\frac{1}{x(2y-1)}$$

(iii) Show that if $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x}+\ldots\ldots}}$, then

$$(2y-1)\frac{dy}{dx}=\cos x.$$

CHAPTER SIX

SECOND ORDER DERIVATIVE

§ 6·1 We have seen that the derivative of a function $y=f(x)$ of x, when it exists, is denoted by $\frac{dy}{dx}$ or $f'(x)$. We have also seen that $\frac{dy}{dx}=f'(x)$ is a function of x. This new function $\frac{dy}{dx}=f'(x)$ may be differentiated with respect to x once again. The derivative of $\frac{dy}{dx}=f'(x)$ when it exists is denoted as $\frac{d^2y}{dx^2}$ or $f''(x)$ and is called the second order derivative of $y=f(x)$. So $\frac{d^2y}{dx^2}=\frac{d}{dx}\left\{\frac{dy}{dx}\right\}$ or $f''(x)=\{f'(x)\}'$. Similarly derivatives of higher orders can be defined. $\frac{dy}{dx}=f'(x)$ it self is frequently called the derivative of the first order of $y=f(x)$. $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots\ldots, \frac{d^ny}{dx^n}$ or $f'''(x), f^{iv}(x), \cdots f^n(x)$ respectively denote derivatives of third, fourth,……, nth order of $y=f(x)$. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ etc. are also denoted by the symbols y_1, y_2, y_3 etc.

Examples

1. Let $f(x)=x^5$

$$\therefore\ f'(x)=\frac{d}{dx}(x^5)=5x^4$$

$$f^{ii}(x)=\frac{d}{dx}\{f'(x)\}=\frac{d}{dx}\{5x^4\}=20x^3.$$

$$f^{iii}(x)=\frac{d}{dx}\{f^{ii}(x)\}=\frac{d}{dx}\{20x^3\}=60x^2$$

$$f^{iv}(x)=\frac{d}{dx}\{f^{iii}(x)\}=\frac{d}{dx}\{60x^2\}=120x$$

$$f^{v}(x)=\frac{d}{dx}\{f^{iv}(x)\}=\frac{d}{dx}\{120x\}=120$$

$$f^{vi}(x)=\frac{d}{dx}\{f^{v}(x)\}=\frac{d}{dx}(120)=0.$$

clearly $f^{vii}(x)=f^{viii}(x),=\ldots\ldots=0.$

2. Let $y=e^{ax}$.

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(y)=\frac{d}{dx}(e^{ax})=ae^{ax}.$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(ae^{ax})=a\frac{d}{dx}(e^{ax})$$

$$=a.ae^{ax}=a^2e^{ax}.$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}(a^2e^{ax})=a^2\frac{d}{dx}(e^{ax})$$

$$=a^2.ae^{ax}=a^3e^{ax}.$$

3. Let $y=\sin x$ $\quad\therefore\ \frac{dy}{dx}=\cos x,$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(\cos x)=-\sin x.$$

Examples 6

Example 1. Find the second order derivatives of the following functions with respect to x.

(i) $\sin(2x+3)$ (ii) $x^4 5^x$ (iii) $\sin x\cos 3x$

(iv) $x^5\log x$ (v) $e^{2x}\cos 5x$

(i) Let $y=\sin(2x+3)$ $\quad\therefore\ \frac{dy}{dx}=2\cos(2x+3)$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\{2\cos(2x+3)\}=-4\sin(2x+3)$$

(ii) Let, $y=x^4 5^x$ $\quad\therefore\ \frac{dy}{dx}=\frac{d}{dx}(x^4 5^x)$

$$=4x^3 5^x+x^4 5^x\log_e 5=5^x(4x^3+x^4\log_e 5)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\{5^x(4x^3+x^4\log_e 5)\}$$

$$=5^x\log_e 5\{4x^3+x^4\log_e 5\}+5^x(12x^2+4x^8\log_e 5)$$

$$=5^x\{\log_e 5(8x^3+x^4\log_e 5)+12x^2\}.$$

(iii) Let $y=\sin x\cos 3x=\frac{1}{2}.2\sin x\cos 3x=\frac{1}{2}(\sin 4x-\sin 2x)$

$\therefore\ \frac{dy}{dx}=\frac{1}{2}(4\cos 4x-2\cos 2x)$

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(2\cos 4x-\cos 2x)=-8\sin 4x+2\sin 2x$

(iv) Let $y=x^5\log x$

$\therefore\ \frac{dy}{dx}=5x^4\log x+x^5.\frac{1}{x}=x^4(5\log x+1)$

$\therefore\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\{x^4(5\log x+1)\}=x^4.\frac{5}{x}+4x^3(5\log x+1)$

$=5x^3+4x^3.5\log x+4x^3=x^3(9+20\log x)$

(v) Let $y=e^{2x}\cos 5x$

$\therefore\ \frac{dy}{dx}=e^{2x}(-5\sin 5x)+2e^{2x}\cos 5x$

$=e^{2x}(2\cos 5x-5\sin 5x)$

$\therefore\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\{e^{2x}(2\cos 5x-5\sin 5x)\}$

$=e^{2x}(-10\sin 5x-25\cos 5x)+2e^{2x}(2\cos 5x-5\sin 5x)$

$=-e^{2x}(21\cos 5x+20\sin 5x)$

Ex. 2. Find $\frac{d^2y}{dx^2}$ when

(i) $(x+y)^{m+n}=x^my^n$ (ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (iii) $y=\tan(x+y)$

(i) $(x+y)^{m+n}=x^my^n$

or, $\log(x+y)^{m+n}=\log(x^my^n)$

or, $(m+n)\log(x+y)=m\log x+n\log y.$

Differentiating both sides with respect to x we get

$$(m+n)\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)=\frac{m}{x}+\frac{n}{y}\frac{dy}{dx}$$

or, $\frac{dy}{dx}\left(\frac{m+n}{x+y}-\frac{n}{y}\right)=\frac{m}{x}-\frac{m+n}{x+y}$

or, $\frac{dy}{dx}\left\{\frac{my+ny-nx-ny}{y(x+y)}\right\}=\frac{mx+my-mx-nx}{x(x+y)}$

or, $\frac{dy}{dx}\left\{\frac{my-nx}{y(x+y)}\right\}=\frac{my-nx}{x(x+y)}$ $\frac{dy}{dx}=\frac{y}{x}$.

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{y}{x}\right)=\frac{x\frac{dy}{dx}-y}{x^2}=\frac{x.\frac{y}{x}-y}{x^2}$$

$$=\frac{xy-xy}{x^3}=0.$$

(ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \therefore \quad \frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$

or, $\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0 \quad \therefore \quad \frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$.

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{b^2}{a^2}\frac{x}{y}\right)=-\frac{b^2}{a^2}\frac{y-x\frac{dy}{dx}}{y^2}$$

$$=-\frac{b^2}{a^2}\frac{y-x\left(-\frac{b^2}{a^2}\frac{x}{y}\right)}{y^2}=-\frac{b^2}{a^2}\cdot\frac{a^2y^2+b^2x^2}{a^2y^3}$$

$$=-\frac{b^2}{a^2}\cdot\frac{a^2b^2}{a^2y^3}=-\frac{b^4}{a^2y^3}.$$

(iii) $y=\tan(x+y)$

$$\therefore \quad \frac{dy}{dx}=\sec^2(x+y)\left\{1+\frac{dy}{dx}\right\}$$

or, $\frac{dy}{dx}\{1-\sec^2(x+y)\}=\sec^2(x+y)$

or, $\frac{dy}{dx}=\frac{\sec^2(x+y)}{1-\sec^2(x+y)}=\frac{1+\tan^2(x+y)}{1-1-\tan^2(x+y)}$

$$=\frac{1+y^2}{-y^2}=-\frac{1+y^2}{y^2}$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(-\frac{1+y^2}{y^2}\right)=\frac{d}{dy}\left(-\frac{1}{y^2}-1\right)\frac{dy}{dx}$$

$$=\frac{2}{y^3}\cdot\left(-\frac{1+y^2}{y^2}\right)=-\frac{2(1+y^2)}{y^5}.$$

Ex. 3. (i) If $y = 4 \cos 5x$, show that $\frac{d^2y}{dx^2} = -25y$

[H. S. 1978]

(ii) If $S = \sin 4t - \frac{6}{7} \sin 7t$ show that

$$\frac{d^2s}{dt^2} = 33 \sin 4t - 49s.$$

(i) $y = 4 \cos 5x \quad \therefore \quad \frac{dy}{dx} = -20 \sin 5x$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-20 \sin 5x) = -100 \cos 5x$$

$$= -25.4 \cos 5x = -25y.$$

(ii) $s = \sin 4t - \frac{6}{7} \sin 7t$

$$\therefore \quad \frac{ds}{dt} = 4 \cos 4t - \tfrac{6}{7}.7 \cos 7t$$

$$= 4 \cos 4t - 6 \cos 7t$$

$$\therefore \quad \frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}(4 \cos 4t - 6 \cos 7t)$$

$$= -16 \sin 4t + 42 \sin 7t$$

$$= 33 \sin 4t - 49 \sin 4t + 42 \sin 7t$$

$$= 33 \sin 4t - 49(\sin 4t - \tfrac{6}{7} \sin 4t)$$

$$= 33 \sin 4t - 49s.$$

Ex. 4. Find $\frac{d^2y}{dx^2}$ when

(i) $x = a \cos^3 t,\ y = b \sin^3 t$

(ii) $x = a \cos 2t,\ y = b \sin^2 t$

(i) $x = a \cos^3 t \quad \therefore \quad \frac{dx}{dt} = 3a \cos^2 t(-\sin t)$

$y = b \sin^3 t \quad \therefore \quad \frac{dy}{dt} = 3b \sin^2 t \cos t$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3b \sin^2 t \cos t}{3a \cos^2 t(-\sin t)} = -\frac{b}{a} \tan t$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{b}{a} \tan t\right)$$

$$= -\frac{b}{a}\,\frac{d}{dt}(\tan t)\,\frac{dt}{dx} = -\frac{b}{a}\sec^2 t.\,\frac{1}{\dfrac{dx}{dt}}$$

$$= -\frac{b}{a}.\frac{1}{\cot^2 t}\;\frac{1}{3a\cos^2 t(-\sin t)}$$

$$= \frac{b}{3a^2}\sec^4 t\ \operatorname{cosec} t$$

(ii) $x = a\cos 2t \quad \therefore \quad \dfrac{dx}{dt} = -2a\sin 2t$

$y = b\sin^2 t \quad \therefore \quad \dfrac{dy}{dt} = 2b\sin t\cos t = b\sin 2t$

$$\therefore \quad \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{b\sin 2t}{-2a\sin 2t} = -\frac{b}{2a}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = 0 \quad \left[\because \ -\frac{b}{2a} \text{ constant}\right]$$

Ex. 5. If $x = f(t)$ and $y = \phi(t)$, then prove that

$$\frac{d^2y}{dx^2} = \frac{f_1\phi_2 - \phi_1 f_2}{f_1{}^3}.$$

where suffixes denote differentiation with respect to t.

[Joint Entrance 1989]

$x = f(t) \quad \therefore \quad \dfrac{dx}{dt} = f_1 \ ; \ y = \phi(t) \quad \therefore \quad \dfrac{dy}{dt} = \phi_1$

$$\therefore \quad \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{\phi_1}{f_1}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\phi_1}{f_1}\right) = \frac{d}{dt}\left(\frac{\phi_1}{f_1}\right)\frac{dt}{dx}$$

$$= \frac{f_1\phi_2 - f_2\phi_1}{f_1{}^2}\left(\frac{1}{f_1}\right) \quad \left[\because \ \frac{dx}{dt} = f_1\right]$$

$$= \frac{f_1\phi_2 - f_2\phi_1}{f_1{}^3}$$

Ex. 6. If $x=\cos t$ and $y=\log t$, prove that at $t=\frac{\pi}{2}$,

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{ax}\right)^2=0$$

[Joint Entrance 1985]

$$x=\cos t \quad \therefore \quad \frac{dx}{dt}=-\sin t$$

$$y=\log.t \quad \therefore \quad \frac{dy}{dt}=\frac{1}{t}$$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{1}{t}}{-\sin t}=-\frac{1}{t\sin t}$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{1}{t\sin t}\right)=-\frac{d}{dt}\left(\frac{1}{t\sin t}\right)\frac{dt}{dx}$$

$$=\frac{1}{(t\sin t)^2}\,\frac{d}{dt}\,(t\sin t).\ \frac{1}{\frac{dx}{dt}}$$

$$=\frac{1}{(t\sin t)^2}\,(\sin t+t\cos t)\ \frac{1}{-\sin t}$$

$$=-\frac{\sin t+t\cos t}{t^2\sin^3 t}$$

Now, $\left[\frac{dy}{dx}\right]_{t=\frac{\pi}{2}}=-\frac{1}{\frac{\pi}{2}\sin\frac{\pi}{2}}=-\frac{2}{\pi}$;

$$\left[\frac{d^2y}{dx^2}\right]_{t=\frac{\pi}{2}}=-\frac{\sin\frac{\pi}{2}+\frac{\pi}{2}\cos\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)^2\left(\sin\frac{\pi}{2}\right)^3}=-\frac{4}{\pi^2}$$

$\therefore$ when $t=\frac{\pi}{2}$.

$$\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=-\frac{4}{\pi^2}+\frac{4}{\pi^2}=0.$$

Ex. 7. If $F(x)=f(x)\ \phi\ (x)$ and $f'(x)\,\phi'\ (x)=a$ (a, constant) show that $\frac{F''}{F}=\frac{f''}{f}+\frac{\phi''}{\phi}+\frac{2a}{f\phi}$

where $''\equiv\frac{d^2}{dx^2}$ and $'\equiv\frac{d}{dx}$, $F(x)\neq 0$ [H. S. 1984]

$$F(x)=f(x)\ \phi\ (x)$$

$$\therefore \quad \log F(x) = \log \{ f(x)\, \phi(x) \} = \log f(x) + \log \phi(x)$$

differentiating both sides with respect to x we get,

$$\frac{F'}{F} = \frac{f'}{f} + \frac{\phi'}{\theta} \qquad \text{or,} \quad F' = F\left\{\frac{f'}{f} + \frac{{}'}{\phi}\right\}$$

$$\text{or,} \quad F' = f\phi\left\{\frac{f'}{f} + \frac{\phi'}{\phi}\right\} = f'\,\phi + f\phi'$$

$$\therefore \quad \frac{d}{dx}\left(F'\right) = \frac{d}{dx}\,(f'\,\phi + f\phi')$$

$$\text{or,} \quad F'' = f''\phi + f'\phi' + f'\phi' + f\phi''$$

$$\text{or,} \quad \frac{F''}{F} = \frac{f''\phi + 2f'\phi' + f\phi''}{f\phi} \quad [\because \quad F = f\phi)]$$

$$= \frac{f''\phi + f\phi''}{f\phi} + 2\,\frac{f'\phi'}{f\phi}$$

$$\text{or,} \quad \frac{F''}{F} = \frac{f''}{f} + \frac{\phi''}{\phi} + \frac{2a}{f\phi}$$

Ex. 8. If $ax^2 + 2hxy + by^2 = 1$, show that

$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx(+ by^3}$ [Joint Entrance 1980]

$ax^2 + 2hxy + by^2 = 1.$

differentiating both sides with respect to x we get,

$$2ax + 2h\left(y + x\frac{dy}{dx}\right) + 2by\;\frac{dy}{dx} = 0$$

$$\text{or,} \quad 2\,\frac{dy}{dx}\,(hx + by) = -2\,(ax + hy)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{ax + hy}{hx + by}\right)$$

$$= -\frac{(hx + by)\;\frac{d}{dx}\,(ax + hy) - (ax + hy)\;\frac{d}{dx}\,(hx + by)}{(hx + by)^2}$$

$$= -\frac{(hx+by)\left(a+h\dfrac{dy}{dx}\right)-(ax+hy)\left(h+b\dfrac{dy}{dx}\right)}{(hx+by)^2}$$

$$= -\frac{(ahx+aby-ahx-h^2y)+\dfrac{dy}{dx}(h^2x+bhy-abx-bhy)}{(hx+by)^2}$$

$$= -\frac{(ab-h^2)\,y+\dfrac{dy}{dx}(h^2-ab)\,x}{(hx+by)^2}$$

$$= \frac{(h^2-ab)\left(y-x\dfrac{dy}{dx}\right)}{(hx+by)^2}$$

$$= \frac{(h^2-ab)\left\{y-x\left(-\dfrac{ax+hy}{hx+by}\right)\right\}}{(hx+by)^2}$$

$$= \frac{(h^2-ab)\left\{\dfrac{hxy+by^2+ax^2+hxy}{hx+by}\right\}}{(hx+by)^2}$$

$$= \frac{(h^2-ab)\,(ax^2+2hxy+by^2)}{(hx+by)^3}$$

$$= \frac{(h^2-ab).\,1}{(hx+by)^3} = \frac{h^2-ab}{(hx+by)^3}$$

Ex. 9. (i) $y=x\sin x$, prove that

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+(2+x^2)\,y=0$$ [H. S. 1980]

(ii) If $y=ae^{mx}+b\cos mx$, show that

$$\frac{d^2y}{dx^2}+m^2y=2am^2e^{mx}$$

(iii) If $y=a\cos\log x+b\sin\log x$, where a, b are constants, show that $x^2\ \dfrac{d^2y}{dx^2}+x\ \dfrac{dy}{dx}+y=0$

[H. S. 1982 '86 ; Joint Entrance 1987]

(iv) If $x=\sin t$, $y=\sin kt$, show that

$$(1-x^2)\frac{d^2y}{dx^2}-x\ \frac{dy}{dx}+k^2\ y=0\ (k=\text{constant})$$ [H. S. 1983]

(v) If $y=x^{n-1}\log x$, prove that
$x^2y_2+(3-2n)xy_1+(n-1)^2y=0$ [Joint Entrance 1987]

(vi) If $2x=y^{\frac{1}{5}}+y^{-\frac{1}{5}}$, prove that
$(x^2-1)y_2+xy_1=25y$ where $y_r=\frac{d^r}{dx^r}$

[Joint Entrance 1982]

(vii) If $y=\sin(2\sin^{-1}x)$, show that
$(1-x^2)\frac{d^2y}{dx^2}=x\frac{dy}{dx}-4y$ [Joint Entrance 1983]

(viii) If $y=(\cos^{-1}x)^2$ prove that
$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$ [Joint Entrance 1986]

Hence find the value of $\frac{d^2y}{dx^2}$ when $x=0$

(ix) If $\log_e y=\sin^{-1}x$, prove that
$(1-x^2)\frac{d^2y}{dx^2}=x\frac{dy}{dx}+y$ [H. S. 1989]

(i) $y=x\sin x$ $\therefore$ $\frac{dy}{dx}=\sin x+x\cos x.$

or, $\frac{dy}{dx}=\frac{y}{x}+x\cos x$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{y}{x}+x\cos x\right)$$

$$=-\frac{y}{x^2}+\frac{\frac{dy}{dx}}{x}+\cos x-x\sin x$$

$$\therefore x^2\frac{d^2y}{dx^2}=-y+x\frac{dy}{dx}+x^2\cos x-x^3\sin x$$

$$=-y+x\frac{dy}{dx}+x\frac{dy}{dx}-y-x^3\sin x$$

$$=2x\frac{dy}{dx}-2y-x^2.\,x\sin x$$

$$=2x\frac{dy}{dx}-2y-x^2y$$

or, $$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+(2+x^2)\,y=0.$$

(ii) $y=ae^{mx}+b\cos mx$

$$\therefore\ \frac{dy}{dx}=ame^{mx}-bm\sin mx.$$

$$\therefore\ \frac{d^2y}{dx^2}=am^2e^{mx}-bm^2\cos mx.$$

$$=-am^2e^{mx}-bm^2\cos mx+2am^2e^{mx}$$

$$=-m^2(ae^{mx}+b\cos mx)+2am^2e^{mx}$$

$$=-m^2y+2am^2e^{mx}$$

$$\therefore\ \frac{d^2y}{dx^2}+m^2y=2am^2e^{mx}.$$

(iii) $y=a\cos\log x+b\sin\log x$

$$\therefore\ \frac{dy}{dx}=-\frac{a\sin\log x}{x}+\frac{b\cos\log x}{x}$$

or, $$x\frac{dy}{dx}=-a\sin\log x+b\cos\log x.$$

differentiating both sides once again with respect to x we get,

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-a\frac{\cos(\log x)}{x}-b\frac{\sin(\log x)}{x}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-\{a\cos(\log x)+b\sin(\log x)\}=-y.$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$

(iv) $x=\sin t\quad\therefore\ \frac{dx}{dt}=\cos t$

$y=\sin kt\quad\therefore\ \frac{dy}{dt}=k\cos kt$

$$\therefore\ \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{k\cos kt}{\cos t}$$

$$\cos t\frac{dy}{dx}=k\cos kt$$

$\therefore\quad \cos^2 t\left(\frac{dy}{dx}\right)^2=k^2\cos^2 kt$ [Squaring both sides]

or, $(1-\sin^2 t)\left(\frac{dy}{dx}\right)^2=k^2(1-\sin^2 kt)$

or, $(1-x^2)\left(\frac{dy}{dx}\right)^2=k^2(1-y^2)$

differentiating both sides once again with respect to x we get

$$(1-x^2)\,2\frac{dy}{dx}\,\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=k^2\left(-2y\frac{dy}{dx}\right)$$

or, $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+k^2y=0$

$\left[\text{dividing both sides by } 2\frac{dy}{dx}\right]$

(v) $y=x^{n-1}\log x.$

or, $\frac{dy}{dx}=x^{n-1}\cdot\frac{1}{x}+(n-1)\,x^{n-2}\log x$

or, $x\frac{dy}{dx}=x^{n-1}+(n-1)\,x^{n-1}\log x$

$=x^{n-1}+(n-1)\,y \quad \cdots \quad \text{(i)}$

differentiating both sides with respect to x we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=(n-1)\,x^{n-2}+(n-1)\,\frac{dy}{dx}.$$

or, $x^2\frac{d^2y}{dx}+x\frac{dy}{dx}=(n-1)\,x^{n-1}+(n-1)\,x\,\frac{dy}{dx}$

or, $x^2\frac{d^2y}{dx^2}=(n-1)\,x^{n-1}+(n-2)\,x\,\frac{dy}{dx}$

$=(n-1)\left\{x\frac{dy}{dx}-(n-1)\,y\right\}+(n-2)\,x\frac{dy}{dx}$ [From (i)]

or, $x^2\frac{d^2y}{dx^2}=x\frac{dy}{dx}\,(n-1+n-2)-(n-1)^2\,y$

or, $x^2\frac{d^2y}{dx^2}+(3-2n)\,x\frac{dy}{dx}+(n-1)^2\,y=0$

or, $x^2y_2+(3-2n)\,xy_1+(n-1)^2\,y=0$

(vi) $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}} = y^{\frac{1}{5}} + \dfrac{1}{y^{\frac{1}{5}}}$

or, $2xy^{\frac{1}{5}} = y^{\frac{2}{5}} + 1$

or, $y^{\frac{2}{5}} - 2xy^{\frac{1}{5}} = -1$

or, $y^{\frac{2}{5}} - 2xy^{\frac{1}{5}} + x^2 = x^2 - 1$

or, $(y^{\frac{1}{5}} - x)^2 = x^2 - 1$

or, $y^{\frac{1}{5}} - x = \pm \sqrt{x^2 - 1}$

or, $y^{\frac{1}{5}} = x \pm \sqrt{x^2 - 1}$

or, $\log (y^{\frac{1}{5}}) = \log (x \pm \sqrt{x^2 - 1})$.

or, $\frac{1}{5} \log y = \log (x \pm \sqrt{x^2 - 1})$.

differentiating both sides with respect to x we get

$$\frac{1}{5}\,\frac{1}{y}\,\frac{dy}{dx} = \frac{1}{x \pm \sqrt{x^2 - 1}}\left\{1 \pm \frac{2x}{2\sqrt{x^2 - 1}}\right\}$$

$$= \frac{1}{x \pm \sqrt{x^2 - 1}}\left\{\frac{(x \pm \sqrt{x^2 - 1})}{\sqrt{x^2 - 1}}\right\}$$

$$= \pm \frac{1}{\sqrt{x^2 - 1}}$$

or, $y_1 = \pm \dfrac{5y}{\sqrt{x^2 - 1}}$

or, $y_1{}^2 = \dfrac{25y^2}{x^2 - 1}$ or, $(x^2 - 1)\, y_1{}^2 = 25y^2$

differentiating both sides with respect to x once again we get,

$$(x^2 - 1)\, 2y_1 y_2 + 2xy_1{}^2 = 50yy_1$$

or, $(x^2 - 1)\, y_2 + xy_1 = 25y$.

[Dividing both sides by $2y_1$]

(vii) $y = \sin (2 \sin^{-1} x)$

$\therefore \dfrac{dy}{dx} = \cos (2 \sin^{-1} x) \,.\, \dfrac{2}{\sqrt{1 - x^2}}$

or, $\sqrt{1-x^2}\,\dfrac{dy}{dx}=2\cos(2\sin^{-1}x)$.

differentiating both sides with respect to x once again we get,

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2}+\frac{dy}{dx}\,\frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}(-2x)$$
$$=-2\sin(2\sin^{-1}x)\frac{2}{\sqrt{1-x^2}}$$

or, $\sqrt{1-x^2}\,\dfrac{d^2y}{dx^2}=\dfrac{x\dfrac{dy}{dx}}{\sqrt{1-x^2}}-\dfrac{4y}{\sqrt{1-x^2}}$

or, $(1-x^2)\dfrac{d^2y}{dx^2}=x\dfrac{dy}{dx}-4y$.

(viii) $y=(\cos^{-1}x)^2$

$\therefore\ \dfrac{dy}{dx}=2\cos^{-1}x\left(-\dfrac{1}{\sqrt{1-x^2}}\right)$

or, $\sqrt{1-x^2}\,\dfrac{dy}{dx}=-2\cos^{-1}x$ ... (i)

$(1-x^2)\left(\dfrac{dy}{dx}\right)^2=4(\cos^{-1}x)^2=4y$

[Squaring both sides]
differentiating both sides again with respect to x we get

$(1-x^2)2\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}-2x\left(\dfrac{dy}{dx}\right)^2=4\dfrac{dy}{dx}$.

or, $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=2$ ...... (ii)

[dividing both sides by $2\dfrac{dy}{dx}$]

Now putting $x=0$ in
equation (ii) $\left[\dfrac{d^2y}{dx^2}\right]_{x=0}=2$.

(ix) $\log_e y=\sin^{-1}x$ or, $y=e^{\sin^{-1}x}$

$\therefore\ \dfrac{dy}{dx}=e^{\sin^{-1}x}\dfrac{1}{\sqrt{1-x^2}}$ or, $\sqrt{1-x^2}\,\dfrac{dy}{dx}=e^{\sin^{-1}x}=y$.

differentiating both sides again with respect to x we get,

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2}+\frac{1}{2}\frac{1}{\sqrt{1-x^2}}(-2x)\frac{dy}{dx}=\frac{dy}{dx}$$

or, $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=\sqrt{1-x^2}\,\dfrac{dy}{dx}$

or, $(1-x^2)\dfrac{d^2y}{dx^2}=x\dfrac{dy}{dx}+y$ $\left[\because\ \sqrt{1-x^2}\,\dfrac{dy}{dx}=y\right]$

Ex. 10. If $y=(\sin^{-1}x)^2$, then find the value of $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+4.$ [H. S. 1985]

$y=(\sin^{-1}x)^2$

$\therefore\ \dfrac{dy}{dx}=2\sin^{-1}x\dfrac{1}{\sqrt{1-x^2}}$

or, $\sqrt{1-x^2}\,\dfrac{dy}{dx}=2\sin^{-1}x$ ......(i)

squaring both sides we get

$$(1-x^2)\left(\frac{dy}{dx}\right)^2=4(\sin^{-1}x)^2$$

differentiating both sides with respect to x we get,

$$(1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=8\sin^{-1}x\ \frac{1}{\sqrt{1-x^2}}$$

or, $(1-x^2)2\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}-2x\left(\dfrac{dy}{dx}\right)^2=4\dfrac{dy}{dx}$ [From (i)]

or, $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=2$...... (ii)

$\left[\text{Dividing both sides by } 2\dfrac{dy}{dx}\right]$

$\therefore\ (1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+4=2+4=6.$

Ex. 11. (*a*) If $x=e^t\sin t$ and $y=e^t\cos t$, show that $\dfrac{d^2y}{dx^2}(x+y)^2=2\left(x\dfrac{dy}{dx}-y\right)$ [H. S. '87]

(*b*) If $x=\sin t$, $y=\sin 2t$, then show that

(i) $(1-x^2)\left(\dfrac{dy}{dx}\right)^2=4(1-y^2)$

(ii) $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+4y=0$

(a) $x=e^t\sin t \quad \therefore \quad \frac{dy}{dx}=e^t\sin t+e^t\cos t=e^t(\sin t+\cos t)$

$y=e^t\cos t \quad \therefore \quad \frac{dy}{dt}=-e^t\sin t+e^t\cos t=e^t(\cos t-\sin t)$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{e^t(\cos t-\sin t)}{e^t(\cos t+\sin t)}=\frac{\cos t-\sin t}{\cos t+\sin t}$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{\cos t-\sin t}{\cos t+\sin t}\right)$$

$$=\frac{d}{dt}\left(\frac{\cos t-\sin t}{\cos t+\sin t}\right)\frac{dt}{dx}$$

$$=\frac{(\cos t+\sin t)\ \frac{d}{dt}\ (\cos t-\sin t)-(\cos t-\sin t)\ \frac{d}{dt}\ (\cos t+\sin t)}{(\cos t+\sin t)^2}\cdot\frac{1}{\frac{dx}{dt}}$$

$$=\frac{(\cos t+\sin t)\ (-\sin t-\cos t)-(\cos t-\sin t)\ (\cos t-\sin t)}{(\cos t+\sin t)^2}\cdot\frac{1}{\frac{dx}{dt}}$$

$$=\frac{-(\cos t+\sin t)^2-(\cos t-\sin t)^2}{(\cos t+\sin t)^2}\cdot\frac{1}{e^t(\sin t+\cos t)}$$

$$=\frac{-2(\cos^2 t+\sin^2 t)}{e^{2t}(\cos t+\sin t)^3}=-\frac{2e^t}{e^{2t}(\cos t+\sin t)^3}$$

$$=-\frac{2e^t}{\{e^t(\cos t+\sin t)\}^2(\cos t+\sin t)}$$

$$=-\frac{2e^t}{(e^t\cos t+e^t\sin t)^2(\cos t+\sin t)}=-\frac{2e^t}{(y+x)^2(\cos t+\sin t)}$$

$$\therefore \quad \frac{d^2y}{dx^2}\ (x+y)^2=-\frac{2e^t}{\cos t+\sin t}$$

Again, $2\left(x\frac{dy}{dx}-y\right)=2\left(e^t\sin t\ \frac{\cos t-\sin t}{\cos t+\sin t}-e^t\cos t\right)$

$$=2e^{t}\left(\frac{\sin t\cos t-\sin^{2} t-\cos^{2} t-\sin t\cos t}{\cos t+\sin t}\right)$$

$$=-2e^{t}\left(\frac{\sin^{2} t+\cos t}{\cos t+\sin t}\right)=\frac{-2e^{t}}{\cos t+\sin t}$$

$$\therefore\ \frac{d^{2}y}{dx^{2}}(x+y)^{2}=2\left(x\frac{dy}{dx}-y\right)$$

(b) $x=\sin t \quad \therefore \quad \dfrac{dx}{dt}=\cos t.$

$y=\sin 2t \quad \therefore \quad \dfrac{dy}{dt}=2\cos 2t$

$$\therefore\ \frac{dy}{dx}=\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}=\frac{2\cos 2t}{\cos t}$$

$$\therefore\ \left(\frac{dy}{dx}\right)^{2}=\frac{4\cos^{2} 2t}{\cos^{2} t}=\frac{4(1-\sin^{2} 2t)}{1-\sin^{2} t}=\frac{4(1-y^{2})}{1-x^{2}}$$

$$\therefore\ (1-x^{2})\left(\frac{dy}{dx}\right)^{2}=4(1-y^{2}) \quad \dots \quad \text{(i)}$$

Again, differentiating both sides with respect to x we get

$$(1-x^{2})\,2\frac{dy}{dx}\frac{d^{2}y}{dx^{2}}-2x\left(\frac{dy}{dx}\right)^{2}=-8y\frac{dy}{dx}.$$

or, $(1-x^{2})\dfrac{d^{2}y}{dx^{2}}-x\dfrac{dy}{dx}=-4y \quad \left[\text{Dividing both sides by } 2\dfrac{dy}{dx}\right]$

or, $(1-x^{2})\dfrac{d^{2}y}{dx^{2}}-x\dfrac{dy}{dx}+4y=0.$

Ex. 12. If $x=Ae^{\frac{-kt}{2}}\cos(pt+\epsilon)$, show that

$$\frac{d^{2}x}{dt^{2}}+k\frac{dx}{dt}+n^{2}x=0$$

[Joint Entrance 1984]

A, k, p, ϵ are constants where $n^{2}=p^{2}+\frac{1}{4}k^{2}$

$$x=Ae^{\frac{-kt}{2}}\cos(pt+\epsilon)$$

$$\therefore \quad \frac{dx}{dt} = -A\frac{k}{2}e^{\frac{-kt}{2}}\cos(pt+\epsilon) + Ae^{\frac{-kt}{2}}\{-p\sin(pt+\epsilon)\}.$$

$$= -Ae^{\frac{-kt}{2}}\left\{\frac{k}{2}\cos(pt+\epsilon) + p\sin(pt+\epsilon)\right\}$$

$$\therefore \quad \frac{d^2x}{dt^2} = -A\left[-\frac{k}{2}e^{\frac{-kt}{2}}\left\{\frac{k}{2}\cos(pt+\epsilon) + p\sin(pt+\epsilon)\right\}\right.$$

$$\left. + e^{\frac{-kt}{2}}\left\{-\frac{kp}{2}\sin(pt+\epsilon) + p^2\cos(pt+\epsilon)\right\}\right.$$

$$= Ae^{\frac{-kt}{2}}\left[\frac{k^2}{4}\cos(pt+\epsilon) + \frac{pk}{2}\sin(pt+\epsilon) + \frac{kp}{2}\sin(pt+\epsilon) - p^2\cos(pt+\epsilon)\right]$$

$$= Ae^{\frac{-kt}{2}}\left[\left(\frac{k^2}{4} - p^2\right)\cos(pt+\epsilon) + pk\sin(pt+\epsilon)\right]$$

$$\therefore \quad \frac{d^2x}{dt^2} + k\frac{dx}{dt} + n^2x$$

$$= Ae^{\frac{-kt}{2}}\left[\left(\frac{k^2}{4} - p^2\right)\cos(pt+\epsilon) + pk\sin(pt+\epsilon)\right]$$

$$-Ae^{\frac{-kt}{2}}\left[\frac{k^2}{2}\cos(pt+\epsilon) + pk\sin(pt+\epsilon)\right] + n^2x$$

$$= Ae^{\frac{-kt}{2}}\left[\left(\frac{k^2}{4} - p^2 - \frac{k^2}{2}\right)\cos(pt+\epsilon)\right] + n^2x$$

$$= -Ae^{\frac{-kt}{2}}\left(\frac{k^2}{4} + p^2\right)\cos(pt+\epsilon) + n^2x$$

$$= -n^2x + n^2x \quad \left[\because \left(\frac{k^2}{4} + p^2\right) = n^2\right]$$

$$= 0$$

Ex. 13. If $x=\frac{1}{z}$ and $y=f(x)$, show that

$$\frac{d^2f}{dx^2}=2z^3\ \frac{dy}{dz}+z^4\ \frac{d^2y}{dz^2}$$

$$\therefore\quad x=\frac{1}{z}\qquad\therefore\quad z=\frac{1}{x}\qquad\therefore\quad\frac{dz}{dx}=-\frac{1}{x^2}=-z^2$$

Now, $\frac{df}{dx}=\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=-z^2\ \frac{dy}{dz}$

$$\therefore\quad\frac{d^2f}{dx^2}=\frac{d}{dx}\left(\frac{df}{dx}\right)=\frac{d}{dx}\left(-z^2\frac{dy}{dz}\right)=\frac{d}{dz}\left(-z^2\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$=\left(-z^2\frac{d^2y}{dz^2}-2z\frac{dy}{dz}\right)\left(-z^2\right)$$

$$=z^4\frac{d^2y}{dz^2}+2z^3\frac{dy}{dz}$$

Ex. 14. If $y=(x+\sqrt{1+x^2})^m$, prove that

$$(1+x^2)y_2+xy_1=m^2y.$$

Hence find the value of $[y_2]_{x=0}$

$$y=(x+\sqrt{1+x^2})^m$$

$$\therefore\quad\frac{dy}{dx}=m(x+\sqrt{1+x^2})^{m-1}\left\{1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right\}$$

$$=m(x+\sqrt{1+x^2})^{m-1}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$=\frac{m(x+\sqrt{1+x^2})^m}{\sqrt{1+x^2}}=\frac{my}{\sqrt{1+x^2}}$$

or, $\sqrt{1+x^2}\ \frac{dy}{dx}=my$ … …(i)

Now differentiating both sides of the relation $\sqrt{1+x^2}\ \frac{dy}{dx}=my$ with respect to x we get

$$\sqrt{1+x^2}\ \frac{d^2y}{dx^2}+\frac{1}{2}\ \frac{2x}{\sqrt{1+x^2}}\ \frac{dy}{dx}=m\ \frac{dy}{dx}$$

or, $(1+x^2)\ \frac{d^2y}{dx^2}+x\frac{dy}{dx^2}=m\ \sqrt{1+x^2}\ \ \frac{dy}{dx}=m\ \sqrt{1+x^2}\ \ \frac{my}{\sqrt{1+x^2}}$

or, $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=m^2y$ … …(ii)

Now when $x=0$,

$$y=1,\ \frac{dy}{dx}=\frac{m(y)_0}{1}=m(y)_0=1 \quad \text{[From (i)]}$$

$$\therefore\quad (1+0)\left[\frac{d^2y}{dx^2}\right]_{x=0}+0.\frac{dy}{dx}=m^2.1 \quad \text{[From (ii)]}$$

$$\text{or,}\quad \left[\frac{d^2y}{dx^2}\right]_{x=0}=m^2$$

Ex. 15. If $p^2=a^2\cos^2\theta+b^2\sin^2\theta$, show that

$$p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}$$

$$p^2=a^2\cos^2\theta+b^2\sin^2\theta$$

$$\therefore\quad \frac{d}{d\theta}(p^2)=\frac{d}{d\theta}(a^2\cos^2\theta+b^2\sin\theta)$$

$$\text{or,}\quad 2p\frac{dp}{d\theta}=-2a^2\cos\theta\sin\theta+2b^2\sin\theta\cos\theta$$

$$=(b^2-a^2)\sin 2\theta.$$

differentiating both sides once again with respect to x we get

$$2\left(\frac{dp}{d\theta}\right)^2+2p\frac{d^2p}{d\theta^2}=2(b^2-a^2)\cos 2\theta.$$

$$\therefore\quad p\frac{d^2p}{d\theta^2}=(b^2-a^2)\cos 2\theta-\left(\frac{dp}{d\theta}\right)^2$$

$$\therefore\quad p\frac{d^2p}{d\theta^2}+p^2=(b^2-a^2)\cos 2\theta-\left(\frac{dp}{d\theta}\right)^2+p^2$$

$$\text{or,}\quad p\frac{d^2p}{d\theta^2}=(b^2-a^2)\cos 2\theta+a^2\cos^2\theta+b^2\sin^2\theta-\left(\frac{dp}{d\theta}\right)^2$$

$$=b^2(\cos^2\theta-\sin^2\theta+\sin^2\theta)+a^2(\cos^2\theta-\cos^2\theta+\sin^2\theta)-\left(\frac{dp}{d\theta}\right)^2$$

$$=b^2\cos^2\theta+a^2\sin^2\theta-\frac{(b^2-a^2)^2\sin^2 2\theta}{4p^2}$$

$$=b^2\cos^2\theta+a^2\sin^2\theta-\frac{(b^2-a^2)^2\,4\sin^2\theta\cos^2\theta}{4p^2}$$

$$=\frac{p^2(a^2\sin^2\theta+b^2\cos^2\theta)-(b^2-a^2)^2\sin^2\theta\cos^2\theta}{p^2}$$

$$=\frac{(a^2\cos^2\theta+b^2\sin^2\theta)(a^2\sin^2\theta+b^2\cos^2\theta)-(b^2-a^2)^2\sin^2\theta\cos^2\theta}{p^2}$$

(ii) $\frac{d^2s}{dt^2}=\frac{d}{dt}\left(\frac{ds}{dt}\right)=\frac{d}{dt}\left(\frac{1}{1-t^2}\right)=\frac{2t}{(1-t^2)^2}$

$$\frac{d^2t}{ds^2}=\frac{d}{ds}\left(\frac{dt}{ds}\right)=\frac{d}{ds}\,(1-t^2)=\frac{d}{dt}\,(1-t^2)\,\frac{dt}{ds}$$

$$=-2t\,(1-t^2).$$

$$\therefore\quad -\left(\frac{ds}{dt}\right)^3\,\frac{d^2t}{ds^2}=-\frac{1}{(1-t^2)^3}\,\{-2t\,(1-t^2)\}=\frac{2t}{(1-t^2)^2}$$

$$\therefore\quad \frac{d^2s}{dt^2}=-\left(\frac{ds}{dt}\right)^3\,\frac{d^2t}{ds^2}.$$

Ex. 17. If $(a+bx)\,e^{\frac{y}{x}}=x$, show that

$$x^3\frac{d^2y}{dx^2}\left(x\frac{dy}{dx}-y\right)^2.$$ [I. I. T. 1983]

$$(a+bx)\,e^{\frac{y}{x}}=x \qquad \therefore\ \log\{(a+bx)\,e^{\frac{y}{x}}\}=x$$

or, $\log(a+bx)+\log e^{\frac{y}{x}}=\log x$

or, $\log(a+bx)+\frac{y}{x}=\log x$

or, $x\log(a+bx)+y=x\log x$ ... (1)

differentiating both sides with respect to x we get,

$$\frac{x}{a+bx}\,.\,b+\log(a+bx)+\frac{dy}{dx}=x\frac{1}{x}+1.\log x$$

or, $x\frac{dy}{dx}+\frac{bx^2}{a+b^2}+x\log(a+bx)=x+x\log x$

or, $x\frac{dy}{dx}+\frac{bx^2}{a+bx}=x+x\log x-x\log(a+bx)$

$=x+y.$ [from (1)]

or, $x\frac{dy}{dx}-y=x-\frac{bx^2}{a+bx}=\frac{ax}{a+bx}$ ... (2)

differentiating both sides with respect to x we get,

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{dy}{dx}=a\left\{\frac{(a+bx)-x.b}{(a+bx)^2}\right\}$$

or, $x\frac{d^2y}{dx^2}=\frac{a^2}{(a+bx)^2}$

or, $x^3\frac{d^2y}{dx^2}=\frac{a^2x^2}{(a+bx)^2}=\left(x\frac{dy}{dx}-y\right)^2$ [from (2)]

$$=\frac{a^2b^2(\sin^4\theta+\cos^4\theta)+(a^4+b^4)\sin^2\theta\cos^2\theta-(a^2-b^2)^2\sin^2\theta\cos^2\theta}{p^2}$$

$$=\frac{a^2b^2(\sin^4\theta+\cos^4\theta)+(a^4+b^4)\sin^2\theta\cos^2\theta-(a^4+b^4-2a^2b^2)\sin^2\theta\cos^2\theta}{p^2}$$

$$=\frac{a^2b^2(\sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta)}{p^2}$$

$$=\frac{a^2b^2(\sin^2\theta+\cos^2\theta)^2}{p^2}=\frac{a^2b^2}{p^2}.$$

Ex. 16. If $s=\frac{1}{2}\log\left(\frac{1+t}{1-t}\right)$, determine $\frac{ds}{dt}$ and $\frac{dt}{ds}$ and hence verify the following two relations.

(i) $\frac{ds}{dt}\cdot\frac{dt}{ds}=1$ (ii) $\frac{d^2s}{dt^2}=-\left(\frac{ds}{dt}\right)^3\cdot\frac{d^2t}{ds^2}$

$$s=\tfrac{1}{2}\log\left(\frac{1+t}{1-t}\right)=\tfrac{1}{2}\{\log(1+t)-\log(1-t)\}$$

$$\therefore\quad \frac{ds}{dt}=\tfrac{1}{2}\left\{\frac{1}{1+t}+\frac{1}{1-t}\right\}=\tfrac{1}{2}\frac{2}{1-t^2}=\frac{1}{1-t^2}.$$

Again $s=\frac{1}{2}\log\left(\frac{1+t}{1-t}\right)$ or, $\log\left(\frac{1+t}{1-t}\right)=2s.$

or, $\frac{1+t}{1-t}=\frac{e^{2s}}{1}$

or, $\frac{(1+t)-(1-t)}{(1+t)+(1-t)}=\frac{e^{2s}-1}{e^{2s}+1}$ or, $t=\frac{e^{2s}-1}{e^{2s}+1}$

$$\therefore\quad \frac{dt}{ds}=\frac{(e^{2s}+1)\frac{d}{ds}(e^{2s}-1)-(e^{2s}-1)\frac{d}{ds}(e^{2s}+1)}{(e^{2s}+1)^2}$$

$$=\frac{(e^{2s}+1).\,2e^{2s}-(e^{2s}-1)\,2e^{2s}}{(e^{2s}+1)^2}=\frac{4e^{2s}}{(e^{2s}+1)^2}$$

Now $1-t^2=1-\left(\frac{e^{2s}-1}{e^{2s}+1}\right)^2=\frac{(e^{2s}+1)^2-(e^{2s}-1)^2}{(e^{2s}+1)^2}=\frac{4e^{2s}}{(e^{2s}+1)^2}.$

$$\therefore\quad \frac{dt}{ds}=1-t^2$$

$$\therefore\quad \frac{ds}{dt}\cdot\frac{dt}{ds}=\frac{1}{1-t^2}(1-t^2)=1.$$

Ex. 18. If $f(x)=x^3 \sin\frac{1}{x}$ when $x\neq 0$,

$=0$ when $x=0$ show that

$f'(0)=0$ but $f''(0)$ does not exist

From definition,

$$'(0)=\underset{h\to 0}{Lt}\ \frac{f(0+h)-f(0)}{h}=\underset{h\to 0}{Lt}\ \frac{f(h)\quad f(0)}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{h^3 \sin\frac{1}{h}-0}{h}=\underset{h\to 0}{Lt}\ \left(h^2 \sin\frac{1}{h}\right)=0$$

Again, $f''(0)=\underset{h\to 0}{Lt}\ \dfrac{f'(h)-f'(0)}{h}$

$$=\underset{h\to 0}{Lt}\ \frac{3h^2 \sin\left(\frac{1}{h}\right)+h^3 \cos\left(\frac{1}{h}\right)\left(-\frac{1}{h^2}\right)-0}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{3h^2 \sin\left(\frac{1}{h}\right)-h \cos\left(\frac{1}{h}\right)}{h}$$

$$=\underset{h\to 0}{Lt}\ \left\{3h \sin\left(\frac{1}{h}\right)-\cos\left(\frac{1}{h}\right)\right\}$$

clearly this limit does not exist as $\underset{h\to 0}{Lt}\ \cos\left(\frac{1}{h}\right)$ does not exist,

$\therefore$ $f''(0)$ does not exist

Exercise 6

1. Determine the second order derivatives of the following functions with respect to x

(i) e^{2x+3} (ii) e^{mx} (iii) $\cos(3-7x)$ (iv) $x^4 e^{6x}$

(v) $\sin 4x \cos 3x$ (vi) 10^x (vii) $\cos(\log x)$

(viii) $\log(\sin x)$ (ix) $\sin^2 x$ (x) $x^2 \log x$.

2. Find $\dfrac{d^2y}{dx^2}$ when

(i) $y^2=4ax$ (ii) $y=\sin(x+y)$ (iii) $x^3+y^3=3axy$

(iv) $\dfrac{1}{x^2-3x+2}$ (v) $\tan y=\dfrac{\sqrt{1+x^2}-1}{x}$

(vi) $\sin x+\cos y=1$ (vii) $x^2+y^2=4$

3. Find $\frac{d^2y}{dx^2}$ when

(i) $x=at,\ y=\frac{a}{t}$ (ii) $x=at^2,\ y=2at$

(iii) $x=a(\theta+\sin\theta),\ y=a(1-\cos\theta)$

(iv) $x=a(\cos\theta+\theta\sin\theta),\quad y=a(\sin\theta-\cos\theta)$

(v) $x=a\left(t+\frac{1}{t}\right),\ y=a\left(t-\frac{1}{t}\right)$

(vi) $x=\frac{3at}{1+t^2},\ y=\frac{3at^2}{1+t^2}$

4. If $x=2\cos\theta-\cos 2\theta,\ y=2\sin\theta-\sin 2\theta$ show that at

$$x=\frac{\pi}{2},\ \frac{d^2y}{dx^2}=-\frac{3}{2}.$$

5. $y=f(a+bx)+\phi(a-bx)$, Show that

$$\frac{d^2y}{dx^2}=b^2\{f''(a+bx)+\phi''(a-bx)\}.$$

6. $y=5\sin 4x$ Show that $\frac{d^2y}{dx^2}+16y=0$

7. If $x=a\sin wt$, show that $\frac{d^2x}{dt^2}=-w^2x$

8. If $2y=x\left(1+\frac{dy}{dx}\right)$ prove that y_2 is a constant

9. If $y=\frac{\log x}{x}$, prove that $\frac{d^2y}{dx^2}=\frac{2\log x-3}{x^3}$

10. If $ax^2+2hxy+by^2+2gx+2fy+c=0$, prove that

$$\frac{d^2y}{dx^2}=\frac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^3}$$

11. Prove that

(i) If $y=c_1e^{mx}+c_2e^{-mx}$, then $\frac{d^2y}{dx^2}=m^2y$

(ii) If $y=\tan^{-1}x$, then $(1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0$

(iii) If $y=(x-2)^{-1}$, then $\frac{d^2y}{dx^2}=\frac{2}{(x-2)^3}$

(iv) If $y=x^2\cos x$ then $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(x^2+6)y=0$

(v) If $y = \sec x$ then $\dfrac{d^2y}{dx^2} = y(2y^2 - 1)$

(vi) If $y = a \sin mx + b \cos mx$ then, $\dfrac{d^2y}{dx^2} + m^2y = 0$.

(vii) If $y = \sin(\log x)$ then $x^2y_2 + xy_1 + y = 0$

(viii) If $y = \cos(\log x)$, then $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

(ix) If $y = \operatorname{cosec} x + \cot x$, then $\dfrac{d^2y}{dx^2} + y\ \dfrac{dy}{dx} = 0$

(x) If $y = \sin(\sin x)$ then, $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y \cos^2 x = 0$

(xi) If $y = x + \tan x$ then $\cos^2 x\ \dfrac{d^2y}{dx^2} + 2x = 2y$

(xii) If $y = \cos(m \sin^{-1} x)$ then $(1 - x^2)\ \dfrac{d^2y}{dx^2} - x\ \dfrac{dy}{dx} + m^2y = 0$.

(xiii) If $y = e^{ax} \sin bx$ then $y_2 - 2ay_1 + (a^2 + b^2)y = 0$

(xiv) If $y = \dfrac{1 + \sin x}{\cos x}$ then $\dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1 - \sin x)^2}$

(xv) If $x = \sin(\log y)$ then $(1 - x^2)\ \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - y = 0$

(xvi) If $x = \cos(\log y)$ then $(1 - x^2)\ \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - y = 0$

12. If $x = \sin t$, $y = t^2$ prove that
$(1 - x^2)y_2 - xy_1 - 2 = 0$

13. If $y = (\tan^{-1} x)^2$, prove that

$$(1 + x^2)^2\ \frac{d^2y}{dx^2} + 2x(1 + x^2)\ \frac{dy}{dx} - 2 = 0$$

14. If $y = \log\left(\dfrac{x}{a + bx}\right)^x$, prove that $x^3\ \dfrac{d^2y}{dx^2} = \left(y - x\ \dfrac{dy}{dx}\right)^2$

15. If $ky = \sin(x + y)$, (k is a constant) prove that

$$\frac{d^2y}{dx^2} = -y\left(1 + \frac{dy}{dx}\right)^3$$

16. If $y = e^x \log x$ prove that $xy_2 - (2x - 1)y_1 + (x - 1)y = 0$.

17. If $y = Ae^{-kt} \cos(pt + \epsilon)$ prove that

$y_2 + 2ky_1 + x^2y = 0$ where $n^2 = p^2 + k^2$ and the suffixes denote differentiation with respect to t.

18. If $y=\sin(m\sin^{-1}x)$ prove that $(1-x^2)y_2-xy_1+m^2y=0$

19. If $y=ax^{n-1}+bx^{-n}$, prove that $x^2y_2=n(n+1)y$

20. If $2x=y^{\frac{1}{n}}+y^{-\frac{1}{n}}$, prove that $(x^2-1)y_2+xy_1=n^2y$

21. If $y=e^{a\sin^{-1}x}$ prove that $(1-x^2)y_2-xy_1-a^2y=0.$

22. If $x=\cos\propto\left(\dfrac{\log y}{m}\right)$ prove that $(x^2-1)y_2+xy_1-m^2y=0.$

23. If $\dfrac{dx}{dy}=u$, $\dfrac{d^2x}{dy^2}=v$ prove that $\dfrac{d^2y}{dx^2}=-\dfrac{v}{u^3}$.

Hence prove that if $y=x\sin(a+y)$, then

$$\frac{d^2y}{dx^2}=\frac{2\sin^3(a+y)\cos(a+y)}{\sin^2a}$$

24. If $\cos x=y\cos(a+x)$ then show that

$$\frac{d^2y}{dx^2}=2\sin a\sec^2(a+x)\tan(a+x)$$ [H. S. 1986]

25. If $\sqrt{x}+\sqrt{y}=\sqrt{a}$ then show that

$$\left[\frac{d^2y}{dx^2}\right]_{x=a}=\frac{1}{2a}$$

26. $f(x)$ is a twice differentiable function and

$f''(x)=-f(x)$; $f'(x)=g(x)$, $h(x)=[f(x)]^2+[g(x)]^2$

and $h(5)=11$. Find the value of $h(10)$.

27. If $(x-a)^2+(y-b)^2=r^2$, prove that

$$\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$$ is a constant

28. $f(x)=x^3\cos\dfrac{1}{x}$ when $x\neq0$; $=0$ when $x=0$.

Prove that $f''(0)$ does not exist.

ANSWERS

Exercise 1

1. (i) correct ; (ii) correct ; (iii) In-correct ; (iv) correct ; (v) correct ; (vi) In-correct ; (vii) In-correct. 2. (i) irrational (ii) recurring, (iii) irrational, (iv) infinite. 4. 1·5, 1·6, 1·65, 1·7. 5. Not possible. 7. $\frac{2355}{999}$. 11. 1·70. 12. Not true 13. (i), (ii), (v) rational ; (vi) rational if $x \neq 2$. 14. (a) (i) $1 \leq x \leq 5$ (ii) $-1 \leq x \leq 7$ (iii) $a-1 \leq x \leq a+1$ (iv) $-2<x<2$

15. (i) $|x-4| \leq 1$. (ii) $|x-1| \leq 6$ (iii) $|x+6| \leq 5$ (iv) $\left|x-\frac{a}{2}\right| \leq \frac{2\delta-a}{2}$.

17. (i) $x=0$; (ii) $x=0$, (iii) $x=0$ (iv) $x \leq 1$ (v) $1 \leq x < 2$ (vi) $x=1$ or -2 (vii) $(2n+1)\frac{\pi}{2}$ [$n=0, \pm 1, \pm 2, \cdots$]

18. (B) $x \leq -2$ or $x \geq 4$ and (C) $x \leq 0$ or $x \geq 4$. [In the question no. 18 read all less than signs as less than or equal to signs.]

Exercise 2

1. (i) $-\infty < x \leq \infty (x \neq 1)$ (ii) $-\infty \leq x < \infty (x \neq 3)$ (iii) $-2 \leq x \leq 2$ (iv) $-\propto < x \leq 5$. (v) All real numbers other than those given by $-1 \leq x \leq 3$. (vi) $-\infty \leq x \leq \infty (x \neq 1, 2)$ (vii) $-\infty < x < \infty$. 10. $2x^2-3x-24$. 11. 1. 12. $\phi(y)=\frac{1+y}{2y-1}$.

13. $f(-5)=\frac{8}{11}$; $f(6)$ undefined. 16. Not indentical. 18. 0. 19. 4·0401. 25. (i) correct ; (ii) In correct ; (iii) In-correct. (iv) In-correct ; (v) In-correct 26. (b) None of the two.

31. Function (i) even (ii) monotonic. 32. (i) $y \pm \frac{\sqrt{5}}{x}$ (ii) $y=-\frac{x+1}{x+1}$ or -1 (iii) $y=\pm\sqrt{a^2-x^2}$.

33. (i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ii) $xy^2=(x-5)^2$ (iii) $3x+7y+1=2xy$

34. (a) $x=\pm\sqrt{y-3}$; No ; Domain of definition of the given function is $0 \leq x \leq \infty$. Domain of definition of the inverse function $3 \leq x < \infty$. (b) $x=\pm 2 \tan^{-1}\sqrt{y}$.

35. Bounded ; lower bound : $-\sqrt{2}$; upper bound : $+\sqrt{2}$.

36. lower bound $\frac{1}{3}$; upper bound 3.

37. (i) discontinuous at $x=0$. (ii) discontinuous at $x=1$.

(iii) continuous. 44. y is a function of x. 45. $A=\frac{\sqrt{3}}{4}x^2$.

46. $y=5$ when $0\le x\le 0{\cdot}5$, $y=8$ when $0{\cdot}5\le x\le 0{\cdot}7$; $y=11$ when $0{\cdot}7<x\le 0{\cdot}9$. $y=14$ when $0{\cdot}9<x\le 1{\cdot}1$ $y=69$ when $4{\cdot}9<x\le 5$ [y in rupees and x in kg.] 47. $y=(a-2x)^2.x$.

48. $x=75t$ when $0\le t<7$
$=525$ when $7\le t\le 12$
$=525-(t-12)\,70$ when $12\le t\le 19{\cdot}5$.

[In the question read 525 instead of 520].

49. (D). 50. $\frac{2n+n^2}{(n+1)^2}$.

Exercise 3

1. (i) 41 ; (ii) $\frac{3}{4}$ (iii) -5 ; (iv) $\frac{b}{d}$

2. (i) 1 (ii) 0 (iii) 8 (iv) 3 (v) -2. 3. (i) $\frac{1}{2}$ (ii) 0 (iii) $\frac{5}{3}$

4. (i) $\frac{c}{r}$ (ii) $\frac{a}{p}$ 5. (i) $\frac{1}{2\sqrt{x}}$ (ii) $-\frac{1}{2x\sqrt{x}}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{2a}$ (v) $\frac{5}{2}$.

6. (i) 1 (ii) 1 (iii) 1. 7. (i) 3, (ii) $\frac{2\sqrt{2}}{3}$.

8. (i) $\frac{2}{3}$ (ii) 3 (iii) $\frac{1}{4}$. 9. (i) 75 (ii) $\frac{3}{20}$ (iii) $\frac{m}{n}$ (iv) $\frac{7}{6}a^{\frac{9}{21}}$ (v) 0 (vi) $\frac{3}{20}$ (vii) 5 (viii) -4 (ix) $6x^5$ (x) $\frac{1}{9}$ (xi) $2ax+b$

10. (i) 2 (ii) 4 (iii) $\frac{1}{4}$ (iv) $2a$ (v) $\frac{1}{6}$ 11. (i) $\frac{\pi}{180}$ (ii) 1 (iii) $\frac{\pi}{180}$ (iv) 1. 12. (i) $\frac{2}{3}$ (ii) 0 (iii) $\frac{1}{3}$ (iv) $\frac{1}{3}$ (v) $\frac{1}{2}$ (vi) $\frac{2}{3}$.

13. (i) 2 (ii) 0 (iii) $\frac{1}{2}$ (iv) 0 (v) $\frac{4}{3}$ (vi) 2 (vii) $\frac{b^2-a^2}{2}$ (viii) 10 (ix) 1 (x) $\frac{a}{b}$ (xi) 1 (xi) 1 (xii) -1 (xiii) 1. 14. (i) 1 (ii) -1

(iii) $\frac{1}{2}$ (iv) 0 (v) $\sqrt{2}$ (vi) $-(\pi^2+1)$ (vii) $\frac{3}{2}$ [In the question read $(1-\sin^2 x)$.] (viii) $-\frac{3}{2}$ (ix) $\frac{1}{2}$ (x) 3. [Read (ix)—(xi) as (viii)—(x) in the question] **15.** (i) 5 (ii) $\frac{1}{2}$ (iii) $\frac{m}{n}$ (iv) 1 (v) 1.

16. (i) a (ii) $\frac{2}{3}$ (iii) $\frac{3}{2}$ (iv) e^2 (v) e^a (vi) 1 (vii) $\log \frac{2}{3}$.

17. (i) 1 (ii) $\frac{a}{b}$ (iii) 1. **18.** (i) 0 (ii) 0.

19. The limit does not exist. **20.** The limit does not exist.

21. 1. **22.** The limit does not exist.

2

23. $\underset{x\to 0}{\text{Lt}} f(x)=0$; $\underset{x\to 1}{\text{Lt}} f(x)=1$. **29.** 0.

31. 0 (when x is not an integer) ; 1 (when x is an integer).

32. -1 when $|x|<1$; 0 when $x=1$. 1 when $|x|>1$. does not exist when $x=-1$. [Here $n\to\infty$]. **34.** (i) π (ii) $\frac{1}{\sqrt{2}}$

35. (i) 2 (ii) $\frac{1}{2}$ **36.** (i) log 4, log 3 (ii) $2\sqrt{2}\cos a$.

Exercise 4

1. Discontinuous **2.** Discontinuous **3.** Continuous **7.** 8

8. (i) $x=1, x=2$ (ii) $x=0, x=(2n+1)\frac{\pi}{2}$ [n is an integer.]

9. (i) continuous (ii) discontinuous (iii) discontinuous. (iv) continuous **11.** continuous **12.** continuous **15.** 0

19. $a=-1, b=1$ **22.** discontinuous **23.** continuous.

Exercise 5A

1. (i) $4x^3$ (ii) -3 (iii) $2{\cdot}5x^{1{\cdot}5}$ (iv) $2ax+b$ (v) $-\frac{2}{(2x+3)^2}$ (vi) $-\frac{2}{x^3}$ **2.** (i) $-\frac{1}{16}$ (ii) $-\frac{3}{16}$ (iii) $\frac{1}{12}$ (iv) 13

3. (ii) $5\cos 5x$ (ii) $-\sin 2x$ (iii) $2\sin 4x$ (iv) $\frac{1}{2}\sec^2\frac{x}{2}$ (v) $\cos\frac{x}{a}$ (vi) $\frac{\sec^2 x}{2\sqrt{\tan x}}$.

4. (i) $2e^{2x}$ (ii) me^{mx} (iii) $e^{\sin x}\cos x$

5. (i) $\log_{10} e.\frac{1}{x}$ (ii) $\frac{1}{a}\cot\frac{x}{a}$ (iii) $\cot x$

6. (i) $-2x\sin x^2$ (ii) $\frac{\cos(\log x)}{x}$ (iii) 0 (iv) $3x^2\cos x^3$

(v) $-3\operatorname{cosec} 3x\cot 3x$ 7. (i) $x^{\cos x}\left(\frac{\cos x}{x}-\sin x\log x\right)$

(ii) $-\frac{1}{\sqrt{1-x^2}}$ (iii) $\cos x - x\sin x$ (iv) $3x^2-\frac{3}{x^4}$ 8. (i) 0

(ii) 1 (iii) $7x^6$ (iv) $\frac{-5}{x^6}$ (v) $30x^5$ (vi) x^5 (vii) x^{n-1}

(viii) $\frac{1}{3.\sqrt[3]{x^2}}$ (ix) $-\frac{1}{3.\sqrt[3]{x^4}}$ (x) amx^{m-1} (xi) $\frac{13}{6}\sqrt[6]{x^7}$

(xii) $-\frac{13}{6}\frac{1}{\sqrt[6]{x^{19}}}$ 9. (i) $-\frac{5}{x}$ (ii) 2 (iii) $2.$

10. (i) $2(ax+b)$ (ii) $n\alpha^n x^{n-1}$ (iii) $8x-5\sin x.$

(iv) $1+2e^x+\frac{3}{x}+4\cos x+5\sin x+6\sec^2 x+7\operatorname{cosec}^2 x+8\sec x\tan x$
$+9\operatorname{cosec} x\cot x.$ (v) $3a^2b+6ab^2x+3b^3x^2$ (vi) $-2ab+2b^2x.$

(vii) $-\frac{1}{x^2}+3+2x$ (viii) $-\frac{2}{x^3}+\frac{3}{x^2}-1$ (ix) $1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$

(x) $\frac{1}{2\sqrt{x}}+3\sqrt{x}+4\sqrt[3]{x}$ 11. (i) $\log_2 e.\frac{1}{x}$ (ii) $\frac{n}{x}$ (iii) $\frac{55}{x}$

(iv) $\log_a e\frac{6}{x}$ 12. (i) $e^x(x^5+5x^4)$ (ii) $e^x(\sin x+\cos x)$

(iii) $3^x(\log_e 3\sin x+\cos x)$ (iv) $5^x(\log_e 5\, x^5+5x^4)$

(v) $2x\cos x-(x^2+1)\sin x$ (vi) $2x\log x+x$ (vii) $3(x+2x\log x)$

(viii) $e^x(x+3)$ (ix) $\frac{x^2+2}{x}+2x\log x$; (x) $e^x\left(\log x+\frac{1}{x}\right)$

13. (i) $2e^x\cos x$ (ii) $e^x(x^3+3x^2)-\operatorname{cosec}^2 x$

(iii) $e^x\{x\cot x(1+2\log x+x\log)-x^2\operatorname{cosec}^2 x\log x\}$

(iv) $x\left(\frac{\cot x}{x}-\log x\operatorname{cosec}^2 x\right)+\log x\cot x$ (v) $3x^2-12x+11$

14. $2 \cos 2x$ **16.** (i) $\dfrac{nx^{n-1} \log x - x^{n-1}}{(\log x)^2}$ (ii) $\dfrac{x \cos x - \sin x}{x^2}$

(iii) $\dfrac{\sin x - x \cos x}{\sin^2 x}$ (iv) $\dfrac{-\sin x \log x - \dfrac{\cos x}{x}}{(\log x)^2}$

(v) $\dfrac{(\sin x + \cos x)6x^2 - (3x^2+2)(\cos x - \sin x)}{(\sin x + \cos x)^2}$

(vi) $\dfrac{a^{\frac{1}{2}} x^{-\frac{1}{2}}}{(a + x - 2\sqrt{ax})}$ (viii) $\dfrac{-\operatorname{cosec}^2 x(x+e^x) + (1+e^x)\cot x}{(x+e^x)^2}$

(vii) $\dfrac{2^x(\cos x - \sin x \log_e 2)}{2^{2x}}$ **18.** (i) $\dfrac{e^x(x-1)}{x^2} + 2x + 2$

(ii) $\dfrac{\sin x - x \cos x}{\sin^2 x} + 3e^x + x \sec^2 x + \tan x$

(iii) $10x + 7(\cos x - x \sin x) + \dfrac{8(x^2 \cos x - 2x \sin x)}{x^4}$

19. (i) $\dfrac{2 \sin x}{(1+\cos x)^2}$ (ii) $\dfrac{1}{\sqrt{x}(1-\sqrt{x})^2}$ (iii) $-2 \sin x$

20. (i) $2x+1$, (ii) $4x^3 + 3x^2 + 2x + 1$

21. (i) $y\left[\dfrac{2x}{x^2-4} - \dfrac{2}{x} - \dfrac{1}{x+4}\right]$ (ii) $\dfrac{x^2}{(x \cos x - \sin x)^2}$

(iii) $\dfrac{x^2}{(x \sin x - \cos x)^2}$ (iv) $\sec^2 x$ **22.** (i) 0 (ii) 0

30. Not differentiable at $x=1$ but differentiable at $x=2$.

31. (i) continuous but not differentiable

(ii) continuous and differentiable.

Exercise 5B

1. (i) $5(x+11)^4$ (ii) $22(2x-7)^{10}$ (iii) $-45(3-5x)^8$

(iv) $\dfrac{2ax+b}{2\sqrt{ax^2+bx+c}}$ (v) $-\dfrac{2ax+b}{2(ax^2+bx+c)^{\frac{3}{2}}}$

2. (i) $4 \cos 4x$ (ii) $-5 \sin 5x$ (iii) $6 \sec^2 6x$

(iv) $-7 \operatorname{cosec} 7x \cot 7x$ (v) $\dfrac{1}{x}$

3. (i) $3e^{3x}$ (ii) $8e^{8x}$ (iii) $3e^{3x+5}$ (iv) $-2.e^{5-2x}$

(v) $e^{ax^2+bx+c}(2ax+b)$

4. (i) $3x^2 \cos x^3$ (ii) $-4x^3 \sin x^4$ (iii) $3\cos(3x+2)$

(iv) $-2\sec^2(5-2x)$ (v) $-e^{\cos x} \sin x$ (vi) $\cos\{\phi(x)\}\phi'(x)$

(vii) $e^x \sec(e^x)\tan(e^x)$ (viii) $2e^{2x}\sec(e^{2x})\tan(e^{2x})$

5. (i) $6\sin^5 x\cos x$ (ii) $\dfrac{\cos x}{2\sqrt{\sin x}}$ (iii) $\dfrac{\sec^2 2x}{\sqrt{\tan 2x}}$

(iv) $7\sec^7 x\tan x$ (v) $14\sec^7 2x\tan 2x$

6. (i) $-\tan x$ (ii) $\sec x \operatorname{cosec} x$ (iii) $\dfrac{2ax+b}{ax^2+bx+c}$ (iv) $\dfrac{2}{2x+3}$

(v) $-\dfrac{2}{3-2x}$ (vi) $\dfrac{7x^6}{x^7+5}$ (vii) $\dfrac{\cos x}{1+\sin x}$ (viii) $\dfrac{1}{x\log x}$

7. (i) $-3x^2 e^{\cos x^3}.\sin(x^3)$ (ii) $n\{f(x)\}^{n-1} f'(x)$

(iii) $-4\sin(e^{\tan^2 2x})\, e^{\tan^2 2x}\tan 2x.\sec^2 2x$

(iv) $-3x^2(\sin x^3)\cos\{\cos(x^3)\}$

(v) $\dfrac{1}{x^2}\operatorname{cosec}^2\left\{e^{\sin^2 \frac{1}{x}}\right\}\sin\dfrac{2}{x}$ (vi) $e^{e^x}.e^x$ (vii) $\dfrac{1}{e^x+1}$

8. (i) $\dfrac{\sin x^7\, e^{\sin x}\cos x - e^{\sin x}\, 7x^6\cos(x^7)}{(\sin x^7)^2}$

(ii) $(\cos\sqrt{x})\cot x - \dfrac{\sin(\sqrt{x})\log\sin x}{2\sqrt{x}}$

ii) $\dfrac{3x^2(ax+b)\sec^2(x^3) - a\tan(x^3)}{(ax+b)^2}$ (iv) $\dfrac{-3}{(5x+6)(2x+3)}$

9. (i) $\dfrac{1}{x}\left\{e^{(1+\log x)} + \sec^2(\log x)\right\}$ (ii) $-e^x(\operatorname{cosec}^2 e^x) - \dfrac{20ax^3}{ax^4+b}$

10. (i) $-\sec x$ (ii) $\operatorname{cosec} x$ (iii) $\dfrac{3}{\sqrt{x^2-a^2}}$

11. (i) $-\dfrac{b^2x}{a^2y}$ (ii) $\dfrac{2a}{y}$ (iii) $\dfrac{b^2x}{a^2y}$ (iv) $-\sqrt[3]{\dfrac{y}{x}}$

(v) $\dfrac{2x(6x^2-y)}{x^2-6y^2}$ (vi) $-\dfrac{ax+hy++g}{hx+by+f}$ (viii) $-\dfrac{x(2x^2+y^2)}{y(x^2+2y^2)}$

12. (i) $\dfrac{\cos(x+y)-y}{x-\cos(x+y)}$ (ii) $\dfrac{y\sec^2(xy)-1}{1-x\sec^2(xy)}$ (iv) $-\dfrac{a\cos ax}{b\sin bx}$

13. (i) $e^{ax}(b\cos bx + a\sin bx)$ (ii) $e^{ax}(a\cos bx - b\sin bx)$

15. (i) $\frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$ (ii) $\frac{-a}{x^2+a^2}$ (iii) $-\frac{1}{\sqrt{-(2x+2)}}\frac{1}{\sqrt{2x+3}}$

(iv) $\frac{x}{\sqrt{1+x^2}}$ (v) $\frac{1}{2}$ (vi) $\frac{1}{2\cos\frac{x}{2}\sqrt{\cos x}}$

16. (i) $\frac{1}{\sqrt{x}(1+x)}$ (ii) $-\frac{2}{1+x^2}$ (iii) $-\frac{3}{\sqrt{1-x^2}}$ (iv) $-\frac{1}{x\sqrt{x^2-1}}$

(v) $\frac{3}{1+x^2}$ (vi) $\frac{2}{1+x^2}$ (vii) $\frac{1}{\sqrt{1-x^2}}$ (viii) $\frac{1}{1+x^2}$

(ix) $\frac{1}{2}\cdot\frac{1}{1+x^2}$ (x) $\frac{1}{\sqrt{(x-a)(b-x)}}$

17. (i) $\frac{2a}{\sqrt{1-a^2x^2}}$ (ii) -1 (iii) $-\frac{x}{\sqrt{1+x^2}}$ (iv) $\frac{\sqrt{(a^2-b^2)}}{2(a+b\cos x)}$

(v) $\frac{2.e^{2x+1}}{1+e^{4x+2}}$ (vi) $-\frac{2}{1+x^2}$ (vii) $-\frac{1}{1+x^2}$

18. (i) $\frac{1}{2\sqrt{1-x^2}}$ (ii) $\frac{2a^2x}{x^4+a^4}$ (iii) $\frac{1}{2}$.

(iv) $e^{e^x}.e^x$ (v) $(\sin x)^{\tan x}\{1+\sec^2 x\log(\sin x)\}$

19. (i) $\frac{1}{t}$ (ii) $\tan\frac{1}{2}\theta$ (iii) $-\frac{1}{t^2}$ (iv) $\frac{b}{a}\operatorname{cosec}\theta$

(v) $\frac{1}{2}\sec^3\theta\operatorname{cosec}\theta$ **20.** (i) $-\frac{1}{2}$ (ii) 1 (iii) 1 (iv) $-\frac{4}{3}$

21. (i) $2\log x.\,x^{(\log x-1)}$ (ii) $x^{\cos^{-1}x}\left\{-\frac{\log x}{\sqrt{1-x^2}}+\frac{\cos^{-1}x}{x}\right\}$

(iii) $x^{1+x+x^2}\left\{(2x+1)\log x+1+x+\frac{1}{x}\right\}$

(iv) $e^{e^x}\cdot e^x$ (v) $(\sin x)^{\tan x}\{1+\sec^2 x\log(\sin x)\}$

(vi) $(\sin x)^{\cos x}\{\cos x\cot x-\sin x\log\sin x\}$

$+(\cos x)^{\sin x}\{\cos x\log\cos x-\sin x\tan x\}$

(vii) $x^{e^x}e^x\left\{\frac{4}{x}+\log x\right\}$ (viii) $a^{x^x}.x^x\log a(1+\log x)$

(ix) $(x^x)^x\{x+2x\log x\}$ (x) $(\log x)^{\cos x}\left\{\frac{\cos x}{x\log x}-\sin x\log(\log x)\right\}$

22. (i) $\dfrac{y^2}{x\{1-y\log x\}}$ (ii) $-\dfrac{y^2(1-\log x)}{x^2(1-\log y)}$ (iii) $\dfrac{y}{x}$

(iv) $\dfrac{y\tan x+\log\sin y}{\log\cos x-x\cot y}$

(v) $-\dfrac{y(x^{\sin y}.\sin y+xy^{\sin x}\log y\cos x)}{y(x^{\sin y}y\log x\cos y+y^{\sin x}\sin x)}$

25. (i) $\sin x$ (ii) 1

Exercise 6

1. (i) $4e^{2x+3}$ (ii) m^2e^{mx} (iii) $49\sin(3-7x)$

(iv) $e^{6x}(36x^4+48x^3+12x^2)$ (v) $-\frac{1}{2}(49\sin 7x+\sin x)$

(vi) $10^x(\log_e 10)^2$ (vii) $\dfrac{\sin(\log x)-\cos(\log x)}{x^2}$

(viii) $-\operatorname{cosec}^2 x$ (ix) $2\cos 2x$ (x) $3+2\log x$

2. (i) $-\dfrac{4a^2}{y^3}$ (ii) $-\dfrac{\sin(x+y)}{\{1-\cos(x+y)\}^3}$

(iii) $-\dfrac{2a^3x}{(y^2-ax)^3}$ (iv) $2\left\{\dfrac{1}{(x-2)^3}-\dfrac{1}{(x-1)^3}\right\}$

(v) $-\dfrac{x}{(1+x^2)^2}$ (vi) $-\dfrac{\sin^2 x+\cos y}{\sin^3 y}$ (vii) $-\dfrac{4}{y^3}$

3. (i) $\dfrac{2}{at^3}$ (ii) $-\dfrac{1}{2at^2}$ (iii) $\dfrac{1}{4a}\sec^4\dfrac{\theta}{2}$ (iv) $\dfrac{\sec^3\theta}{a\theta}$

(v) $-\dfrac{4t}{a(t^2-1)^3}=-\dfrac{4a^2}{y^3}$ (vi) $\dfrac{2(1+t^2)^3}{3a(1-t^2)^3}$ **26.** 11

INTEGRAL CALCULUS

CHAPTER ONE

INDEFINITE INTEGRAL

§ 1·1. Aim of Integral Calculus.

The term **Integration** means "finding the sum of". The aim of integral calculus is to find the sum of special types of those infinite series, each term of which tends to zero as the number of terms of the series tends to infinity. In fact the subject of integral calculus came into being in the attempt of finding the area enclosed by closed curves. In such an attempt, finding the sum of above mentioned infinite series was required. Integral calculus has another view-point. This view-point is to find the original function (primitive) when its differential coefficient or derivative is known. In this respect the process of integration can be regarded as the inverse process of differentiation. These two view-points have been bridged by the *Fundamental Theorem of Integral Calculus* to be discussed in a later Chapter.

Integral Calculus has various applications in Applied Mathematics, Physics and other branches of science. You shall find applications of Integral Calculus in the determination of area and some other problems discussed in this book.

Historically Integral Calculus was discovered from the first view point. But we shall, in this book, discuss first, the second view-point i.e., we shall first discuss integration as the inverse process of differentiation.

§ 1·2. Integration as the inverse process of differentiation : Indefinite Integration.

You know that $\frac{d}{dx}(x^2)=2x$. This is also written as $\int 2xdx=x^2$ which is read as integral $2xdx$ is equal to x^2 or integral of $2x$ with respect to x is x^2. You know that differentiation of x^2 gives $2x$. The inverse process of getting x^2 from $2x$ is called the process of *integration* with respect to x. $2x$ is called the *Integrand* and $\int 2xdx$ or x^2 is the *Indefinite Integral* with respect to x of the Integrand $2x$. In $\int 2xdx$, the quantity dx is the differential of x. As the operation of differentiation with respect to x is indicated by the symbol $\frac{d}{dx}$, so is the process of integration with respect to x indicated by the symbol $\int dx$.

Example. As, $\frac{d}{dx}(\sin x)=\cos x$,

$\therefore \int \cos x\, dx=\sin x$

Def. If $\frac{d}{dx}\{f(x)\}=g(x)$, then $\int g(x)dx=f(x)$.

Hence by definition, if $\int g(x)dx=f(x)$ then,

$\frac{d}{dx}\{f(x)\}=g(x)$, or, $\frac{d}{dx}\{\int g(x)dx\}=g(x)$.

§ 1·3. Integral of a function with respect to a variable is not unique : Constant of Integration :

Let $\int f(x)dx=g(x)$. $\therefore \frac{d}{dx}\{g(x)\}=f(x)$.

Again, $\frac{d}{dx}\{g(x)+c\}=f(x)$ [c, is any constant]

$\therefore$ By definition, $\int f(x)dx=g(x)+c$.

So, $g(x)+c$, is an integral of $f(x)$ with respect to x.

Hence, integrals of $f(x)$ with respect to x are more than one. Again, if $(g)x$ and $h(x)$, be two integrals of $f(x)$ with respect to x, then $g(x)$ and $h(x)$ differ by a constant.

For, since $\int f(x)dx=g(x)$ and $\int f(x)dx=h(x)$

$\therefore \frac{d}{dx}\{g(x)\}=f(x)$ and $\frac{d}{dx}\{h(x)\}=f(x)$.

$\therefore \frac{d}{dx}\{g(x)\}-\frac{d}{dx}\{h(x)\}=0$, or, $\frac{d}{dx}\{g(x)-h(x)\}=0$.

$\therefore g(x)-h(x)$ is a constant.

For, you have learnt in differential calculus that the derivative with respect to a variable or the rate of change of a function cannot be zero, if the function is not a constant.

Now, it is clear from the above discussion that if $g(x)$ be an integral of $f(x)$ with respect to x, then $g(x)+c$; where c is a constant, will also be an integral of $f(x)$ with respect to x. For different values of c, one can obtain different integrals of $f(x)$. When $c=0$, the integral $g(x)$ is obtained. Hence $g(x)+c$ is the general form of integral of $f(x)$ with respect to x. The constant c is called an arbitrary constant of integration. The general form of the indefinite integral of a function with respect to a variable is expressed by adding with the integral of the function, an arbitrary constant of integration.

From the above discussion it is now evident that every function possesses more than one integral with respect to a variable. So, no function can be definitely called that, it is the only integral of a given function with respect to a variable and hence the name *Indefinite Integral.*

Note. For convenience of printing we have not added in this book, the arbitrary constant of integration on many a occasion. But you must always remember its presence. To find the indefinite integral is to find the general form of the integral. If the constant of integration is absent, then one cannot get the general form.

§ 1·4. Standard Forms :

(1) $\int x^n\,dx=\dfrac{x^{n+1}}{n+1}+c\ (n\neq-1)$

Proof. $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}+c\right)=\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right)+\dfrac{d}{dx}(c)$

$$=\frac{1}{n+1}\,\frac{d}{dx}\left(x^{n+1}\right)+0=\frac{1}{n+1}(n+1)x^n=x^n.$$

$\therefore\ \int x^n dx=\dfrac{x^{n+1}}{n+1}+c,\ (n\neq-1).$

Examples.

1. $\int x^5 dx=\dfrac{x^{5+1}}{5+1}+c=\dfrac{x^6}{6}+c.$

2. $\int\sqrt{x}dx=\int x^{\frac{1}{2}}dx=\dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c=\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{2}{3}x^{\frac{3}{2}}+c.$

3. $\int dx=\int 1.dx=\int x^0 dx=\dfrac{x^{0+1}}{0+1}+c=\dfrac{x^1}{1}+c=x+c.$

(2) $\int x^{-1}dx$ or, $\int\dfrac{dx}{x}=\log x+c.$

Proof. $\dfrac{d}{dx}\left(\log x+c\right)=\dfrac{1}{x},\ \therefore\ \int\dfrac{dx}{x}=\log x+c.$

(3) $\int e^{ax}dx=\dfrac{e^{ax}}{a}+c.$

Proof. $\frac{d}{dx}\left(\frac{e^{ax}}{a}+c\right)=\frac{d}{dx}\left(\frac{e^{ax}}{a}\right)+\frac{d}{dx}(c)$

$=\frac{1}{a}\cdot\frac{d}{dx}\left(e^{ax}\right)+0=\frac{1}{a}\left(a.e^{ax}\right)=e^{ax}$ $\therefore$ $\int e^{ax}dx=\frac{e^{ax}}{a}+c.$

Examples :

1. $\int e^{3x}dx=\frac{e^{3x}}{3}+c.$

2. $\int e^{-5x}dx=\frac{e^{-5x}}{-5}+c=-\frac{e^{-5x}}{5}+c.$

(4) $\int a^{mx}dx=\frac{1}{m}\frac{a^{mx}}{\log_e{}^a}+c.$

Proof, $\frac{d}{dx}\left(\frac{1}{m}\frac{a^{mx}}{\log_e{}^a}+c.\right)$

$=\frac{1}{m\log_e{}^a}\frac{d}{dx}\left(a^{mx}\right)+\frac{d}{dx}(c)=\frac{1}{m\log_e{}^a}ma^{mx}.\log_e{}^a=a^{mx}.$

$\therefore$ By definition, $\int a^{mx}\,dx=\frac{1}{m}\frac{a^{mx}}{\log_e{}^a}+c.$

Examples :

1. $\int 2^{5x}dx=\frac{1}{5}\frac{2^{5x}}{\log_e{}^2}+c.$

(5) **Integration of Trigonometric functions :**

(i) $\int\frac{d}{dx}\left(\frac{\sin ax}{a}\right)=\frac{1}{a}\frac{d}{dx}\left(\sin ax\right)=\frac{1}{a}a\cos ax=\cos ax$

$\therefore$ $\int\cos ax\,dx=\frac{\sin ax}{a}+c.$

(ii) $\frac{d}{dx}\left(-\frac{\cos ax}{a}\right)=-\frac{1}{a}\frac{d}{dx}\left(\cos ax\right)$

$=-\frac{1}{a}\left(-a\sin ax\right)=\sin ax,$

$\therefore$ $\int\sin ax\,dx=-\frac{\cos ax}{a}+c$

(iii) $\frac{d}{dx}\left(\frac{\tan ax}{a}\right)=\frac{1}{a}\frac{d}{dx}(\tan ax)=\frac{1}{a}a\sec^2 ax=\sec^2 ax$

$\therefore$ $\int\sec^2 ax\,dx=\frac{\tan ax}{a}+c.$

(iv) $$\frac{d}{dx}\left(-\frac{\cot ax}{a}\right)=-\frac{1}{a}\frac{d}{dx}(\cot ax)$$

$$=-\frac{1}{a}(-a\operatorname{cosec}^2 ax)=\operatorname{cosec}^2 ax.$$

$$\therefore\quad \int \operatorname{cosec}^2 ax\,dx=-\frac{\cot ax}{a}+c.$$

(v) $$\frac{d}{dx}\left(-\frac{\operatorname{cosec} ax}{a}\right)=-\frac{1}{a}\frac{d}{dx}(\operatorname{cosec} ax)$$

$$=-\frac{1}{a}(-a\operatorname{cosec} ax\cot ax)=\operatorname{cosec} ax\cot ax.$$

$$\therefore\quad \int \operatorname{cosec} ax\cot ax\,dx=-\frac{\operatorname{cosec} ax}{a}+c.$$

(vi) $$\frac{d}{dx}\left(\frac{\sec ax}{a}\right)=\frac{1}{a}\frac{d}{dx}(\sec ax)$$

$$=\frac{1}{a}(a\sec ax\tan ax)=\sec ax\tan ax.$$

$$\therefore\quad \int \sec ax\tan ax\,dx=\frac{\sec ax}{a}+c.$$

Cor. Putting $a=1$ in each of the above formulas,

$\int \cos x\,dx=\sin x+c$; $\int \sin x\,dx=-\cos x+c$.

$\int \sec^2 x\,dx=\tan x+c$; $\int \operatorname{cosec}^2 x\,dx=-\cot x+c$.

$\int \sec x\tan x\,dx=\sec x+c$; $\int \operatorname{cosec} x\cot x\,dx=-\operatorname{cosec} x+c$.

Examples :

1. $\int \cos 5x\,dx=\dfrac{\sin 5x}{5}+c$.

2. $\int \sin(-3x)\,dx=-\dfrac{\cos(-3x)}{-3}+c=\dfrac{\cos 3x}{3}+c$.

Examples 1A

Ex. 1. Integrate :

1. (i) $\displaystyle\int\frac{dx}{\sqrt[3]{x}}$ (ii) $\int x^{-6}\,dx$

(i) $$\int\frac{dx}{\sqrt[3]{x}}=\int x^{-\frac{1}{3}}dx=\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+C=\frac{x^{\frac{2}{3}}}{\frac{2}{3}}+C=\tfrac{3}{2}x^{\frac{2}{3}}+C.$$

(ii) $\int x^{-6}\,dx=\dfrac{x^{-6+1}}{-6+1}+C=\dfrac{x^{-5}}{-5}+C=-\dfrac{1}{5x^5}+C.$

Ex. 2. Integrate :

(i) $\int\dfrac{dx}{e^{6x}}$ (ii) $\sqrt[3]{e^x}\,dx.$ (iii) $\int\dfrac{dx}{\sqrt{e^x}}$

(i) $\int\dfrac{dx}{e^{6x}}=\int e^{-6x}dx=\dfrac{e^{-6x}}{-6}+C=-\dfrac{1}{6e^{6x}}+C,$

(ii) $\int\sqrt[3]{e^x}\,dx=\int e^{\frac{x}{3}}dx=\dfrac{e^{\frac{x}{3}}}{\frac{1}{3}}+C=3e^{\frac{x}{3}}+C=3.\sqrt[3]{e^x}+C$

(iii) $\int\dfrac{dx}{\sqrt{e^x}}=\int e^{-\frac{x}{2}}dx=\dfrac{e^{-\frac{x}{2}}}{-\frac{1}{2}}+C=-2\dfrac{1}{\sqrt{e^x}}+C.$

Ex. 3. Integrate :

(i) $\int 7^x\,dx$ (ii) $\int e^x\,dx.$

(i) $\int 7^x\,dx=\dfrac{7^x}{\log_e^7}+C.$

(ii) $\int e^x\,dx=\dfrac{e^x}{\log_e^e}+C=e^x+C$ $[\because \log_e^e=1]$

Ex. 4. Integrate :

(i) $\int\sec^2 4x\,dx$ (ii) $\int\dfrac{\sin\theta}{\cos^2\theta}d\theta$

(i) $\int\sec^2 4x\,dx=\dfrac{\tan 4x}{4}+C$

(ii) $\int\dfrac{\sin\theta}{\cos^2\theta}\,d\theta=\int\sec\theta\tan\theta\,d\theta=\sec\theta+C.$

Exercise 1A

1. Integrate :

(i) $\int x^{100}\,dx$ (ii) $\int x^7\,dx$ (iii) $\int\dfrac{dx}{x^2}$ (iv) $\int x^{-3}\,dx$

(v) $\int\dfrac{dx}{\sqrt[4]{x^5}}$ (vi) $\int x^2\sqrt{x}\,dx$ (vii) $\int\dfrac{dx}{\sqrt{x}}$ (viii) $\int\dfrac{dx}{x^{\frac{n-1}{n}}}$

2. Integrate :—

(i) $\int e^{2x}\,dx$ (ii) $\int e^{17x}\,dx$ (iii) $\int e^{cx}\,dx$ (iv) $\int e^{\frac{4}{5}x}dx.$

(v) $\int \sqrt{e^x}\,dx$ (vi) $\int e^{-70x}\,dx$ (vii) $\int \frac{dx}{\sqrt[5]{e^x}}$ (viii) $\int e^{\log x}\,dx.$

3. Integrate :—

(i) $\int 3^x\,dx$ (ii) $\int (\frac{1}{2})^x\,dx$ (iii) $\int a^x\,dx$ (iv) $\int 6^{2x}\,dx.$

(v) $\int 10^x\,dx$ (vi) $6^{10x}\,dx.$

4. Integrate :—

(i) $\int \sin 7x\,dx.$ (ii) $\int \sin(-2x)\,dx.$ (iii) $\int \cos 6x\,dx.$

(iv) $\int \cos(-4x)\,dx$ (v) $\int \operatorname{cosec}^2 3x\,dx.$

(vi) $\int -\operatorname{cosec} 2x \cot 2x\,dx.$

§ 1·5. General rules of Integration :—

(1) $\int a.f(x)dx = a.\int f(x)dx.$ **(a is a constant)**

Proof. Let $\int f(x)dx = g(x) + c.$

$$\therefore \frac{d}{dx}\Big(g(x)+c\Big) = f(x), \quad \text{or,} \quad \frac{d}{dx}\Big(g(x)\Big) + \frac{d}{dx}(c) = f(x),$$

$$\text{or,} \quad \frac{d}{dx}\Big(g(x)\Big) = f(x) \quad \left[\because \frac{d}{dx}(c) = 0\right]$$

$$\text{Now,} \quad \frac{d}{dx}\Big(a\{g(x)\}+ac\Big) = \frac{d}{dx}\Big(a\{g(x)\}\Big) + \frac{d}{dx}(ac)$$

$$= a\frac{d}{dx}\Big(g(x)\Big) + 0 = a.f(x)$$

$$\therefore \int af(x)dx = a.g(x) + a.c = a.\{g(x)+c\} = a\int f(x)dx.$$

Cor. Let $f(x)=1$, $\therefore \int a dx = a\int dx = ax + c.$

(2) $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx.$

Proof. Let, $\int f(x)dx = h_1(x)$ and $\int g(x)dx = h_2(x)$

$$\therefore \frac{d}{dx}\Big(h_1(x)\Big) = f(x) \text{ and } \frac{d}{dx}\Big(h_2(x)\Big) = g(x).$$

$$\therefore \frac{d}{dx}\Big(h_1(x) \pm h_2(x)\Big) = \frac{d}{dx}h_1(x) \pm \frac{d}{dx}h_2(x) = (f)x \pm g(x)$$

$$\therefore \int \{f(x) \pm g(x)\}dx = h_1(x) \pm h_2(x) = \int f(x)dx \pm \int g(x)dx.$$

Cor. 1. By repeated application of the above rule, it can be proved that if each of $\int f_1(x)dx$, $\int f_2(x)dx, \cdots$, $\int f_n(x)dx$ can be determined, then

$$\int \{\pm f_1(x) \pm f_2(x) \pm f_3(x) \pm \cdots f_n(x)\} dx.$$
$$= \pm \int f_1(x)dx \pm \int f_2(x)dx \pm \int f_3(x)dx \pm \cdots \pm \int f_n(x)dx.$$

[n is a finite positive integer]

Cor. 2. From Cor. 1 and rule 1 we get that if each of $\int f_1(x)dx$, $\int f_2(x)dx$, $\cdots$, $\int f_n(x)dx$, can be determined and if a_1, $a_2, \cdots a_n$ be constants (n is a finite positive integer).

$$\int \{\pm a_1 f_1(x) \pm a_2 f_2(x) \pm \cdots \pm a_n f_n(x)\} dx$$
$$= \pm a_1 \int f_1(x)dx \pm a_2 \int f_2(x)dx \pm \cdots \pm a_n \int f_n(x)dx.$$

Examples :

1. $\int (x^2 + e^x)dx = \int x^2 dx + \int e^x dx = \dfrac{x^3}{3} + e^x + c.$

2. $\int (x^3 + \cos x)dx = \int x^3 dx + \int \cos x \, dx = \dfrac{x^4}{4} + \sin x + c.$

3. $\int 2x \, dx = 2 \int x dx = 2. \dfrac{x^2}{2} + c = x^2 + c$

4. $\int \sin(-2x)dx = \int -\sin 2x \, dx = -\int \sin 2x dx.$

$$= -\left(-\frac{\cos 2x}{2}\right) + c = \frac{\cos 2x}{2} + c.$$

§ 1·6. Determination of integrals of powers and products of sine and co-sine functions of a variable by reducing it to functions of multiple angles.

From the formula (i) $1 + \cos 2x = 2\cos^2 x$ and (ii) $1 - \cos 2x = 2\sin^2 x$, $\cos^2 x$ and $\sin^2 x$ can be expressed in the forms $\frac{1}{2}(1 + \cos 2x)$ and $\frac{1}{2}(1 - \cos 2x)$ respectively.

Hence $\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2x)dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx$

$$= \tfrac{1}{2}x + \frac{\sin 2x}{4} + c.$$

Similarly, $\int \sin^2 x dx = \frac{1}{2}x - \dfrac{\sin 2x}{4} + c.$

(ii) $\sin 3x = 3\sin x - 4\sin^3 x$

$\therefore\ \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

or, $\int \sin^3 x dx = \int \frac{1}{4}(3\sin x - \sin 3x)dx = -\frac{3}{4}\cos x + \frac{\cos 3x}{12} + c.$

(iii) Again, $\cos 3x = 4\cos^3 x - 3\cos x$ gives $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$.

or, $\int \cos^3 x dx = \frac{1}{4}\int (\cos 3x + 3\cos x)dx = \frac{\sin 3x}{12} + \frac{3}{4}\sin x + c.$

(iv) Products of sines and co-sines can be integrated by expressing them as sums of sines and cosines by the formulas $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ etc. Follow the illustrations carefully.

Examples :

1. $\int \cos 2x \cos 4x dx = \int \frac{1}{2}.2\cos 4x \cos 2x dx$

$= \frac{1}{2}\int (\cos 6x + \cos 2x)dx = \frac{1}{2}\int \cos 6x dx + \frac{1}{2}\int \cos 2x dx$

$= \frac{1}{2}.\frac{\sin 6x}{6} + \frac{1}{2}.\frac{\sin 2x}{2} + c = \frac{\sin 6x}{12} + \frac{\sin 2x}{4} + c.$

2. $\int 4\sin 2x \cdot \cos 3x dx = \int 2.2\cos 3x \sin 2x dx$

$= 2\int (\sin 5x - \sin x)dx = 2\int \sin 5x dx - 2\int \sin x dx$

$= 2\left(-\frac{\cos 5x}{5}\right) - 2(-\cos x) + c = 2\left(\cos x - \frac{\cos 5x}{5}\right) + c$

Examples 1B

Ex. 1. Integrate :

(i) $\int (x^2 - \sin x)dx$ (ii) $\int (x+2)^2 dx$ (iii) $\int (2+x)^3 dx.$

(iv) $\int (x+2)(x+3)dx$ (v) $\int \frac{1}{\sqrt{x}}(x^4 + 7\sqrt{x})dx$

(vi) $\int x^2(1-x)^2 dx$ [Tripura '78] (vii) $\int \left(2x - \frac{1}{x}\right)^2 dx$ [Tripura '81]

(viii) $\int \left(x + \frac{1}{x}\right)^2 dx$ [Tripura '82]

(i) $\int (x^2 - \sin x)dx = \int x^2 dx - \int \sin x\, dx$

$= \frac{x^3}{3} - (-\cos x) + c = \frac{x^3}{3} + \cos x + c.$

(ii) $\int (x+2)^2 dx = \int (x^2+4x+4)dx = \int x^2 dx + \int 4x\,dx + \int 4dx$

$= \int x^2 dx + 4\int x\,dx + 4\int dx = \frac{x^3}{3} + 4.\frac{x^2}{2} + 4x + c$

$= \frac{x^3}{3} + 2x^2 + 4x + c.$

(iii) $\int (2+x)^3 dx = \int (8+12x+6x^2+x^3)dx$

$= \int 8dx + \int 12x\,dx + \int 6x^2 dx + \int x^3 dx$

$= 8\int dx + 12\int x\,dx + 6\int x^2 dx + \int x^3 dx$

$= 8x + 12.\frac{x^2}{2} + 6.\frac{x^3}{3} + \frac{x^4}{4} + c$

$= 8x + 6x^2 + 2x^3 + \frac{x^4}{4} + c.$

(iv) $\int (x+2)(x+3)dx = \int (x^2+5x+6)dx$

$= \int x^2 dx + \int 5x\,dx + \int 6dx = \int x^2 dx + 5\int x\,dx + 6\int dx$

$= \frac{x^3}{3} + 5.\frac{x^2}{2} + 6.x + c = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + c$

(v) $\int \frac{1}{\sqrt{x}}(x^4 + 7\sqrt{x})dx = \int (x^{\frac{7}{2}} + 7)dx = \int x^{\frac{7}{2}} dx + \int 7dx$

$= \frac{x^{\frac{9}{2}}}{\frac{9}{2}} + 7x + c = \frac{2}{9}x^{\frac{9}{2}} + 7x + c.$

(vi) $\int x^2 (1-x)^2\, dx = \int x^2\, (1-2x+x^2)\, dx.$

$= \int x^2 dx - 2\int x^3\, dx + \int x^4\, dx$

$= \frac{x^3}{3} - 2.\frac{x^4}{4} + \frac{x^5}{5} + C = \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} + C.$

(vii) $\int \left(2x - \frac{1}{x}\right)^2 dx = \int 4x^2\, dx - \int 4dx + \int \frac{1}{x^2} dx$

$= 4\frac{x^3}{3} - 4x - \frac{1}{x} + C.$

(viii) $\int \left(x + \frac{1}{x}\right)^2 dx = \int x^2\, dx + 2\int dx + \int \frac{1}{x^2} dx$

$= \frac{x^3}{3} + 2x - \frac{1}{x} + C.$

Ex. 2. Integrate : $\int \tan^2 x\, dx$

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \int \sec^2 x\, dx - \int dx$$
$$= \tan x - x + c.$$

Ex. 3. (*a*) $\int \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2 dx = \int (e^x + 2 + e^{-x}) dx$

$$= \int e^x\, dx + 2\int dx + \int e^{-x}\, dx = e^x + 2x - e^{-x} + C.$$

(*b*) $\int \frac{e^{4x} + e^{6x}}{e^x + e^{-x}} dx = \int \frac{e^{5x}(e^{-x} + e^x)}{e^x + e^{-x}} dx = \int e^{5x} dx = \frac{e^{5x}}{5} + C.$

Ex. 4. Integrate : $\int e^{n \log x} dx = \int e^{\log x^n} dx = \int x^n\, dx$

$= \frac{x^{n+1}}{n+1} + c$ when $n \neq -1$

$= \log x + c$ when $n = -1$

Ex. 5. Integrate :

(*a*) $\int \sin 3x \cos 2x\, dx.$ [Tripura 1986]
(*b*) $\int \sin^3 x\, dx.$ [Tripura 1983]
(*c*) $\int \sqrt{1 + \sin 2x}\, dx.$ [Tripura 1981]

(*a*) $\int \sin 3x \cos 2x\, dx = \frac{1}{2} \int 2 \sin 3x \cos 2x\, dx$
$= \frac{1}{2} \int (\sin 5x + \sin x) dx = \frac{1}{2} \int \sin 5x\, dx + \frac{1}{2} \int \sin x\, dx$
$= -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + c.$

(*b*) $\int \sin^3 x\, dx = \frac{1}{4} \int (3 \sin x - \sin 3x) dx$
$= \frac{3}{4} \int \sin x\, dx - \frac{1}{4} \int \sin 3x\, dx = -\frac{3}{4} \cos x + \frac{1}{12} \cos 3x + c.$

(*c*) $\int \sqrt{1 + \sin 2x}\, dx = \int \sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}\, dx$
$= \int \sqrt{(\sin x + \cos x)^2} = \int \sin x\, dx + \int \cos x\, dx$
$= -\cos x + \sin x + c.$

Ex. 6. $\int \sin x^0 dx = \int \sin \frac{\pi x}{180} dx = -\frac{\cos \frac{\pi x}{180}}{\frac{\pi}{180}} + \text{A}$

$$= -\frac{180}{\pi} \cos \frac{\pi x}{180} + \text{A} = -\frac{180}{\pi} \cos x^0 + \text{A}$$

Ex. 7. $\int \frac{\cot x}{\tan x} dx = \int \cot^2 x\, dx = \int (\text{cosec}^2 x - 1) dx$

$$= \int \text{cosec}^2 x\, dx - \int dx = -\cot x - x + c.$$

Ex. 8. $\int\frac{2\sin^3x+3\cos^3x}{\sin^2x\cos^2x}dx=\int\left(\frac{2\sin^3x}{\sin^2x\cos^2x}+\frac{3\cos^3x}{\sin^2x\cos^2x}\right)dx$

$$=\int(2\sec x\tan x+3\cot x\,\mathrm{cosec}\,x)dx$$

$$=2\int\sec x\tan x\,dx+3\int\cot x\,\mathrm{cosec}\,x\,dx$$

$$=2\sec x-3\,\mathrm{cosec}\,x+c.$$

Ex. 9. $\int\sec^2x\,\mathrm{cosec}^2x\,dx=\int\frac{dx}{\sin^2x\cos^2x}$

$$=\int\frac{\sin^2x+\cos^2x}{\sin^2x\cos^2x}dx=\int(\sec^2x+\mathrm{cosec}^2x)\,dx$$

$$=\int\sec^2x\,dx+\int\mathrm{cosec}^2x\,dx=\tan x-\cot x+c.$$

Ex. 10. $\int\frac{\cos x+\sin x}{\cos x-\sin x}(1-\sin 2x)dx$

$$\int\frac{\cos x+\sin x}{\cos x-\sin x}(\sin^2x+\cos^2x-2\sin x\cos x)dx$$

$$=\int\frac{\cos x+\sin x}{\cos x-\sin x}\left(\cos x-\sin x\right)^2dx$$

$$=\int(\cos x+\sin x)(\cos x-\sin x)dx$$

$$=\int(\cos^2x-\sin^2x)dx=\int\cos 2x\,dx=\frac{\sin 2x}{2}+c.$$

Ex. 11. $\int\frac{dx}{1-\sin x}=\int\frac{1+\sin x}{(1-\sin x)(1+\sin x)}dx=\int\frac{1+\sin x}{\cos^2x}dx$

$$=\int\left(\frac{1}{\cos^2x}+\frac{\sin x}{\cos^2x}\right)dx=\int(\sec^2x+\sec x\tan x)dx$$

$$=\int\sec^2x\,dx+\int\sec x\tan x\,dx=\tan x+\sec x+c.$$

Note. $\int\frac{dx}{1+\sin x}=\int\frac{1-\sin x}{\cos^2x}dx=\int\sec^2x\,dx-\int\sec x\tan x\,dx$

$$=\tan x-\sec x+c.$$

Ex. 12. $\int\frac{9^{1+x}+3^{1+x}}{3^x}dx=\int\frac{9.9^x+3.3^x}{3^x}dx$

$$=\int\frac{9.3^{2x}+3.3^x}{3^x}dx=\int\frac{3^x(9.3^x+3)dx}{3^x}=\int(9.3^x+3)dx$$

$$=9\int 3^x dx+3\int dx=9.\frac{3^x}{\log_e 3}+3x+c.$$

Ex. 13. $\int e^{x\log 2}dx=\int e^{\log 2^x}dx=\int 2^x dx=\dfrac{2^x}{\log_e 2}+c.$

14. $\int \dfrac{\cos^4 x}{\sin^2 x}dx=\int \dfrac{(\cos^2 x)^2}{\sin^2 x}dx=\int \dfrac{(1-\sin^2 x)^2}{\sin^2 x}dx$

$=\int \dfrac{1-2\sin^2 x+\sin^4 x}{\sin^2 x}dx=\int(\operatorname{cosec}^2 x-2+\sin^2 x)dx$

$=\int \operatorname{cosec}^2 x\, dx-2\int dx+\frac{1}{2}\int(1-\cos 2x)dx$

$=-\cot x-2x+\frac{1}{2}x-\dfrac{\sin 2x}{4}+c.$

$=-\cot x-\frac{3}{2}x-\dfrac{\sin 2x}{4}+c.$

Ex. 15. $\int \cos x.\cos 2x\cos 3x\, dx$

$=\frac{1}{2}\int(2\cos x\cos 2x)\cos 3x\, dx$

$=\frac{1}{2}\int(\cos 3x+\cos x)\cos 3x\, dx$

$=\frac{1}{2}\int(\cos^2 3x+\cos 3x\cos x)dx$

$=\frac{1}{4}\int 2\cos^2 3x\, dx+\frac{1}{4}\int 2\cos 3x\cos x\, dx$

$=\frac{1}{4}\int(1+\cos 6x)dx+\frac{1}{4}\int(\cos 4x+\cos 2x)dx$

$=\frac{1}{4}x+\dfrac{\sin 6x}{24}+\dfrac{\sin 4x}{16}+\dfrac{\sin 2x}{8}+c.$

Ex. 16. Integrate : $\int \dfrac{\cos 5x+\cos 4x}{1-2\cos 3x}dx$

Let $I=\int \dfrac{\cos 5x+\cos 4x}{1-2\cos 3x}dx$

$=\int \dfrac{\sin 3x\cos 5x+\sin 3x\cos 4x}{\sin 3x-2\sin 3x\cos 3x}dx$

[Multiplying both numerator and denominator by $\sin 3x$]

$=\int \dfrac{\sin 3x\cos(3x+2x)+\sin 3x\cos(3x+x)}{\sin 3x-\sin 6x}dx$

$$=\int \frac{\begin{array}{l}\sin 3x\cos 3x\cos 2x-\sin 3x\sin 3x\sin 2x\\ +\sin 3x\cos 3x\cos x-\sin 3x\sin 3x\sin x\end{array}}{\sin 3x-\sin 6x}dx$$

$$=\frac{1}{2}\int \frac{\begin{array}{l}2\sin 3x\cos 3x\cos 2x-2\sin 3x\sin 3x\sin 2x\\ +2\sin 3x\cos 3x\cos x-2\sin 3x\sin 3x\sin x\end{array}}{\sin 3x-\sin 6x}dx$$

$$=\frac{1}{2}\int \frac{\begin{array}{l}2\sin 3x\cos 3x(\cos 2x+\cos x)-\sin 3x(2\sin 3x\sin 2x\\ +2\sin 3x\sin x)\end{array}}{\sin 3x-\sin 6x}dx$$

$$=\tfrac{1}{2}\int\frac{\sin 6x(\cos 2x+\cos x)-\sin 3x(\cos x-\cos 5x+\cos 2x-\cos 4x)}{\sin 3x-\sin 6x}$$

$$=\tfrac{1}{2}\int\frac{(\cos 2x+\cos x)\ (\sin 6x-\sin 3x)+\sin 3x\ (\cos 5x+\cos 4x)}{\sin 3x-\sin 6x}dx$$

$$=-\tfrac{1}{2}\int(\cos 2x+\cos x)\ dx+\tfrac{1}{2}\int\frac{\cos 5x+\cos 4x}{1-2\cos 3x}\ dx$$

$$=-\tfrac{1}{2}\left(\frac{\sin 2x}{2}+\sin x\right)+c'+\tfrac{1}{2}\mathrm{I}$$

$$\therefore\quad \tfrac{1}{2}\mathrm{I}=-\tfrac{1}{2}\left(\frac{\sin 2x}{2}+\sin x\right)+c'$$

or, $\mathrm{I}=-\left(\dfrac{\sin 2x}{2}+\sin x\right)+c.$

Exercise 1B

Integrate

1. (*a*) (i) $\int (x^2+3^x)\ dx$ (ii) $\int (2+3x)^3\ dx$ (iii) $\int\dfrac{x^4+x^2+1}{x^2+x+1}dx$

(iv) $\int(1+e^{2x})dx$ (v) $\int(x^7+e^x+a^x)dx$ (vi) $\int\left(\dfrac{x-1}{\sqrt{x}}-\dfrac{e^x}{e^{-x}}\right)dx$

(vii) $\int \cot^2 x dx$ (viii) $\int (2\cos x+\tan^2 x)\ dx$

(*b*) (i) $\int \sin x \sin 2x dx$ (ii) $\int \sin 10\ x \cos 6x\ dx$

(iii) $\int 2\cos 6x \cos 4x dx$ (iv) $\int \sin^2 x\ dx$ (v) $\int \cos^2 2x dx$

(vi) $\int \cos^2 \dfrac{x}{2}\ dx$ (vii) $\int \cos^3 x dx.$ (viii) $\int \sin^3 3x dx$

(ix) $\int \cos x \cos 2x \cos 3x\ dx$ (*x*) $\int \sin x \sin 2x \sin 4x dx.$

2. (i) $\int (2x-1)\ (x-2) dx$ (ii) $\int \sqrt[3]{x}\left(x^6-\dfrac{2}{x^2}\right)dx$

(iii) $\int\left(x+\dfrac{2}{x}\right)\left(2x+\dfrac{1}{x}\right)dx$ (iv) $\int\dfrac{(1+x)^3}{x^5}dx$ [C. U. Int. '60]

3. (i) $\int\dfrac{x^2-7x+12}{x-4}dx$ (ii) $\int\dfrac{x^3+3x^2-4x-12}{x+2}dx.$

4. (i) $\int\dfrac{(e^x+1)^2}{e^{2x}}dx$ (ii) $\int\dfrac{e^{3x}-4e^x-1}{e^{2x}}dx.$

5. $\int\dfrac{e^{6x}-1}{e^{2x}-1}.$ **6.** $\int \cos x^{\circ} dx.$ **7.** $\int \sin^2 ax\ dx.$

8. $\int \cos^2 6x\, dx$. 9. $\int \sin^2 \frac{x}{3} dx$. 10. $\int \cot^2 2x\, dx$.

11. $\int \frac{1-\tan^2 x}{1+\tan^2 x} dx$. 12. $\int \frac{\cos\theta}{\sin^2\theta} d\theta$

13. $\int 2 \sin 3x \sin 4x\, dx$. 14. $\int \cos 4x \cos 5x\, dx$.

15. $\int \sin mx \cos nx\, dx$. 16. $\int \sin^3 \frac{x}{2}\, dx$.

17. $\int \frac{\tan x}{\cot x} dx$. 18. $\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x} dx$.

19. (i) $\int (\tan^2 x + 2) dx$. (ii) $\int (\tan^2 x + 2)(\cot^2 x + 3) dx$.

20. $\int \frac{1+\sin 2x}{\sin x + \cos x} dx$.

21. (i) $\int \frac{(a^x+1)^2}{a^x} dx$ (ii) $\int \frac{a^{3x}+a^x}{a^{2x}} dx$.

22. (i) $\int \frac{dx}{1+\cos 2x}$ (ii) $\int \frac{dx}{1-\cos x}$

23. $\int \frac{\sin x - \cos 2x}{1-\sin x} dx$.

24. $\int \sqrt{1+\cos 2x}\, dx$

25. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta}$; where θ is a constant.

26. $\int (\sin^6 x + \cos^6 x)\, dx$.

27. (i) $\int \cos^4 x\, dx$ (ii) $\int \sin^4 x\, dx$.

28. $\int \sin ax \sin bx \cos cx\, dx$.

29. $\int \sin 2x\, (1+\cos 2x)\, dx$. [C. U. '60 Int.]

30. $\int (\cos 3x - 2 \sin x + x^2)\, dx$ [C. U. '61 Int.]

31. $\int \frac{a \sin^2 x + b \cos^2 x}{\sin^2 x \cos^2 x}\, dx$. 32. $\int \frac{\cos x}{\sin^2 x}(1-3\cos^2 x)\, dx$

33. $\int \frac{\cos 8x - \cos 7x}{1+2\cos 5x}\, dx$.

CHAPTER TWO

INTEGRATION BY SUBSTITUTION OF VARIABLES

§ **2·1.** You have seen in Differential Calculus that there are certain general rules for differentiation of functions (of course, which are differentiable and most of the elementary functions are differentiable). But in case of integration, there is no definite rule for integration of integrable functions. The processes of integration is mainly tentative. So integration is more difficult than differentiation. In chapter one you have seen that many functions which are not of the standard forms have been integrated by expressing the integrand as the sum or difference of more than one integrands of standard forms with the help of Trigonometric and Algebraic formulas. But all integrands cannot be reduced to standard forms with the help of those formulas. In those cases different other methods are followed. In this book we shall discuss about two methods, viz., (i) Integration by substitution and (ii) Integration by parts. The subject matter of the present chapter is integration by substitution of variables.

§ **2·2.** Let $\int f(x)dx = g(x)$

$$\therefore \ \frac{d}{dx}\{g(x)\} = f(x).$$

Now if $x = F(z)$, then $\frac{dx}{dz} = F'(z)$

$$\therefore \ \frac{d}{dz}\{g(x)\} = \frac{d}{dx}\{g(x)\}.\frac{dx}{dz} = f(x)F'(z) = f\{F(z)\}F'(z)$$

$\therefore$ By definition, $g(x) = \int f\{F(z)\}F'(z)dz$

or, $\int f(x)dx = \int f\{F(z)\}F'(z)dz$

Now the variable of integration is z instead of x. Hence the integral (if it can be determined) will be expressed in terms of z. Express the final result in terms of x from the relation of x and z.

Note : You know that $\frac{dx}{dz}$ is the ratio of the two differentials dx and dz. $\therefore$ $\frac{dx}{dz} = F'(z)$ or $dx = F'(z)dz$ and use of this form is more convenient.

Ex. 1. Integrate : $\int (2x+3)^5 dx.$
Let $I=\int (2x+3)^5 dx.$

$$\therefore \quad \frac{dI}{dx}=(2x+3)^5.$$

Let $2x+3=z$, or, $x=\frac{z}{2}-\frac{3}{2}$, $\therefore \quad \frac{dx}{dz}=\frac{1}{2}$

Now, $\frac{dI}{dz}=\frac{dI}{dx}\cdot\frac{dx}{dz}=(2x+3)^5\cdot\frac{1}{2}=\frac{1}{2}z^5$

$$\therefore \quad I=\int\tfrac{1}{2}z^5 dz=\tfrac{1}{2}\int z^5 dz=\tfrac{1}{2}\,\frac{z^6}{6}+c=\tfrac{1}{12}(2x+3)^6+c.$$

Ex. 2. Integrate : $\int\frac{dt}{t\sqrt{t^2-1}}.$

Let $t=\sec\theta$. $\therefore \quad dt=\frac{dt}{d\theta}\,d\theta=\sec\theta\tan\theta\, d\theta$

and $t\sqrt{t^2-1}=\sec\theta\sqrt{\sec^2\theta-1}=\sec\theta\tan\theta$

$$\therefore \quad \int\frac{dt}{t\sqrt{t^2-1}}=\int\frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\tan\theta}=\int d\theta=\theta+c$$

Now, $\because \sec\theta=t$, $\therefore \theta=\sec^{-1}t$

$$\therefore \quad \int\frac{dt}{t\sqrt{t^2-1}}=\sec^{-1}t+c.$$

Sometimes variables are substituted in the form $\phi(x)=z$.

$$\therefore \quad \phi(x)=z, \quad \therefore \quad \frac{dz}{dx}=\phi'(x), \quad \therefore \quad dx=\frac{dz}{\phi'(x)}.$$

Now from $\phi(x)=z$, express $f(x)$ in terms of z and also $dx=\frac{dz}{\phi'(x)}$

$\therefore \quad \int f(x)dx$ will be of the form $\int g(z)dz$.

Ex. 3. $\int \sin^2\theta\cos\theta\, d\theta.$
Let $\sin\theta=z$, $\therefore \quad \cos\theta\, d\theta=dz$

and $\int\sin^2\theta\cos\theta\, d\theta=\int z^2dz=\frac{z^3}{3}+c=\frac{\sin^3\theta}{3}+c.$

§ 2·3. Rules of substitution : There is no general rule for substitution of variables for integration of functions. Variables are generally substituted by inspection. In the next few articles we discuss some convenient rules of substitution of variables.

§ 2·4. Integration of integrals of the form $\int f(ax+b)dx$

Ex. 1. Integrate : $\int \sin(ax+b)\,dx$.

Let $ax+b=z$. $\therefore$ $adx=dz$ or, $dx=\dfrac{dz}{a}$

$$\therefore \int \sin(ax+b)\,dx=\int \frac{1}{a}\sin z dz=\frac{1}{a}\int \sin z dz=-\frac{\cos z}{a}+c.$$

$$=-\frac{\cos(ax+b)}{a}+c.$$

Ex. 2. Integrate : $\displaystyle\int \frac{dx}{3x+4}$.

Let $3x+4=t$. $\therefore$ $3dx=dt$ or, $dx=\dfrac{dt}{3}$.

$$\therefore \int \frac{dx}{3x+4}=\int \frac{dt}{3t}=\frac{1}{3}\int \frac{dt}{t}=\frac{1}{3}\log t+c$$

$$=\tfrac{1}{3}\log(3x+4)+c.$$

§ 2·5. The form : $\int \{f(x)\}^n f'(x)dx$

To integrate $\int \{f(x)\}^n f'(x)dx$ let, $f(x)=z$

$\therefore$ $f'(x)=\dfrac{dz}{dx}$, or, $f'(x)\,dx=dz$.

$$\text{Hence given integral}=\int z^n dz=\frac{z^{n+1}}{n+1}+c \quad [\text{ If } n\neq -1\ ]$$

$$\text{and}=\log z+c\ [\text{ If } n=-1\ ]$$

$$\text{Hence given integral}=\frac{\{f(x)\}^{n+1}}{n+1}+c \quad [\text{ If } n\neq -1\ ]$$

$$\text{and }=\log\{f(x)\}+c \quad [\text{ If } n=-1\ ]$$

Example For, integration of $\int (ax^2+bx+c)^3(2ax+b)dx$ notice that if $f(x)=(ax^2+bx+c)$, then $f'(x)=2ax+b$. Hence the given integral is of the form

$$\int \{f(x)\}^3 f'(x)dx=\frac{\{f(x)\}^4}{4}+c=\frac{(ax^2+bx+c)^4}{4}+c.$$

§ 2·6. The form : $\int \phi\{f(x)\}\,f'(x)\,dx$

If $\int \phi(x)dx=g(x)$, then, $\int \phi\{f(x)\}.\,f'(x)dx=g\{f(x)\}$

Proof. $\int \phi\{f(x)\}f'(x)\,dx=\int \phi(z)dz$, [putting $z=f(x)$, $dz=f'(x)dx$.]

$$=g(z), [\because \int \phi(x)\,dx=g(x)]$$
$$=g\{f(x)\}.$$

Hence if the integrand be the product of a function of an integrable function of the form $\phi\{f(x)\}$ and the derivative $f'(x)$ of the second function, then substitute z for the second function $f(x)$.

Note. The form $\int\{f(x)\}^n f'(x)dx$ discussed in the last article is a special form of $\int\phi\{f(x)\}\{f'(x)\}\,dx$.

Example 1. Integrate : $\int\frac{e^{\tan^{-1}x}}{1+x^2}dx$. [C. U. 1939]

Let $\tan^{-1} x=z$. $\therefore \frac{dx}{1+x^2}=dz$

$\therefore$ Given integral$=\int e^z\,dz=e^z+c=e^{\tan^{-1}x}+c$.

Examples 2A

Ex. 1. Integrate : $\int(4-3x)^{100}dx$.

Let $4-3x=z$. $\therefore -3dx=dz$, $\therefore dx=-\frac{dz}{3}$.

$\therefore \int(4-3x)^{100}\,dx=-\int z^{100}\frac{dz}{3}=-\frac{1}{3}\cdot\frac{z^{101}}{101}+c$

$=-\frac{(4-3x)^{101}}{303}+c$.

Ex. 2. Integrate :

$$\int\frac{dx}{x^2-5x+6}=\int\frac{dx}{(x-2)(x-3)}=\int\left(\frac{1}{x-3}-\frac{1}{x-2}\right)dx$$

$$=\int\frac{dx}{x-3}-\int\frac{dx}{x-2}.$$

Now, let $x-3=u$ and $x-2=v$

$\therefore$ $dx=du$ and $dx=dv$ (respectively)

$\therefore$ Given integral$=\int\frac{du}{u}-\int\frac{dv}{v}=\log u-\log v+c$

$$=\log\frac{u}{v}+c=\log\frac{x-3}{x-2}+c.$$

Ex. 3. Integrate :

(*a*) $\int \frac{dx}{ax+b}$ (*b*) $\int \frac{dx}{b-ax}$ (*c*) $\int \frac{a'x+b'}{ax+b}dx$

(*a*) Let $ax+b=z$. $\therefore$ $adx=dz$ or $dx=\frac{dz}{a}$

$$\therefore \int \frac{dx}{ax+b}=\int \frac{1}{z}\frac{dz}{a}=\frac{1}{a}\int \frac{dz}{z}=\frac{1}{a}\log z+c.$$

$$=\frac{1}{a}\log(ax+b)+c$$

(*b*) Let $b-ax=z$ $\therefore$ $-adx=dz$ or, $dx=-\frac{dz}{a}$

$$\text{So, } \int \frac{dx}{b-ax}=\int \frac{1}{z}\left(-\frac{dz}{a}\right)=-\frac{1}{a}\int \frac{dz}{z}=-\frac{1}{a}\log z+c$$

$$=-\frac{1}{a}\log(b-ax)+c.$$

(*c*) Let $ax+b=z$ $\therefore$ $adx=dz$ and $x=\frac{z-b}{a}$

$$\therefore \int \frac{a'x+b'}{ax+b}dx=\int \frac{a'\frac{z-b}{a}+b'}{z}\cdot\frac{dz}{a}$$

$$=\frac{1}{a^2}\int \frac{a'z+ab'-a'b}{z}dz=\frac{a'}{a^2}\int dz+\frac{ab'-a'b}{a^2}\int \frac{dz}{z}$$

$$=\frac{a'}{a^2}z+\frac{ab'-a'b}{a^2}\log z+c'$$

$$=\frac{a'}{a^2}(ax+b)+\frac{ab'-a'b}{a^2}\log(ax+b)+c'$$

$$=\frac{a'}{a}x+\frac{ab'-a'b}{a^2}\log(ax+b)+c$$

$$\left[\frac{a'b}{a^2}+c'=c\text{ (say)}\right]$$

Alt. method : $\int \frac{a'x+b'}{ax+b}dx=a'\int \frac{x+\frac{b'}{a'}}{ax+b}dx$

$$=\frac{a'}{a}\int \frac{ax+\frac{ab'}{a'}}{ax+b}dx=\frac{a'}{a}\int \frac{ax+b+\frac{ab'}{a'}-b'}{ax+b}dx$$

$$=\frac{a'}{a}\int\frac{ax+b}{ax+b}dx+\frac{a'}{a}\left(\frac{ab'-a'b}{a'}\right)\int\frac{dx}{ax+b}$$

$$=\frac{a'}{a}\int dx+\frac{ab'-a'b}{a}\int\frac{dx}{ax+b}$$

$$=\frac{a'}{a}x+\frac{ab'-a'b}{a^2}\log(ax+b)+c \qquad \text{[See (i) above]}$$

Ex. 4. Integrate : (*a*) $\int\frac{\cos x\,dx}{\cos(x+a)}$. (*b*) $\int\frac{\sin x\,dx}{\sin(x-\alpha)}$.

[H. S. '84, '87]

(*a*) Let $x+a=z$ $\therefore$ $dx=dz$ and $x=z-a$.

$$\therefore \int\frac{\cos x\,dx}{\cos(x+a)}=\int\frac{\cos(z-a)dz}{\cos z}=\int\frac{\cos z\cos a+\sin z\sin a}{\cos z}dz$$

$$=\cos a\int dz+\sin a\int\frac{\sin z}{\cos z}dz$$

$$=\cos a\int dz-\sin a\int\frac{d(\cos z)}{\cos z} \quad [\text{ as } d(\cos z)=-\sin z\,dz\ ]$$

$$=z\cos a-\sin a\log(\cos z)+c'$$

$$=(x+a)\cos a+\sin a\log\sin(x+a)+c'$$

$$=x\cos a+\sin a\log\sec(x+a)+c$$

[as $a\cos a$ is constant, $a\cos a+c'=\text{constant}=c$ (say)]

(*b*) Let $x-\alpha=z$ $\therefore$ $dx=dz$ and $x=z+\alpha$.

$$\therefore \int\frac{\sin x}{\sin(x-\alpha)}dx=\int\frac{\sin(z+\alpha)}{\sin z}dz$$

$$=\int\frac{\sin z\cos\alpha+\cos z\sin\alpha}{\sin z}dz$$

$$=\cos\alpha\int dz+\sin\alpha\int\frac{\cos z}{\sin z}dz$$

$$=\cos\alpha\int dz+\sin\alpha\int\frac{d(\sin z)}{\sin z} \quad [\because\ d(\sin z)=\cos z\,dz]$$

$$=\cos\alpha.z+\sin\alpha\log(\sin z)+c'$$

$$=(x-\alpha)\cos\alpha+\sin\alpha\log\sin(x-\alpha)+c'$$

$$=x\cos\alpha+\sin\alpha\log\sin(x-\alpha)+c.$$

Ex. 5. Integrate : $\int x^2\sqrt{x^3+1}\,dx$.

Let, $x^3+1=z$. $\therefore$ $3x^2dx=dz$ $\therefore$ $x^2dx=\frac{dz}{3}$.

So, the given integral$=\int \sqrt{z}\dfrac{dz}{3}=\dfrac{1}{3}\int z^{\frac{1}{2}}dz$

$=\frac{1}{3}.\dfrac{z^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{2}{9}(x^3+1)^{\frac{3}{2}}+c.$

Note. Here if we take $f(x)=x^3+1$, then $f'(x)=3x^2$.

But in the given integrand we have x^2 in place of $3x^2$. We can write $x^2=\frac{1}{3}.3x^2$.

Ex. 6. Integrate :

(*a*) $\int\dfrac{x\,dx}{\sqrt{3x^2+4}}$ [H. S. 1980] (*b*) $\int\dfrac{x\,dx}{\sqrt{3x^2+1}}$ [H. S. 1978]

(*a*) Let, $3x^2+4=z$ $\therefore$ $6x\,dx=dz$ or, $x\,dx=\dfrac{dz}{6}$

$\therefore$ Given integral$=\int\dfrac{dz}{6\sqrt{z}}=\frac{1}{6}\int z^{-\frac{1}{2}}dz=\frac{1}{6}.\dfrac{z^{\frac{1}{2}}}{\frac{1}{2}}+c$

$=\frac{1}{3}\sqrt{z}+c=\frac{1}{3}\sqrt{3x^2+4}+c.$

(*b*) **Alternative method :** Let $3x^2+1=z^2$ $\therefore$ $6x\,dx=2zdz$ or, $xdx=\frac{1}{3}zdz.$

$\therefore$ Given integral$=\int\dfrac{z\,dz}{3z}=\frac{1}{3}\int dz=\frac{1}{3}z+c$

$=\frac{1}{3}\sqrt{3x^2+1}+c.$

Ex. 7. Integrate :

(*a*) $\int\dfrac{e^x dx}{e^x+1}$ (*b*) $\int\dfrac{e^x-e^{-x}}{e^x+e^{-x}}dx$ [Tripura, '84]

(*c*) $\int\dfrac{e^{2x}+1}{e^{2x}-1}dx$ [Tripura '85] (*d*) $\int\dfrac{dx}{e^x+1}$ [H. S. 1985]

(*e*) $\int\dfrac{e^{2x}dx}{e^x+1}$. [H. S. 1983] (*f*) $\int\dfrac{dx}{(e^x-1)^2}$.

(*a*) Let $e^x+1=z$ $\therefore$ $e^x dx=dz.$

$\therefore$ Given integral$=\int\dfrac{dz}{z}=\log z+c=\log(e^x+1)+c.$

(*b*) Let $e^x+e^{-x}=z$ $\therefore$ $(e^x-e^{-x})\,dx=dz$

So, given integral$=\int\dfrac{dz}{z}=\log z+c=\log(e^x+e^{-x})+c.$

(*c*) $\int \frac{e^{2x}+1}{e^{2x}-1}dx = \int \frac{e^x(e^x+e^{-x})}{e^x(e^x-e^{-x})} = \int \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$

Let $e^x - e^{-x} = z$. $\therefore$ $(e^x+e^{-x})dx = dz$

$\therefore$ Given integral$=\int \frac{dz}{z} = \log z + c = \log(e^x - e^{-x}) + c.$

(*d*) $\int \frac{dx}{e^x+1} = \int \frac{e^{-x}dx}{e^{-x}(e^x+1)} = \int \frac{e^{-x}dx}{1+e^{-x}}$

Let $1+e^{-x}=z$ $\therefore$ $-e^{-x}dx = dz$ or, $e^{-x}dx = -dz$

$\therefore$ Given integral$=\int \frac{-dz}{z} = -\log z + c = -\log(1+e^{-x}) + c.$

(*e*) Let $e^x+1=z$ $\therefore$ $e^x dx = dz$ and $e^x = z-1$.

Now $\int \frac{e^{2x}dx}{e^x+1} = \int \frac{e^x \,.\, e^x dx}{e^x+1} = \int \frac{(z-1)dz}{z}$

$= \int dz - \int \frac{dz}{z} = z - \log z + c'$

$= e^x + 1 - \log(e^x+1) + c' = e^x - \log(e^x+1) + c.$

(*f*) $\int \frac{dx}{(e^x-1)^2} = \int \frac{e^{-2x}\,dx}{e^{-2x}(e^x-1)^2} = \int \frac{e^{-2x}\,dx}{(1-e^{-x})^2} = \int \frac{e^{-x}.e^{-x}\,dx}{(1-e^{-x})^2}$

Let $1-e^{-x}=z$ $\therefore$ $e^{-x}\,dx = dz$. Also $e^{-x} = 1-z$.

$\therefore$ Given integral$=\int \frac{(1-z)dz}{z^2} = \int \frac{1}{z^2}dz - \int \frac{dz}{z}$

$= -\frac{1}{z} - \log z + c = -\frac{1}{1-e^{-x}} - \log(1-e^{-x}) + c$

$= -\frac{1}{1-\frac{1}{e^x}} - \log\left(1 - \frac{1}{e^x}\right) + c = \frac{e^x}{1-e^x} + \log \frac{e^x}{e^x-1} + c$

$= \frac{e^x}{1-e^x} + x - \log(e^x-1) + c.$

Ex. 8. Integrate : (i) $\int \tan x\, dx$ (ii) $\int \cot x\, dx$ (iii) $\int \sec x\, dx$ and (iv) $\int \operatorname{cosec} x\, dx$.

(i) $\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx$

Now let $\cos x = z$. $\therefore$ $-\sin x\, dx = dz$ or $\sin x\, dx = -dz$.

$\therefore$ Given integral$=-\int\frac{dz}{z}=-\log(z)+c=-\log(\cos x)+c$
$=\log(\cos x)^{-1}+c=\log(\sec x)+c.$

(ii) $\int \cot x\,dx=\int\frac{\cos x}{\sin x}dx=\int\frac{dz}{z}$

[where, $z=\sin x$ $\therefore$ $dz=\cos x\,dx$]
$=\log z+c=\log(\sin x)+c.$

(iii) $\int \sec x\,dx=\int\frac{\sec x\,(\sec x+\tan x)}{\sec x+\tan x}dx$

Now let $\sec x+\tan x=z.$ $\therefore$ $(\sec x\tan x+\sec^2 x)dx$
$=dz$ or $\sec x(\sec x+\tan x)dx=dz.$

$\therefore$ $\int \sec x\,dx=\int\frac{dz}{z}=\log z+c=\log(\sec x+\tan x)+c$

Alternative method : $\int \sec x\,dx=\int\frac{dx}{\cos x}=\int\frac{dx}{\sin\left(\frac{\pi}{2}+x\right)}$

$$=\int\frac{dx}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}=\int\tfrac{1}{2}\cdot\frac{\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)dx}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}$$

Now let, $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=z$

$\therefore$ $\frac{1}{2}\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)dx=dz.$

So, $\int \sec x\,dx=\int\frac{dz}{z}=\log|z|+c$

$$=\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+c=\log\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|+c$$

(iv) $\int \operatorname{cosec} x\,dx=\int\frac{dx}{\sin x}=\int\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\int\frac{1}{2}\frac{\sec^2\frac{x}{2}}{\tan\frac{x}{2}}$

Now, let $\tan\frac{x}{2}=z$ $\therefore$ $\frac{1}{2}\sec^2\frac{x}{2}\,dx=dz$

So, $\int \operatorname{cosec} x\,dx=\int\frac{dz}{z}=\log|z|+c=\log\left|\tan\frac{x}{2}\right|+c$

Alternative method : $\int \operatorname{cosec} x\,dx=\int\frac{\operatorname{cosec} x\,(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}\,dz$

Now, let $\operatorname{cosec} x - \cot x = z$

$\therefore \quad (-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x)dx = dz$

or, $\operatorname{cosec} x (\operatorname{cosec} x - \cot x)\, dx = dz$

$\therefore \quad \int \operatorname{cosec} x dx = \int \frac{dz}{z} = \log |z| + c$

$= \log |\operatorname{cosec} x - \cot x| + c.$

[**Note.** Remember the integrals of these examples as formulas.]

Ex. 9. Integrate : $\int \frac{dx}{\sin x \cos x}$ [Tripura, 1978]

$$\int \frac{dx}{\sin x \cos x} = \int \frac{(\sin^2 x + \cos^2 x)\, dx}{\sin x \cos x} = \int \frac{\sin^2 x\, dx}{\sin x \cos x} + \int \frac{\cos^2 x\, dx}{\sin x \cos x}$$

$= \int \tan x\, dx + \int \cot x\, dx = \log |\sec x| + \log |\sin x| + c$

Ex. 10. Integrate : $\int \tan x \sec^2 x\, dx.$

Let $\tan x = z. \quad \therefore \quad \sec^2 x\, dx = dz.$

$\therefore \quad \int \tan x \sec^2 x\, dx = \int z dz = \frac{z^2}{2} + c = \frac{\tan^2 x}{2} + c.$

Ex. 11. Integrate : $\int \frac{\tan^{-1} x\, dx}{1+x^2}.$

Let $\tan^{-1} x = z \quad \therefore \quad \frac{dx}{1+x^2} = dz.$

$\therefore \quad$ Given integral $= \int z dz = \frac{z^2}{2} + c = \frac{1}{2} (\tan^{-1} x)^2 + c.$

Ex. 12. Integrate :

(i) $\int \frac{1+\cos x}{\sqrt{x+\sin x}} dx.$ (ii) $\int \frac{1+\cos x}{x+\sin x} dx.$

Let $x + \sin x = z. \quad \therefore \quad (1+\cos x)\, dx = dz$

Now (i) $\int \frac{1+\cos x}{\sqrt{x+\sin x}} dx = \int \frac{dz}{\sqrt{z}} = 2\sqrt{z} + c = 2\sqrt{x+\sin x} + c$

(ii) $\int \frac{1+\cos x}{x+\sin x} dx = \int \frac{dz}{z} = \log z + c = \log (x+\sin x) + c.$

Ex. 13. Integrate : $\int \frac{\sec x\, dx}{\log (\sec x + \tan x)}$

Let $\log(\sec x+\tan x)=z$

$\therefore\quad \dfrac{1}{\sec x+\tan x}(\sec x\tan x+\sec^2 x)dx=dz$

or, $\dfrac{1}{\sec x+\tan x}\sec x(\tan x+\sec x)dx=dz$

or, $\sec x\,dx=dz.$

$\therefore$ Given integral$=\displaystyle\int\frac{dz}{z}=\log z+c$

$=\log\{\log(\sec x+\tan x)\}+c.$

Ex. 14. Integrate :

(i) $\displaystyle\int\frac{dx}{(\sqrt{1-x^2})\sin^{-1}x}$ (ii) $\displaystyle\int\frac{\sec^2 x}{1+\tan x}dx$

(iii) $\displaystyle\int\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx$ [Tripura, '82]

(i) Let $\sin^{-1}x=z\quad\therefore\quad\dfrac{dx}{\sqrt{1-x^2}}=dz$

$\therefore$ Given integral$=\displaystyle\int\frac{dz}{z}=\log z+c=\log(\sin^{-1}x)+c.$

(ii) Let $1+\tan x=z.\quad\therefore\quad\sec^2 x\,dx=dz.$

$\therefore\quad\displaystyle\int\frac{\sec^2 x\,dx}{1+\tan x}=\int\frac{dz}{z}=\log z+c=\log(1+\tan x)+c.$

(iii) Let $\sin^{-1}x=z\quad\therefore\quad\dfrac{dx}{\sqrt{1-x^2}}=dz$

$\therefore$ Given Integral$=\int e^z\,dz=e^z+c=e^{\sin^{-1}x}+c.$

Ex. 15. Integrate : (i) $\int\sqrt[5]{1+x}\,dx$, (ii) $\int\sqrt[4]{1+\tan x}\sec^2 x\,dx$,

(iii) $\displaystyle\int\frac{dx}{x\sqrt{1+\log x}}$ (iv) $\displaystyle\int\frac{(\sin^{-1}x+3)^2}{\sqrt{1-x^2}}dx.$

(i) Let $1+x=z,\quad\therefore\quad dx=dz$

$\therefore\quad\int\sqrt[5]{1+x}\,dx=\int\sqrt[5]{z}\,dz=\int z^{\frac{1}{5}}dz=\frac{5}{6}z^{\frac{6}{5}}+c=\frac{5}{6}(1+x)^{\frac{6}{5}}+c$

(ii) Let $1+\tan x=z\quad\therefore\quad\sec^2 x\,dx=dz$

and $\int\sqrt[4]{1+\tan x}\sec^2 x\,dx=\int\sqrt[4]{z}dz=\int z^{\frac{1}{4}}dz$

$=\frac{4}{5}z^{\frac{5}{4}}+c=\frac{4}{5}(1+\tan x)^{\frac{5}{4}}+c.$

(iii) Let $1+\log x=u$. $\therefore$ $\frac{1}{x}dx=du$

and $\int\frac{dx}{x\sqrt{1+\log x}}=\int\frac{du}{\sqrt{u}}=\int u^{-\frac{1}{2}}du=2u^{\frac{1}{2}}+c$

$=2\sqrt{u}+c=2\sqrt{1+\log x}+c.$

(iv) Let $\sin^{-1}x+3=z$ $\therefore$ $\frac{1}{\sqrt{1-x^2}}dx=dz$

and $\int\frac{(\sin^{-1}x+3)^2}{\sqrt{1-x^2}}dx=\int z^2dz=\frac{z^3}{3}+c=\frac{(\sin^{-1}x+3)^3}{3}+c.$

Ex. 16. Integrate : (i) $\int\frac{x^2dx}{\sqrt{x+2}}$. [Ranchi, '63 ; P.U. '46]

Let $x+2=t^2$. $\therefore$ $dx=2t\,dt$ and $x=t^2-2$.

$\therefore$ $\int\frac{x^2dx}{\sqrt{x+2}}=\int\frac{(t^2-2)^2 2t\,dt}{t}=\int 2(t^4-4t^2+4)dt$

$=2\left(\frac{t^5}{5}-\frac{4}{3}t^3+4t\right)+c$

$=\frac{2}{5}(x+2)^{\frac{5}{2}}-\frac{8}{3}(x+2)^{\frac{3}{2}}+8(x+2)^{\frac{1}{2}}+c.$

(ii) $\int\frac{\sqrt{x+4}}{x}dx=\int\frac{x+4}{x\sqrt{x+4}}dx.$

$=\int\frac{x}{x\sqrt{x+4}}dx+\int\frac{4dx}{x\sqrt{x+4}}$

$=\int\frac{dx}{\sqrt{x+4}}+4\int\frac{dx}{x\sqrt{x+4}}=I_1+4I_2$

Now, let $x+4=z^2$. $\therefore$ $dx=2zdz$ and $\sqrt{x+4}=z$

$\therefore$ $I_1=\int\frac{2zdz}{z}=2\int dz=2z=2\sqrt{x+4}$

and $I_2=\int\frac{2z\,dz}{(z^2-4)z}=2\int\frac{dz}{z^2-4}=\frac{2}{2.2}\log\frac{z-2}{z+2}$

$=\frac{1}{2}\log\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}$

So, the given integral$=2\sqrt{x+4}+2\log\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}$

Ex. 17. Integrate : (i) $\int \frac{cx+d}{\sqrt{ax+b}}dx$

(ii) $\int (cx+d)\sqrt{ax+b}\,dx$.

Let $ax+b=z^2$, $\therefore$ $adx=2zdz$ and $x=\frac{z^2-b}{a}$

(i) $\int \frac{cx+d}{\sqrt{ax+b}}dx=\int \frac{c\left(\frac{z^2-b}{a}\right)+d}{z}.\ 2\frac{z}{a}dz$

$=\frac{2c}{a^2}\int z^2dz+\frac{2(ad-bc)}{a^2}\int dz=\frac{2c}{a^2}.\frac{z^3}{3}+\frac{2(ad-bc)}{a^2}z+k$

$=\frac{2c}{3a^2}\left(ax+b\right)^{\frac{3}{2}}+\frac{2(ad-bc)}{a^2}\left(ax+b\right)^{\frac{1}{2}}+k.$

(ii) $\int (cx+d)\sqrt{ax+b}\,dx=\int\left(c.\frac{z^2-b}{a}+d\right).z.\frac{2z}{a}dz$

$=\frac{2c}{a^2}\int z^4dz+2\frac{ad-bc}{a^2}\int z^2dz=\frac{2c}{a^2}.\frac{z^5}{5}+2\frac{ad-bc}{a^2}.\frac{z^3}{3}+k.$

$=\frac{2c}{5a^2}\left(ax+b\right)^{\frac{5}{2}}+\frac{2(ad-bc)}{3a^2}\left(ax+b\right)^{\frac{3}{2}}+k.$

Ex. 18. Integrate : $\int\frac{x^3+3x^2+3x+4}{x^2+2x+1}dx.$ [C. U. '63]

$\int\frac{x^3+3x^2+3x+4}{x^2+2x+1}dx=\int\frac{x(x^2+2x+1)+(x^2+2x+1)+3}{x^2+2x+1}dx$

$=\int\left\{x+1+\frac{3}{(x+1)^2}\right\}dx=\int x\,dx+\int dx+\int\frac{3}{(x+1)^2}dx$

$=\frac{x^2}{2}+x-\frac{3}{x+1}+c.$

Ex. 19. $\int \tan^3x\,dx=\int \tan x.\tan^2x\,dx=\int \tan x(\sec^2x-1)dx$
$=\int \tan x\sec^2x\,dx-\int \tan x\,dx$
$=\int z.dz-\log\sec x$, [In the first integral put
$\tan x=z$, $\therefore$ $\sec^2x\,dx=dz$]
$=\frac{z^2}{2}-\log\sec x+c=\frac{1}{2}\tan^2x-\log\sec x+c.$

Ex. 20. Integrate :—

(*a*) $\int x^2\cos x^3\,dx.$ [H. S. 1978]

(*b*) $\int\cos(\log x)\frac{dx}{x}.$ [Tripura, 1980] (*c*) $\int \frac{\sin x\, dx}{(a+b\cos x)^2}$. [Tripura, '83]

(*d*) $\int \frac{\cos x\, dx}{(\cos \frac{x}{2}+\sin \frac{x}{2})^2}$. [Tripura, '82]

(*e*) $\int \frac{\sin 2x}{3+\cos 2x}\, dx$. [H. S. 1988]

(*a*) Let $x^3=z \quad \therefore \quad 3x^2\, dx=dz \quad$ or, $\quad x^2\, dx=\frac{dz}{3}$.

$\therefore$ Given integral$=\int \cos z \frac{dz}{3}=\frac{1}{3}\int \cos z\, dz=\frac{1}{3}\sin z+c$

$$=\tfrac{1}{3}\sin x^3+c.$$

(*b*) Let $\log x=z \quad \therefore \quad \frac{dx}{x}=dz$

$\therefore$ Given integral$=\int \cos z\, dz=\sin z+c=\sin(\log x)+c.$

(*c*) Let $a+b\cos x=z \quad \therefore \quad -b\sin x\, dx=dz$

or, $\sin x\, dx=-\frac{dz}{b}$.

$\therefore$ Given integral$=-\int \frac{dz}{b.z^2}=-\frac{1}{b}\left(-\frac{1}{z}\right)+c=\frac{1}{bz}+c$

$$=\frac{1}{b(a+b\cos x)}+c.$$

(*d*) $\int \frac{\cos dx}{(\cos \frac{x}{2}+\sin \frac{x}{2})^2}=\int \frac{\cos x dx}{\cos^2 \frac{x}{2}+\sin^2 \frac{x}{2}+2\cos \frac{x}{2}\sin \frac{x}{2}}$

$=\int \frac{\cos x\, dx}{1+\sin x}$. Now let $1+\sin x=z \quad \therefore \quad \cos x\, dx=dz$

$\therefore$ Given integral$=\int \frac{dz}{z}=\log z+c=\log(1+\sin x)+c,$

(*e*) Let $3+\cos 2x=z \quad \therefore \quad -2\sin 2x\, dx=dz$

or, $\sin 2x\, dx=-\frac{dz}{2}$.

$\therefore$ Given integral$=-\frac{1}{2}\int \frac{dz}{z}=-\frac{1}{2}\log z+c$

$$=-\tfrac{1}{2}\log(3+\cos 2x)+c.$$

Ex. 21. Integrate :—

(*a*) $\int \frac{\sqrt{\tan x}}{\cos^4 x}\,dx.$ [H. S. 1985]

(*b*) $\int \frac{\sqrt{\tan x}}{\sin 2x}.$ [Joint Entrance, 1981]

(*c*) $\int \frac{\sin 2x\,dx}{a^2 \cos^2 x + b^2 \sin^2 x}.$

(*a*) $\int \frac{\sqrt{\tan x}\,dx}{\cos^4 x} = \int \sqrt{\tan x}\,\sec^4 x\,dx$

Let $\tan x = z^2$ $\therefore$ $\sec^2 x\,dx = 2z\,dz$

$\therefore$ Given integral $= \int \sqrt{\tan x}\,\sec^2 x\,\sec^2 x\,dx$

$= \int \sqrt{\tan x}\,(1+\tan^2 x)\sec^2 x\,dx$

$= \int 2z\,(1+z^4)z\,dz = 2\int z^2\,dz + 2\int z^6\,dz.$

$= 2\frac{z^3}{3} + 2\frac{z^7}{7} + c = \frac{2}{3}\tan^{\frac{3}{2}} x + \frac{2}{7}\tan^{\frac{7}{2}} x + c$

(*b*) $\int \frac{\sqrt{\tan x}\,dx}{\sin 2x} = \int \frac{\sqrt{\tan x}\,\sec^2 x\,dx}{2\sin x \cos x \sec^2 x}$

$= \int \frac{\sqrt{\tan x}\,\sec^2 x\,dx}{2\tan x} = \frac{1}{2}\int \frac{\sec^2 x\,dx}{\sqrt{\tan x}}$

$= \int \frac{2z\,dz}{z} = 2\int dz$ [taking $\tan x = z^2$ or $\sec^2 x\,dx = 2z\,dz$.]

$= 2z + c = 2\sqrt{\tan x} + c.$

(*c*) Let $a^2 \cos^2 x + b^2 \sin^2 x = z$

$\therefore$ $(-2a^2 \cos x \sin x + 2b^2 \sin x \cos x)\,dx = dz$

or, $(b^2 - a^2)\sin 2x\,dx = dz$ or, $\sin 2x\,dx = \frac{dz}{b^2 - a^2}$

$\therefore$ Given integral $= \frac{1}{b^2 - a^2}\int \frac{dz}{z} = \frac{1}{b^2 - a^2}\log z + c$

$= \frac{1}{b^2 - a^2}\log(a^2 \cos^2 x + b^2 \sin^2 x) + c.$

Ex. 22. Integrate :— $\int \frac{x\,dx}{x^2+9}.$ [Tripura 1979]

Let $x^2 + 9 = z$ $\therefore$ $2x\,dx = dz$ or, $x\,dx = \frac{dz}{2}$

$\therefore$ Given integral$=\frac{1}{2}\int\frac{dz}{z}=\frac{1}{2}\log z+c=\frac{1}{2}\log(x^2+9)+c$

Ex. 23. Integrate : $\int\frac{dx}{\sqrt{x}+x\sqrt{x}}$. [Joint Entrance, 1981]

Let $x=\tan^2\theta \quad \therefore \quad dx=2\tan\theta\sec^2\theta\, d\theta.$

$$\therefore \int\frac{dx}{\sqrt{x}+x\sqrt{x}}=\int\frac{dx}{\sqrt{x}(1+x)}=\int\frac{2\tan\theta\sec^2\theta\, d\theta}{\tan\theta\,(1+\tan^2\theta)}$$

$$=2\int\frac{\tan\theta\sec^2\theta\, d\theta}{\tan\theta\sec^2\theta}=2\int d\theta=2\theta+c$$

$$=2\tan^{-1}\sqrt{x}+c$$

$[\because \tan^2\theta=x \quad \therefore \tan\theta=\sqrt{x}$ and $\theta=\tan^{-1}\sqrt{x}\,]$

Ex. 24. Integrate : (*a*) $\int\frac{x-1}{x+2}dx.$ [Tripura, '79]

(*b*) $\int\frac{2x+1}{x(x+3)}dx.$ [H. S. '79]

(*c*) $\int\frac{x\, dx}{(2x+1)(x+1)}.$ [H. S. 1987]

(*a*) $\int\frac{x-1}{x+2}dx=\int\frac{x+2-3}{x+2}dx=\int dx-3\int\frac{dx}{x+2}$

$=x-3\log(x+2)+c$

[Let $x+2=u$ or, $dx=du$]

(*b*) $\int\frac{2x+1}{x(x+3)}dx=\int\frac{2x\, dx}{x(x+3)}+\int\frac{dx}{x(x+3)}$

$=2\int\frac{dx}{x+3}+\int\frac{1}{3}\left\{\frac{1}{x}-\frac{1}{x+3}\right\}dx$

$=(2-\frac{1}{3})\int\frac{dx}{x+3}+\frac{1}{3}\int\frac{dx}{x}=\frac{5}{3}\log(x+3)+\frac{1}{3}\log x+c.$

(*c*) $\int\frac{x\, dx}{(2x+1)(x+1)}=\int\left\{\frac{1}{x+1}-\frac{1}{2x+1}\right\}dx=\int\frac{dx}{x+1}-\int\frac{dx}{2x+1}$

$=\log(x+1)-\frac{1}{2}\log(2x+1)+c$

[Let $2x+1=u$ or, $2dx=du$]

Ex. 25. Integrate :— $\int\frac{dx}{\sqrt{ax+b}+\sqrt{ax-b}}$. [Joint Entrance '82]

Given integral$=\int\frac{(\sqrt{ax+b}-\sqrt{ax-b})\,dx}{(\sqrt{ax+b}+\sqrt{ax-b})(\sqrt{ax+b}-\sqrt{ax-b})}$

$=\int\frac{(\sqrt{ax+b}-\sqrt{ax-b})dx}{(ax+b)-(ax-b)}=\frac{1}{2b}\int(\sqrt{ax+b}-\sqrt{ax-b})\,dx$

$=\frac{1}{2b}\int\sqrt{ax+b}\;dx-\frac{1}{2b}\int\sqrt{ax-b}\;dx$

Let $ax+b=u$ and $ax-b=v$.

$\therefore\quad adx=du$ and $adx=dv$

$\therefore\quad dx=\frac{du}{a}=\frac{dv}{a}.$

$\therefore$ Given integral$=\frac{1}{2b}\int\frac{\sqrt{u}\,du}{a}-\frac{1}{2b}\int\frac{\sqrt{v}\,dv}{a}$

$=\frac{1}{2ab}\cdot\frac{2}{3}u^{\frac{3}{2}}-\frac{1}{2ab}\cdot\frac{2}{3}v^{\frac{3}{2}}+c$

$=\frac{1}{3ab}\{(ax+b)^{\frac{3}{2}}-(ax-b)^{\frac{3}{2}}\}+c$

Ex. 26. Integrate : $\int\frac{\sin x+\cos x}{\sin(x-\alpha)}dx.$ [Joint Entrance 1982]

Let $x-\alpha=z\quad\therefore\quad dx=dz$ and $x=z+\alpha.$

$\therefore$ Given integral$=\int\frac{\sin(z+\alpha)+\cos(z+\alpha)}{\sin z}dz$

$=\int\frac{\sin z\cos\alpha+\cos z\sin\alpha+\cos z\cos\alpha-\sin z\sin\alpha}{\sin z}dz$

$=\int\frac{\sin z(\cos\alpha-\sin\alpha)+\cos z(\cos\alpha+\sin\alpha)}{\sin z}dz$

$=(\cos\alpha-\sin\alpha)\int dz+(\cos\alpha+\sin\alpha)\int\frac{\cos z}{\sin z}dz$

$=(\cos\alpha-\sin\alpha).\,z+(\cos\alpha+\sin\alpha)\log\sin z+c'$

$\left[\because\quad\int\frac{\cos z}{\sin z}=\int\frac{d(\sin z)}{\sin z}=\log\sin z\right]$

$=(\cos\alpha-\sin\alpha)(x-\alpha)+(\cos\alpha+\sin\alpha)\log\sin(x-\alpha)+c'$

$=x(\cos\alpha-\sin\alpha)+(\cos\alpha+\sin\alpha)\log\sin(x-\alpha)+c.$

Ex. 27. Integrate : $\int\frac{\tan\alpha-\tan x}{\tan\alpha+\tan x}dx$

$$\int\frac{\tan\alpha-\tan x}{\tan\alpha+\tan x}dx=\int\frac{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin x}{\cos x}}{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin x}{\cos x}}dx=\int\frac{\sin(\alpha-x)}{\sin(\alpha+x)}dx.$$

Now, let $\alpha+x=z$ $\therefore$ $dx=dz$ and $\alpha-x=\alpha-(z-\alpha)=2\alpha-z.$

So, given Integral$=\int\frac{\sin(2\alpha-z)}{\sin z}dz$

$=\int\frac{\sin 2\alpha\cos z-\cos 2\alpha\sin z}{\sin z}dz$

$=\sin 2\alpha\int\frac{\cos z}{\sin z}dz-\cos 2\alpha\int dz$

$=\sin 2\alpha\log|\sin z|-\cos 2\alpha\, z+c'$
$=\sin 2\alpha\log|\sin(x+\alpha)|-\cos 2\alpha\,(x+\alpha)+c'$
$=\sin 2\alpha\log|\sin(x+\alpha)|-x\cos 2\alpha+c.$

Ex. 28. Integrate : (*a*) $\int(a^2+x^2)\sqrt{a+x}\,dx$
(*b*) $\int\sqrt{x+\sqrt{x^2+2}}\,dx$

(*a*) Let, $a+x=z^2$ $\therefore$ $dx=2z\,dz$ and $x=z^2-a$

$\therefore$ Given Integral$=\int\{a^2+(z^2-a)^2\}\,z.\,2z\,dz$
$=\int(2a^2+z^4-2a\,z^2)\,2z^2\,dz$
$=4a^2\int z^2\,dz+2\int z^6\,dz-4a\int z^4\,dz$

$=4a^2\frac{z^3}{3}+2\frac{z^7}{7}-4a\frac{z^5}{5}+c$

$=\frac{4}{3}a^2\,(a+x)^{\frac{3}{2}}+\frac{2}{7}\,(a+x)^{\frac{7}{2}}-\frac{4a}{5}\,(a+x)^{\frac{5}{2}}+c$

(*b*) $\int\sqrt{x+\sqrt{x^2+2}}\,dx=\int\frac{x+\sqrt{x^2+2}}{\sqrt{x+\sqrt{x^2+2}}}dx$

$=\int\frac{x\sqrt{x^2+2}+x^2+2}{(\sqrt{x+\sqrt{x^2+2}})(\sqrt{x^2+2})}dx$

Let, $x+\sqrt{x^2+2}=z^2$

$\therefore$ $\left(1+\frac{1}{2}\frac{2x}{\sqrt{x^2+2}}\right)dx=2z\,dz$

or, $\frac{x+\sqrt{x^2+2}}{\sqrt{x^2+2}}\,dx=2z\,dz$ or, $\frac{z^2}{\sqrt{x^2+2}}\,dx=2z\,dz$

or, $$\frac{dx}{\sqrt{x^2+2}}=\frac{2dz}{z}$$

So given integral$=\frac{1}{2}\int\frac{(x+\sqrt{x^2+2})^2+2}{\sqrt{(x+\sqrt{x^2+2})}}\cdot\frac{dx}{\sqrt{x^2+2}}$

$\left[\because\ x\sqrt{x^2+2}+x^2+2=\frac{1}{2}(2x\sqrt{x^2+2}+x^2+4)\right.$

$\left.=\frac{1}{2}\{(x+\sqrt{x^2+2})^2+2\}\ \right]$

$$=\frac{1}{2}\int\frac{z^4+2}{z}\cdot\ \frac{2dz}{z}=\int\left(z^2+\frac{2}{z^2}\right)dz$$

$$=\frac{z^3}{3}-\frac{2}{z}+c=\frac{(x+\sqrt{x^2+2})^{\frac{3}{2}}}{3}-\frac{2}{\sqrt{(x+\sqrt{x^2+2})}}+c$$

$$=\frac{(x^2+x^2+2+2x\sqrt{x^2+2}-6)}{3\sqrt{x+\sqrt{x^2+2}}}+c$$

$$=\frac{2}{3}\frac{(x^2+x\sqrt{x^2+2}-2)}{\sqrt{x+\sqrt{x^2+2}}}+c$$

Ex. 29. Integrate :

(*a*) $\displaystyle\int\frac{dx}{\sin x+\tan x}$ (*b*) $\displaystyle\int\sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx$

(*a*) $$\int\frac{dx}{\sin x+\tan x}=\int\frac{dx}{\sin x+\frac{\sin x}{\cos x}}=\int\frac{\cos x\,dx}{\sin x\,(1+\cos x)}$$

$$=\int\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin x.2\cos^2\frac{x}{2}}dx=\frac{1}{2}\int\operatorname{cosec}x\,dx-\frac{1}{2}\int\frac{\sin^2\frac{x}{2}}{\sin x\cos^2\frac{x}{2}}dx$$

$$=\frac{1}{2}\int\operatorname{cosec}x\,dx-\frac{1}{2}\int\frac{\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}\cos^2\frac{x}{2}}dx$$

$$=\frac{1}{2}\int\operatorname{cosec}x\,dx-\frac{1}{2}\int\frac{1}{2}\tan\frac{x}{2}\sec^2\frac{x}{2}\,dx$$

$$=\frac{1}{2}\int\operatorname{cosec}x\,dx-\frac{1}{2}\int z\,dz\quad\left[\tan\frac{x}{2}=z\text{ (say)}\right]$$

$$=\frac{1}{2}\log\tan\frac{x}{2}-\frac{1}{2}\,\frac{z^2}{2}+c$$

$$=\frac{1}{2}\log\tan\frac{x}{2}-\frac{1}{4}\tan^2\frac{x}{2}+c$$

(*b*) Let $x=\cos 2\theta\ \ \therefore\ dx=-2\sin 2\theta\,d\theta.$

Also $\sqrt{\dfrac{1-x}{1+x}}=\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}=\sqrt{\dfrac{2\sin^2\theta}{2\cos^2\theta}}=\tan\theta$

$\therefore\ \sin\left(2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right)=\sin(2\tan^{-1}\tan\theta)=\sin 2\theta$

$\therefore$ Given integral$=\int\sin 2\theta\,(-2\sin 2\theta\,d\theta)$

$=-\int 2\sin^2 2\theta\,d\theta=-\int(1-\cos 4\theta)\,d\theta$

$=-\theta+\dfrac{\sin 4\theta}{4}+c=-\theta+\dfrac{2\sin 2\theta\cos 2\theta}{4}+c$

$=-\frac{1}{2}\cos^{-1}x+\frac{1}{2}x\sqrt{1-x^2}+c$

$=\frac{1}{2}(x\sqrt{1-x^2}-\cos^{-1}x)+c$

Ex. 30. (*a*) Integrate : $\int\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}dx.$ **[I. I. T. 1985]**

Let $x=\cos^2\theta$ $\therefore$ $dx=-2\cos\theta\sin\theta\,d\theta$

$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

So, given integral$=\displaystyle\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(-4\cos\theta\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)d\theta$

$=-4\int\sin^2\dfrac{\theta}{2}\cos\theta\,d\theta=2\int(\cos\theta-1)\cos\theta\,d\theta$

$=\int 2\cos^2\theta\,d\theta-2\int\cos\theta\,d\theta=\int(1+\cos 2\theta)d\theta-2\int\cos\theta\,d\theta$

$=\theta+\dfrac{\sin 2\theta}{2}-2\sin\theta+c$

$=\theta+\sin\theta\cos\theta-2\sin\theta+c=\theta+\sin\theta(\cos\theta-2)+c$

$=\cos^{-1}\sqrt{x}+\sqrt{1-x}\,(\sqrt{x}-2)+c.$

Ex. 30. (*b*) Evaluate : $\int\dfrac{dx}{x^2(x^4+1)^{\frac{3}{4}}}dx$ **[I. I. T. 1984]**

Let $x^2=\tan\theta$ $\therefore$ $2x\,dx=\sec^2\theta\,d\theta$

$\therefore\ dx=\dfrac{\sec^2\theta}{2x}d\theta=\dfrac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta$

$$(x^4+1)^{\frac{3}{4}}=(\tan^2\theta+1)^{\frac{3}{4}}=(\sec^2\theta)^{\frac{3}{4}}=\sec^{\frac{3}{2}}\theta$$

$\therefore$ given Integral $=\int\frac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta\frac{1}{\tan\theta\sec^{\frac{3}{2}}\theta}$

$$=\int\frac{\sqrt{\sec\theta}\,d\theta}{2(\tan\theta)^{\frac{3}{2}}}=\int\frac{(\cos\theta)^{\frac{3}{2}}.(\cos\theta)^{-\frac{1}{2}}}{2(\sin\theta)^{\frac{3}{2}}}d\theta$$

$$=\int\frac{\cos\theta\,d\theta}{2(\sin\theta)^{\frac{3}{2}}}=\tfrac{1}{2}\int\frac{d(\sin\theta)}{(\sin\theta)^{\frac{3}{2}}}=\tfrac{1}{2}\int\frac{dz}{z^{\frac{3}{2}}}\quad[\text{ where } z=\sin\theta\ ]$$

$$=-\tfrac{1}{2}.\frac{z^{-\frac{1}{2}}}{\frac{1}{2}}+c=-\frac{1}{\sqrt{z}}+c=-\frac{1}{\sqrt{\sin\theta}}+c$$

$$=-\frac{1}{(1-\cos^2\theta)^{\frac{1}{4}}}+c=-\frac{1}{\left(1-\frac{1}{1+\tan^2\theta}\right)^{\frac{1}{4}}}+c$$

$$=-\frac{1}{\left(1-\frac{1}{1+x^4}\right)^{\frac{1}{4}}}+c=-\frac{(1+x^4)^{\frac{1}{4}}}{x}+c$$

Ex. 31. Integrate : $\int\frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx$

Let $\sqrt{x}=z$. $\therefore$ $\frac{dx}{2\sqrt{x}}=dz$, or, $\frac{dx}{\sqrt{x}}=2dz$.

$\therefore$ Given integral $=\int 2\sin z\,dz=-2\cos z+c$

$$=-2\cos\sqrt{x}+c.$$

Ex. 32. Integrate : (*a*) $\int\frac{e^x(1+x)}{\cos^2(xe^x)}dx$.

(*b*) $\int\frac{e^x(1+x)}{\cot(xe^x)}dx$. [Tripura '87]

Let $xe^x=z$. $\therefore$ $(e^x+xe^x)\,dx=dz$

or, $e^x(1+x)\,dx=dz$,

So, (*a*) $\int\frac{e^x(1+x)}{\cos^2(xe^x)}dx=\int\frac{dz}{\cos^2 z}=\int\sec^2 z\,dz$

$$=\tan z+c=\tan(xe^x)+c$$

(*b*) $\int\frac{e^x(1+x)dx}{\cot(xe^x)}=\int\frac{dz}{\cot z}=\int\tan z\,dz$

$$=\log(\sec z)+c=\log\sec(xe^x)+c.$$

Exercise 2A

Integrate :

1. (i) $\int (ax+b)^{11}\,dx$ (ii) $\int (4x-5)^6\,dx$ (iii) $\int \frac{dx}{(a-x)^2}$

(iv) $\int \frac{dx}{a-bx}$. (v) $\int (1+x)^5\,dx$

2. (i) $\int \cos(ax+b)\,dx$ (ii) $\int \sec^2(2x+3)\,dx$

(iii) $\int \sin^2(2t+3)dt$ (iv) $\int \cot^2(2-3t)\,dt$ (v) $\int \frac{\tan^2 x}{\cos^2 x}dx.$

3. $\int a^{p+qt}dt.$

4. (i) $\int \frac{dx}{x^2-9}$ (ii) $\int \frac{dx}{4-x^2}$ (iii) $\int \frac{dx}{4x^2-25}$

5. $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

$$\left[\textbf{Hints :} \int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}dx \right.$$

$$\left. = \frac{1}{a-b}\left\{\int \sqrt{x+a}\,dx - \int \sqrt{x+b}\,dx\right\}\right].$$

6. $\int \frac{xdx}{a+bx}$

$$\left[\textbf{Hints :} \int \frac{xdx}{a+bx} = \frac{1}{b}\int \frac{bxdx}{a+bx} = \frac{1}{b}\int \frac{a+bx-a}{a+bx}dx \right.$$

$$\left. = \frac{1}{b}\left\{\int \frac{a+bx}{a+bx}dx - \int \frac{a}{a+bx}dx\right\} = \frac{1}{b}\int dx - \frac{a}{b}\int \frac{dx}{a+bx}\right]$$

7. (i) $\int \frac{dx}{x^2-10x+24}$ (ii) $\int \frac{dx}{x^2-7x+12}$

8. (i) $\int \frac{2ax+b}{ax^2+bx+c}dx$ (ii) $\int (x^3+6x^2+5x+2)(3x^2+12x+5)\,dx$

9. $\int \frac{e^x-1}{e^x+1}dx$ **10.** $\int \frac{2x\,dx}{1+x^2}.$

11. (i) $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$ [C. P. 1933] (ii) $\int \frac{dx}{(1+x^2)\tan^{-1}x}dx$

12. (i) $\int x^3 \sqrt{x^4+a^4}\,dx$ (ii) $\int \frac{x\,dx}{\sqrt{2x^2+3}}$ **13.** $\int \frac{\cos x \sin x}{1+\sin^2 x}dx$

14. $\int (\tan x+\sin x)^2 (\sec^2 x+\cos x)\, dx.$

15. (i) $\int \frac{\sec^2 x}{(1+\tan x)^2}dx.$ (ii) $\int \frac{\operatorname{cosec}^2 x}{1+\cot x}$

(iii) $\int \frac{dx}{\cos^2 x \sqrt{\tan x-1}}$

16. (i) $\int \frac{x\,dx}{\sqrt{x^2-a^2}}$ (ii) $\int \frac{4x^3}{1+x^4}dx$ (iii) $\int \frac{ax^{n-1}}{x^n+b^n}dx.$

17. (i) $\int \frac{dx}{x+x \log x}$ (ii) $\int \frac{\log(\log x)\, dx}{x \log x}$

(iii) $\int \frac{dx}{x \log x \log(\log x)}$

18. (i) $\int \frac{\cot x\,dx}{\log(\sin x)}$ (ii) $\int \frac{\tan x\,dx}{\log(\sec x)}$

19. $\int \frac{\sin x+x\cos x}{x \sin x}dx$ **20.** $\int \frac{\sec x \operatorname{cosec} x}{\log \tan x}dx.$

21. $\int e^x \sec^2(e^x)\,dx.$ **22.** (*a*) $\int \frac{a^{\sin^{-1}x}}{\sqrt{1-x^2}}dx.$ (*b*) $\int x a^{x^2} dx$

23. $\int (\cot e^x)\, e^x\, dx.$ **24.** $\int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx.$

25. $\int e^{\cos^{-1}x} \frac{dx}{\sqrt{1-x^2}}$

26. $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx.$ **27.** (*a*) $\int e^{-\frac{1}{x}}.\frac{1}{x^2}dx.$ (*b*) $\int x^2 e^{x^3} dx$

28. $\int e^{\sin x} \cos x\, dx$ **29.** $\int e^{x-\frac{1}{x}}\left(1+\frac{1}{x^2}\right)dx.$

30. $\int \frac{e^x\,dx}{3+4e^x}.$ **31.** $\int e^{x^2+6x+9}(x+3)\,dx.$

32. $\int \frac{\log \sqrt{x}}{3x}\,dx.$ [C. U. '64]

33. (*a*) $\int \frac{dx}{x\sqrt{x^4-1}}$ [C. U. '62] (*b*) $\int \frac{dx}{x^2\sqrt{1-x^2}}$

(*c*) $\int \frac{dx}{x^2\sqrt{1+x^2}}$ (*d*) $\int \frac{dx}{(1-x^2)\sqrt{1-x^2}}$.

34. $\int \frac{\cos x\, dx}{3+4\sin x}$. **35.** (*a*) $\int x^2 \cos x^3\, dx$. (*b*) $\int \frac{\sin(\log x)}{x} dx$.

36. (*a*) $\int x^{n-1}\cos x^n\, dx$, (*b*) $\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$.

37. (i) $\int x(a+bx^2)dx$ (ii). $\int x^{n-1}\sin(a+bx^n)\, dx$.

(iii) $\int x^5 \sec^2(3+4x^6)\, dx$. (iv) $\int x^{m-1}(2x^m+11)^{100}dx$.

38. $\int (\tan x - x)\tan^2 x\, dx$. **39.** $\int (\tan x \tan 2x \tan 3x)\, dx$.

$\left[\text{Hints : } \tan 3x = \frac{\tan x + \tan 2x}{1-\tan x \tan 2x}\right]$

40. $\int \frac{(a+bx)^2}{(p+qx)^3} dx$. **41.** $\int \frac{\cos 4x - \cos 2x}{\sin 4x - \sin 2x} dx$.

42. $\int \frac{\sin 2x\, dx}{a\sin^2 x - b\cos^2 x}$ **43.** $\int \frac{\sin 2x\, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2}$.

44. $\int \frac{dx}{\{(x-2)+(x-3)\}\sqrt{(x-2)(x-3)}}$.

45. $\int \frac{x^4 dx}{x-1}$. **46.** $\int \frac{3x+4}{4x+5} dx$. **47.** $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}} dx$.

48. $\int \frac{dx}{x^3\sqrt{x^2-1}}$. **49.** $\int (a^2+x^2)\sqrt{a+x}\, dx$.

50. $\int \frac{dx}{x^2(1+x^2)^2}$. **51.** $\int \frac{dx}{x(x^2+1)}$. **52.** $\int \frac{dx}{x(x+1)^2}$.

53. $\int \frac{x^3 dx}{\sqrt{1-x^2}}$. **54.** $\int \frac{x^{\frac{1}{2}}dx}{1+x^{\frac{3}{4}}}$. **55.** $\int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx$.

§ 2·7. A few standard forms :

In this article we shall discuss integrations of integrands of the forms $\frac{1}{x^2+a^2}$, $\frac{1}{x^2-a^2}$ and $\frac{1}{a^2-x^2}$ and their square roots with respect to x. These integrals are taken as standard forms and are to be used as formulas.

Now as $1+\tan^2\theta = \sec^2\theta$, or, $\sec^2\theta - 1 = \tan^2\theta$ and $1-\cos^2\theta$

$=\sin^2\theta$, or, $1-\sin^2\theta=\cos^2\theta$, so if the integrand be a power of x^2+a^2 put $x=a\tan\theta$, if it be a power of x^2-a^2, put $x=a\sec\theta$ and if it be a power of a^2-x^2, put $x=a\sin\theta$. In many cases the alternative methods are useful.

(i) $\int\frac{dx}{a^2+x^2}$.

Let, $x=a\tan\theta$. $\therefore$ $dx=a\sec^2\theta\,d\theta$.

and $a^2+x^2=a^2+a^2\tan^2\theta=a^2(1+\tan^2\theta)=a^2\sec^2\theta$.

$$\therefore\ \int\frac{dx}{a^2+x^2}=\int\frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta}=\frac{1}{a}\int d\theta=\frac{1}{a}\theta+c.$$

Now, $\because$ $x=a\tan\theta$, $\therefore$ $\tan\theta=\frac{x}{a}$, or, $\theta=\tan^{-1}\frac{x}{a}$.

$$\therefore\ \int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c.$$

Ex. $$\int\frac{dx}{x^2+\frac{3}{4}}=\int\frac{dx}{x^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{x}{\frac{\sqrt{3}}{2}}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}$$

(ii) $\int\frac{dx}{x^2-a^2}$ $(x>a)$.

Let $x=a\sec\theta$, $\therefore$ $dx=a\sec\theta\tan\theta\,d\theta$

and $x^2-a^2=a^2\sec^2\theta-a^2=a^2(\sec^2\theta-1)=a^2\tan^2\theta$.

$$\therefore\ \int\frac{dx}{x^2-a^2}=\int\frac{a\sec\theta\tan\theta\,d\theta}{a^2\tan^2\theta}=\frac{1}{a}\int\frac{\sec\theta}{\tan\theta}d\theta$$

$$=\frac{1}{a}\int\left(\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}\right)d\theta=\frac{1}{a}\int\left(\frac{1}{\sin\theta}\right)d\theta=\frac{1}{a}\int\operatorname{cosec}\theta\,d\theta.$$

$$=\frac{1}{a}\log\left(\tan\frac{\theta}{2}\right)+c.$$

$$=\frac{1}{2a}\log\left(\tan^2\frac{\theta}{2}\right)+c=\frac{1}{2a}\log\frac{x-a}{x+a}+c.$$

$$\left[\because\ \cos\theta=\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}},\right.$$

$$\left.\therefore\quad \frac{x}{a}=\sec\theta=\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}},\text{ or, }\frac{x-a}{x+a}=\tan^2\frac{\theta}{2}.\right]$$

Alternative mcthod :

$$\int\frac{dx}{x^2-a^2}=\int\left\{\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)\right\}dx$$

$$=\frac{1}{2a}\left\{\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right\}$$

$$=\frac{1}{2a}[\log u-\log v]+c\ [x-a=u\text{ and }x+a=v\text{ (say)}]$$

$$=\frac{1}{2a}\left[\log\frac{u}{v}\right]+c=\frac{1}{2a}\log\frac{x-a}{x+a}+c.$$

(iii) $\int\frac{dx}{a^2-x^2}(a>x).$

Let. $x=a\sin\theta,\quad\therefore\quad dx=a\cos\theta\,d\theta$

and $a^2-x^2=a^2-a^2\sin^2\theta=a^2(1-\sin^2\theta)=a^2\cos^2\theta.$

Now, $\int\frac{dx}{a^2-x^2}=\int\frac{a\cos\theta\,d\theta}{a^2\cos^2\theta}=\frac{1}{a}\int\sec\theta\,d\theta.$

$$=\frac{1}{a}\log(\sec\theta+\tan\theta)+c$$

$$=\frac{1}{a}\log\left(\frac{1+\sin\theta}{\cos\theta}\right)+c=\frac{1}{a}\log\sqrt{\frac{a+x}{a-x}}+c.$$

$$=\frac{1}{2a}\log\frac{a+x}{a-x}+c.$$

$$\left[\frac{1+\sin\theta}{\cos\theta}=\sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}=\sqrt{\frac{(1+\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}}\right.$$

$$=\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\sqrt{\frac{1+\frac{x}{a}}{1-\frac{x}{a}}}\quad\left(\because\quad x=a\sin\theta,\quad\therefore\quad\sin\theta=\frac{x}{a}\right)$$

$$\left.=\sqrt{\frac{a+x}{a-x}}\right].$$

Alternative method :

$$\int\frac{dx}{a^2-x^2}=\int\frac{1}{2a}\left\{\frac{1}{a+x}+\frac{1}{a-x}\right\}dx$$

$$=\frac{1}{2a}\left\{\int\frac{dx}{a+x}+\int\frac{dx}{a-x}\right\}=\frac{1}{2a}\left\{\int\frac{du}{u}-\int\frac{dv}{v}\right\}$$

[putting $x+a=u$ and $a-x=v$]

$$=\frac{1}{2a}\left\{\log u-\log v\right\}+c=\frac{1}{2a}\log\frac{u}{v}+c=\frac{1}{2a}\log\frac{a+x}{a-x}+c.$$

Note : 1. As, $\frac{dx}{a^2-x^2}=-\frac{dx}{x^2-a^2}$, so it may appear that the integrals $\int\frac{dx}{a^2-x^2}$ and $\int\frac{dx}{x^2-a^2}$ are of the same form. But as logarithms of negative numbers are not defined, so if $x>a$, then $\int\frac{dx}{x^2-a^2}$ and if $x<a$ then $\int\frac{dx}{a^2-x^2}$ can be integrated. For this the two integrations are performed separately.

2. In integrating the above two integrals the formula of integration of $\int\operatorname{cosec}\theta\,d\theta$ and $\int\sec\theta\,d\theta$ have been used. In the alternative methods these two integrals have not been used. Again $\int\sec\theta\,d\theta$ and $\int\operatorname{cosec}\theta\,d\theta$ are often evaluated with the help of the integrals $\int\frac{dx}{x^2-a^2}$ and $\int\frac{dx}{a^2-x^2}$. But there is nothing wrong in any case. For, in the integration of $\int\frac{dx}{x^2-a^2}$ and $\int\frac{dx}{a^2-x^2}$ by the alternative method $\int\sec\theta\,d\theta$ and $\int\operatorname{cosec}\theta\,d\theta$ are not used. Again, $\int\frac{dx}{x^2-a^2}$ or $\int\frac{dx}{a^2-x^2}$ have not been used inthe integration of $\int\sec\theta\,d\theta$ and $\int\operatorname{cosec}\theta\,d\theta$.

Ex. 1. $\int\frac{dx}{x^2-16}=\int\frac{dx}{x^2-(4)^2}=\frac{1}{8}\log\frac{x-4}{x+4}+c$ [$x>4$]

Ex. 2. $\int\frac{dx}{9-x^2}=\int\frac{dx}{(3)^2-x^2}=\frac{1}{6}\log\frac{3+x}{3-x}+c$ [$x<3$]

(iv) $\int\frac{dx}{\sqrt{x^2+a^2}}$

Let $x=a\tan\theta$. $\therefore$ $dx=a\sec^2\theta\,d\theta$.

and $\sqrt{x^2+a^2}=\sqrt{a^2\tan^2\theta+a^2}=\sqrt{a^2(1+\tan^2\theta)}=a\sec\theta$.

$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a \sec^2\theta \, d\theta}{a \sec\theta \, d\theta} = \int \sec\theta \, d\theta$$

$$= \log(\sec\theta + \tan\theta) + c.$$

Now $\because$ $x = a\tan\theta$, $\therefore$ $\tan\theta = \frac{x}{a}$.

and $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a}\sqrt{x^2+a^2}$

$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \log(\sec\theta + \tan\theta) + c$$

$$= \log\left(\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right) = \log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) + c$$

$$= \log(x+\sqrt{x^2+a^2}) - \log a + c$$

$$= \log(x+\sqrt{x^2+a^2}) + k \quad [\, k = c - \log a, \text{ is a constant} \,]$$

(v) $\int \frac{dx}{\sqrt{x^2-a^2}}$ $[\, x > a \,]$

Let, $x = a\sec\theta$, $\therefore$ $dx = a \sec\theta \tan\theta \, d\theta$

and $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = a\tan\theta$.

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta \, d\theta}{a\tan\theta} = \int \sec\theta \, d\theta$$

$$= \log(\sec\theta + \tan\theta) = \log\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) + c$$

$\left[\ \because\ x = a\sec\theta,\ \therefore\ \sec\theta = \frac{x}{a} \text{ and } \tan\theta = \frac{\sqrt{x^2-a^2}}{a}\ \right]$

$$= \log(x + \sqrt{x^2-a^2}) - \log a + c$$

$$= \log(x + \sqrt{x^2-a^2}) + k.$$

Note. Putting $x + \sqrt{x^2 \pm a^2} = z$, the two integrals (iv) and (v) can also be integrated.

(vi) $\int \frac{dx}{\sqrt{a^2-x^2}}$.

Let, $x = a\sin\theta$. $\therefore$ $dx = a\cos\theta \, d\theta$

and $\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$.

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a \cos\theta \, d\theta}{a \cos\theta} = \int d\theta = \theta + c$$

$$= \sin^{-1}\frac{x}{a} + c \left[\because \ x = a \sin\theta, \ \therefore \ \sin\theta = \frac{x}{a}, \text{ or, } \theta = \sin^{-1}\frac{x}{a} \right]$$

Ex. $\int \frac{dx}{\sqrt{x^2+3}} = \log(x+\sqrt{x^2+3}) + c.$

§ 2·8. Integration of integrals of the forms $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{(px+q)dx}{ax^2+b+c}$.

(i) To integrate $\int \frac{dx}{ax^2+bx+c}$, if ax^2+bx+c can be resolved into factors, then follow the methods of partial fractions. Note carefully the following examples.

If ax^2+bx+c cannot be resolved into factors, then express $\int \frac{dx}{ax^2+bx+c}$ in one of the forms (i), (ii) and (iii) of § 2·7 depending on the values of a, b and c.

(ii) $\int \frac{px+q}{ax^2+bx+c} dx.$

You know $\frac{d}{dx}(ax^2+bx+c) = 2ax+b.$

Now, $\int \frac{px+q}{ax^2+bx+c} dx$

$$= \frac{p}{2a} \int \frac{2ax + \frac{2aq}{p}}{ax^2+bx+c} dx = \frac{p}{2a} \int \frac{(2ax+b) + \frac{2aq}{p} - b}{ax^2+bx+c} dx$$

$$= \frac{p}{2a} \int \frac{2ax+b}{ax^2+bx+c} dx + \frac{2aq-bp}{2a} \int \frac{dx}{ax^2+bx+c}$$

$$= \frac{p}{2a} I_1 + \frac{2aq-bp}{2a} I_2.$$

Now, $I_1 = \log(ax^2+bx+c)$ and the form of I_2 has been discussed in (i) above.

Note. If ax^2+bx+c can be resolved into factors, then integration can also be performed by expressing $\dfrac{px+q}{ax^2+bx+c}$ in the form $\dfrac{k_1x+k_2}{c_1x+c_2}+\dfrac{k_3x+k_4}{c_3x+c_4}$.

§ 2·9. (i) $\displaystyle\int\frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ and (ii) $\displaystyle\int\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ $(\beta>\alpha)$

Let, $x-\alpha=z^2$ $\therefore$ $dx=2zdz$

or, $\dfrac{dx}{z}=2dz$ or, $\dfrac{dx}{\sqrt{x-\alpha}}=2dz$

Again, $x-\alpha=z^2$ $\therefore$ $x=\alpha+z^2$, or, $x-\beta=z^2+\alpha-\beta$.

(i) Given integral$=\displaystyle\int\frac{2dz}{\sqrt{z^2+\alpha-\beta}}$

$=2\log(z+\sqrt{z^2+\alpha-\beta})=2\log(\sqrt{x-\alpha}+\sqrt{x-\beta})$.

(ii) Given integral

$$=\int\frac{2dz}{\sqrt{\beta-\alpha-z^2}}=2\sin^{-1}\sqrt{\frac{z}{\beta-\alpha}}=2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}.$$

2·10. (i) $\displaystyle\int\frac{dx}{\sqrt{ax^2+bx+c}}$ and (ii) $\displaystyle\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$

(i) $\displaystyle\int\frac{dx}{\sqrt{ax^2+bx+c}}$

Here there are two possibilities.

(*a*) ax^2+bx+c can be expressed as the product of two linear factors with real coefficients.

Or. (*b*) Factorisation is not possible. In case (*a*) follow the method of § 2·9. In case (*b*), follow the following general method. This general method can also be followed in case (*a*).

If a be positive,

$$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}$$

$$=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}}$$

$$=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2\pm\alpha^2}}\quad\text{where } \alpha^2=\frac{4ac-b^2}{4a^2}$$

[If $4ac-b^2$ be positive, then take the '+' sign. If $4ac-b^2$ be negative, then take the '−' sign.]

Now, $$\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2\pm\alpha^2}}=\int\frac{dt}{\sqrt{t^2\pm\alpha^2}}\left(t=x+\frac{b}{2a},\text{ say}\right)$$

and this is of the form (iv) or (v) of § 2·7.

If a be negative let $a=-d$ (d positive)

$$\therefore\ \int\frac{dx}{\sqrt{ax^2+bx+c}}=\int\frac{dx}{\sqrt{c+bx-dx^2}}$$

$$=\frac{1}{\sqrt{d}}\int\frac{dx}{\sqrt{\frac{4dc+b^2}{4d^2}-\left(x-\frac{b}{2d}\right)^2}}$$

$$=\frac{1}{\sqrt{d}}\int\frac{dt}{\sqrt{\alpha^2-t^2}}\qquad\left[t=x-\frac{b}{2d}\text{ and }\alpha^2=\frac{4dc+b^2}{4d^2}.\text{ (say)}\right]$$

This form has been discussed in § 2·7 (vi).

§ 2·11. Integration of integrals of the forms :

(i) $\displaystyle\int\frac{dx}{(ax+b)(\sqrt{cx+d})}$

and (ii) $\displaystyle\int\frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$

(i) If we put $cx+d=t^2$, then $\displaystyle\int\frac{dx}{(ax+b)\sqrt{cx+d}}$ will be reduced to one of the forms (i), (ii) and (iii) of § 2·7.

(ii) If we put $ax+b=\frac{1}{t}$, $\displaystyle\int\frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ will be reduced to forms discussed in § 2·10.

Examples 2B

Integrate :

Ex. 1. $\displaystyle\int\frac{dx}{1+a^2x^2}.$

Let, $ax=z$. $\therefore$ $adx=dz$ and $dx=\frac{dz}{a}$

$$\therefore \quad \int \frac{dx}{1+a^2x^2}=\int \frac{dz}{a(1+z^2)}=\frac{1}{a}\int \frac{dz}{1+z^2}=\frac{1}{a}\tan^{-1}z=\frac{1}{a}\tan^{-1}(ax)$$

Ex. 2. $\int \frac{e^x dx}{e^{2x}+1}$

Let, $e^x=z$. $\therefore$ $e^x dx=dz$ and $e^{2x}=(e^x)^2=z^2$.

So, $\int \frac{e^x dx}{e^{2x}+1}=\int \frac{dz}{z^2+1}=\tan^{-1}z=\tan^{-1}(e^x)$.

Ex. 3. $\int \frac{\cos x\, dx}{1+\sin^2 x}$ Let, $\sin x=z$; $\therefore$ $\cos x\, dx=dz$.

So, $\int \frac{\cos x\, dx}{1+\sin^2 x}=\int \frac{dz}{1+z^2}=\tan^{-1}z=\tan^{-1}(\sin x)$.

Ex. 4. $\int \frac{dx}{x\{1+(\log x)^2\}}$

Let, $\log x=z$. $\therefore$ $\frac{dx}{x}=dz$ and $1+(\log x)^2=1+z^2$

$$\therefore \quad \int \frac{dx}{x\{1+(\log x)^2\}}=\int \frac{dz}{1+z^2}=\tan^{-1}z=\tan^{-1}(\log x).$$

Ex. 5. $\int \frac{e^x dx}{1-e^{2x}}$.

Let, $e^x=z$; $e^x dx=dz$ and $e^{2x}=(e^x)^2=z^2$

$$\therefore \quad \int \frac{e^x dx}{1-e^{2x}}=\int \frac{dz}{1-z^2}=\tfrac{1}{2}\log \frac{1+z}{1-z}=\tfrac{1}{2}\log \frac{1+e^x}{1-e^x}$$

Ex. 6. $\int \sec\theta\, d\theta \left[\theta<\frac{\pi}{2}\right]$

$$=\int \frac{1}{\cos\theta}d\theta=\int \frac{\cos\theta\, d\theta}{\cos^2\theta}=\int \frac{\cos\theta\, d\theta}{1-\sin^2\theta}$$

Let, $\sin\theta=z$, $\therefore$ $\cos\theta\, d\theta=dz$

$$\therefore \quad \int \sec\theta\, d\theta=\int \frac{dz}{1-z^2}=\tfrac{1}{2}\log \frac{1+z}{1-z}=\tfrac{1}{2}\log \frac{1+\sin\theta}{1-\sin\theta}$$

$$=\log\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\log \frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}}=\log \frac{1+\sin\theta}{\cos\theta}$$

$$=\log(\sec\theta+\tan\theta)$$

Ex. 7. $\int \frac{3x^2 dx}{x^6-1}$ $(x>1)$

Let, $x^3=u$; $3x^2dx=du$

$$\therefore \int\frac{3x^2dx}{x^6-1}=\int\frac{du}{u^2-1}=\tfrac{1}{2}\log\frac{u-1}{u+1}=\tfrac{1}{2}\log\frac{x^3-1}{x^3+1}$$

Ex. 8. $\displaystyle\int\frac{dx}{\sqrt{1-a^2x^2}}$

Let, $ax=z$. $\therefore$ $adx=dz$, or, $dx=\dfrac{dz}{a}$.

$$\therefore \int\frac{dx}{\sqrt{1-a^2x^2}}=\int\frac{dz}{a\sqrt{1-z^2}}=\frac{1}{a}\sin^{-1}z+c=\frac{1}{a}\sin^{-1}(ax)+c.$$

Ex. 9. $\displaystyle\int\frac{dx}{6x^2+17x+12}=\int\frac{dx}{(2x+3)(3x+4)}$

$$=\int\left\{\frac{3}{3x+4}-\frac{2}{2x+3}\right\}dx.$$

$$=3\int\frac{dx}{3x+4}-2\int\frac{dx}{2x+3}=3\frac{\log(3x+4)}{3}-2\left\{\frac{\log(2x+3)}{2}\right\}+c$$

$$=\log\frac{3x+4}{2x+3}+c.$$

Ex. 10. $\displaystyle\int\frac{dx}{x^2+x+1}=\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}=\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

Now, let $x+\frac{1}{2}=z$, $\therefore$ $dx=dz$

$$\therefore \text{ Given integral}=\int\frac{dz}{{}^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{z}{\frac{\sqrt{3}}{2}}+c$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+c=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+c.$$

[In the above example, $\displaystyle\int\frac{dx}{x^2+x+1}$ is not a standard form. Putting $x+\frac{1}{2}=z$ the integral reduces to the standard form $\displaystyle\int\frac{dz}{z^2+\left(\frac{\sqrt{3}}{2}\right)^2}$]

Ex. 11. $\displaystyle\int\frac{dx}{9x^2-12x+8}$ [Rajasthan, 1959]

$$\int\frac{dx}{9x^2-12x+8}=\int\frac{dx}{(3x-2)^2+2^2}$$

Let, $3x-2=z$. $\therefore$ $3dx=dz$. $dx=\dfrac{dz}{3}$.

$\therefore$ Given integral$=\dfrac{1}{3}\displaystyle\int\dfrac{dz}{z^2+(2)^2}=\dfrac{1}{6}\tan^{-1}\dfrac{z}{2}=c.$

$=\dfrac{1}{6}\tan^{-1}\dfrac{3x-2}{2}+c.$

Ex. 12. Integrate $\mathrm{I}=\displaystyle\int\dfrac{\sec^2 x\,dx}{\tan^2 x+2\tan x+3}$

Let, $\tan x=z$, $\therefore$ $\sec^2 x\,dx=dz$.

$\therefore$ Given integral$=\displaystyle\int\dfrac{dz}{z^2+2z+3}=\int\dfrac{dz}{(z+1)^2+(\sqrt{2})^2}$

Now let, $z+1=u$ $\therefore$ $dz=du$

and $\mathrm{I}=\displaystyle\int\dfrac{du}{u^2+(\sqrt{2})^2}=\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{u}{\sqrt{2}}+c$

$=\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{z+1}{\sqrt{2}}+c=\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{\tan x+1}{\sqrt{2}}\right)+c.$

Ex. 13. $\displaystyle\int\dfrac{dx}{2x^2-6x+4}=\int\dfrac{dx}{2(x^2-3x+2)}=\dfrac{1}{2}\int\dfrac{dx}{(x-\frac{3}{2})-(\frac{1}{2})^2}$

$=\dfrac{1}{2}\displaystyle\int\dfrac{dz}{z^2-(\frac{1}{2})^2}$ [$x-\frac{3}{2}=z$ (say)]

$=\dfrac{1}{2}\cdot\dfrac{1}{2.\frac{1}{2}}\log\dfrac{z-\frac{1}{2}}{z+\frac{1}{2}}+c=\dfrac{1}{2}\log\dfrac{x-\frac{3}{2}-\frac{1}{2}}{x-\frac{3}{2}+\frac{1}{2}}+c=\dfrac{1}{2}\log\dfrac{x-2}{x-1}+c.$

Ex. 14. $\displaystyle\int\dfrac{dx}{1+x-x^2}=\int\dfrac{dx}{1-(x^2-x)}=\int\dfrac{dx}{\frac{5}{4}-(x-\frac{1}{2})^2}$

$=\displaystyle\int\dfrac{du}{\left(\dfrac{\sqrt{5}}{2}\right)^2-u^2}$ [$u=x-\frac{1}{2}$ (say)]

$=\dfrac{1}{2.\dfrac{\sqrt{5}}{2}}\log\dfrac{u+\dfrac{\sqrt{5}}{2}}{u-\dfrac{\sqrt{5}}{2}}+c=\dfrac{1}{\sqrt{5}}\log\dfrac{x-\dfrac{1}{2}+\dfrac{\sqrt{5}}{2}}{x-\dfrac{1}{2}-\dfrac{\sqrt{5}}{2}}+c$

$=\dfrac{1}{\sqrt{5}}\log\dfrac{2x-1+\sqrt{5}}{2x-1-\sqrt{5}}+c.$

Ex. 15. $\displaystyle\int\dfrac{7x-9}{x^2-2x+35}dx=\int\dfrac{\frac{7}{2}(2x-2)-2}{x^2-2x+35}dx$ [C. U. '33]

$$=\frac{7}{2}\int\frac{2x-2}{x^2-2x+35}dx-2\int\frac{dx}{x^2-2x+35}$$

$$=\frac{7}{2}\int\frac{du}{u}-2\int\frac{dx}{(x-1)^2+(\sqrt{34})^2}\quad[\,x^2-2x+35=u\text{ (say)}\,]$$

$$=\tfrac{7}{2}\log u-2\int\frac{dt}{t^2+(\sqrt{34})^2}\quad[\,x-1=t\text{ (say)}\,]$$

$$=\tfrac{7}{2}\log(x^2-2x+35)-\frac{2}{\sqrt{34}}\tan^{-1}\frac{t}{\sqrt{34}}+c$$

$$=\tfrac{7}{2}\log(x^2-2x+35)-\frac{2}{\sqrt{34}}\tan^{-1}\frac{x-1}{\sqrt{34}}+c.$$

Ex. 16. Integrate $\int\frac{2x+3}{2x^2+x-1}dx$

$$\int\frac{2x+3}{2x^2+x-1}dx=\int\frac{\frac{1}{2}(4x+1)+\frac{5}{2}}{2x^2+x-1}dx$$

$$=\frac{1}{2}\int\frac{4x+1}{2x^2+x-1}dx+\frac{5}{2}\int\frac{dx}{2x^2+x-1}$$

Now, $\int\frac{4x+1}{2x^2+x-1}dx=\log(2x^2+x-1)+c_1$

$$\left[\because\ \frac{d}{dx}(2x^2+x-1)=4x+1\right]$$

$$\int\frac{dx}{2x^2+x-1}=\tfrac{1}{2}\int\frac{dx}{\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$=\frac{1}{2.2.\frac{3}{4}}\log\frac{x+\frac{1}{4}-\frac{3}{4}}{x+\frac{1}{4}+\frac{3}{4}}+c_2=\frac{1}{3}\log\frac{2x-1}{2(x+1)}+c_2$$

∴ Given integral

$$=\tfrac{1}{2}\log(2x^2+x-1)+\tfrac{5}{6}\log\frac{2x-1}{2(x+1)}+c.$$

Ex. 17. $\int\frac{dx}{\sqrt{x^2-ax}}=\int\frac{dx}{\sqrt{x(x-a)}}=2\log(\sqrt{x}+\sqrt{x-a}).$

Ex. 18. $\int\frac{dx}{\sqrt{x^2+7x+12}}=\int\frac{dx}{\sqrt{(x+3)(x+4)}}$

$$=2\log(\sqrt{x+3}+\sqrt{x+4}).$$

Ex. 19. $\int \frac{dx}{\sqrt{ax-x^2}} = \int \frac{dx}{\sqrt{x(a-x)}}$

$$=2\sin^{-1}\sqrt{\frac{x}{a-0}} = 2\sin^{-1}\sqrt{\frac{x}{a}}.$$

Ex. 20. $\int \frac{dx}{\sqrt{3x-x^2-2}}$ [C. U. '41]

Given integral $= \int \frac{dx}{\sqrt{3x-x^2-\frac{9}{4}+\frac{1}{4}}} = \int \frac{dx}{\sqrt{\frac{1}{4}-(x^2-3x+\frac{9}{4})}}$

$$= \int \frac{dx}{\sqrt{(\frac{1}{2})^2-(x-\frac{3}{2})^2}} = \int \frac{dt}{\sqrt{(\frac{1}{2})-t^2}} \qquad [\, x-\tfrac{3}{2}=t \text{ (say)} \,]$$

$$= \sin^{-1}\frac{t}{\frac{1}{2}} = \sin^{-1} 2t = \sin^{-1} 2(x-\tfrac{3}{2}) = \sin^{-1}(2x-3).$$

Ex. 21. $\int \frac{dx}{\sqrt{3x^2+4x+2}}$ [C. U. 1942]

Given integral $= \int \frac{dx}{\sqrt{3(x^2+\frac{4}{3}x+\frac{2}{3})}}$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{x^2+\frac{4}{3}x+\frac{4}{9}+\frac{2}{9}}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{t^2+\left(\frac{\sqrt{2}}{3}\right)^2}} \qquad [\, x+\tfrac{2}{3}=t \text{ (say)} \,]$$

$$= \frac{1}{\sqrt{3}} \log \left\{ t+\sqrt{t^2+\left(\frac{\sqrt{2}}{3}\right)^2} \right\}$$

$$= \frac{1}{\sqrt{3}} \log \left(x+\tfrac{2}{3}+\sqrt{x^2+\tfrac{4}{3}x+\tfrac{2}{3}} \right)$$

$$= \frac{1}{\sqrt{3}} \log \left\{ 3x+2+\sqrt{3\left(3x^2+4x+2\right)} \right\} + c$$

Ex. 22. $\int \frac{dz}{\sqrt{1+x+x^2}} = \int \frac{dx}{\sqrt{\frac{3}{4}+(x+\frac{1}{2})^2}}$

$$= \int \frac{dz}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+z^2}} \qquad [\, x+\tfrac{1}{2}=z \text{ (say)} \,]$$

$$=\log\left\{z+\sqrt{z^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}=\log\left(x+\tfrac{1}{2}+\sqrt{(x+\tfrac{1}{2})^2+\tfrac{3}{4}}\right)$$

$$=\log\left(\frac{2x+1}{2}+\sqrt{x^2+x+1}\right)=\log\left(\frac{2x+1+2\sqrt{x^2+x+1}}{2}\right)+c'$$

$$=\log\left(2x+1+2\sqrt{x^2+x+1}\right)-\log 2+c',$$

Now as log 2 is a constant,

$$\therefore\ \int\frac{dx}{\sqrt{1+x+x^2}}=\log\left(2x+1+2\sqrt{x^2+x+1}\right)+c.$$

Ex. 23. Integrate $\int\frac{2x+9}{\sqrt{x^2+x+1}}dx$

$\frac{d}{dx}(x^2+x+1)=2x+1$ and $2x+9=(2x+1)+8,$

Now, $\int\frac{2x+9}{\sqrt{x^2+x+1}}dx=\int\frac{2x+1}{\sqrt{x^2+x+1}}dx+\int\frac{8dx}{\sqrt{x^2+x+1}}$

$$=2\sqrt{x^2+x+1}+8\int\frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}+c_1$$

$$=2\sqrt{x^2+x+1}+8\log\left\{(x+\tfrac{1}{2})+\sqrt{(x+\tfrac{1}{2})^2+\tfrac{3}{4}}\right\}+c_1+c_2$$

$$=2\sqrt{x^2+x+1}+8\log\left\{\frac{2x+1}{2}+\sqrt{x^2+x+1}\right\}+c.$$

Ex. 24. Integrate $\int\frac{x+1}{\sqrt{x^2+1}}dx$

$$\int\frac{x+1}{\sqrt{x^2+1}}dx=\tfrac{1}{2}\int\frac{2x}{\sqrt{x^2+1}}dx+\int\frac{dx}{\sqrt{x^2+1}}$$

$$=\sqrt{x^2+1}+\log(x+\sqrt{x^2+1})+c.$$

Ex. 25. Integrate $\int\sqrt{\frac{1+x}{1-x}}dx.$ [C. U. 1925, '28, '59]

$$\int\sqrt{\frac{1+x}{1-x}}dx=\int\frac{1+x}{\sqrt{1-x^2}}dx=\int\frac{dx}{\sqrt{1-x^2}}+\int\frac{xdx}{\sqrt{1-x^2}}$$

$$=I_1+I_2\ \text{(say)}$$

Now, $I_1=\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c_1$

Also $I_2=-\int\frac{dz}{2\sqrt{z}}=-\sqrt{z}+c_2=-\sqrt{1-x^2}+c_2$ where $1-x^2=z$

$\therefore$ Given integral$=\mathrm{I}_1+\mathrm{I}_2=\sin^{-1}x+c_1-\sqrt{1-x^2}+c_2$

$$=\sin^{-1}x-\sqrt{1-x^2}+c \quad [c_1+c_2=c]$$

Ex. 26. Integrate : (*a*) $\int\frac{x^2}{x^2+4}dx$ [H. S. 1982]

(*b*) $\int\frac{1+2x^2}{x^2(1+x^2)}dx.$ [H. S. 1983]

(*a*) $\int\frac{x^2}{x^2+4}dx=\int\frac{x^2+4-4}{x^2+4}dx=\int dx-4\int\frac{dx}{x^2+2^2}$

$=x-\frac{4}{2}\tan^{-1}\frac{x}{2}+c=x-2\tan^{-1}\left(\frac{x}{2}\right)+c.$

(*b*) $\int\frac{1+2x^2}{x^2(1+x^2)}dx=\int\frac{1+x^2+x^2}{x^2(1+x^2)}dx$

$=\int\frac{1+x^2}{x^2(1+x^2)}dx+\int\frac{x^2}{x^2(1+x^2)}dx$

$=\int\frac{dx}{x^2}+\int\frac{dx}{1+x^2}=-\frac{1}{x}+\tan^{-1}x+c.$

Ex. 27. Integrate : (*a*) $\int\frac{dx}{\sqrt{x^2+x-2}}$, [Tripura '83]

(*b*) $\int\frac{dx}{\sqrt{2+x-x^2}}$. [H. S. '83] (*c*) $\int\frac{dx}{\sqrt{1-x-x^2}}$. [H. S. '81]

(*d*) $\int\frac{dx}{x^2-x+1}$. [Tripura '84] (*e*) $\int\frac{dx}{x\sqrt{1+x^2}}$. [Tripura '85]

(*f*) $\int\frac{dx}{\sqrt{2ax-x^2}}\,(x>a)$. [H. S. 1982 ; Joint Entrance 1983 ;
Tripura 1987]

(*a*) $\int\frac{dx}{\sqrt{x^2+x-2}}=\int\frac{dx}{\sqrt{(x^2+x+\frac{1}{4})-\frac{9}{4}}}=\int\frac{dx}{\sqrt{(x+\frac{1}{2})^2-(\frac{3}{2})^2}}$

$=\int\frac{dz}{\sqrt{z^2-(\frac{3}{2})^2}}$ [where $x+\frac{1}{2}=z$ and $dx=dz$]

$=\log\left(z+\sqrt{z^2-(\tfrac{3}{2})^2}\right)+c'$

$=\log\left(x+\frac{1}{2}+\sqrt{x^2+x-2}\right)+c'=\log\left(2x+1+2\sqrt{x^2+x-2}\right)+c$

(*b*) $\int\frac{dx}{\sqrt{2+x-x^2}}=\int\frac{dx}{\sqrt{\frac{9}{4}-(x^2-x+\frac{1}{4})}}=\int\frac{dx}{\sqrt{(\frac{3}{2})^2-(x-\frac{1}{2})^2}}$

$$=\int \frac{dz}{\sqrt{(\frac{3}{2})^2-z^2}} \quad [\text{ where } x-\tfrac{1}{2}=z \quad \therefore \quad dx=dz]$$

$$=\sin^{-1}\frac{z}{\frac{3}{2}}+c=\sin^{-1}\{\tfrac{2}{3}(x-\tfrac{1}{2})\}=\sin^{-1}\left(\frac{2x-1}{3}\right)+c.$$

(*c*) $$\int\frac{dx}{\sqrt{1-x-x^2}}=\int\frac{dx}{\sqrt{\frac{5}{4}-(x^2+x+\frac{1}{4})}}=\int\frac{dx}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-(x+\frac{1}{2})^2}}.$$

$$=\sin^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{5}}{2}}+c=\sin^{-1}\frac{2x+1}{\sqrt{5}}+c.$$

(*d*) $$\int\frac{dx}{x^2-x+1}=\int\frac{dx}{x^2-x+\frac{1}{4}+\frac{3}{4}}=\int\frac{dx}{(x-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}.$$

$$=\int\frac{dz}{z^2+\left(\frac{\sqrt{3}}{2}\right)^2} \quad [\text{ where } x-\tfrac{1}{2}=z \quad \text{or,} \quad dx=dz\ ]$$

$$=\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{z}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+c$$

(*e*) $$\int\frac{dx}{x\sqrt{1+x^2}}$$

Let, $x=\frac{1}{z}$ $\therefore$ $dx=-\frac{1}{z^2}\,dz$; $\sqrt{1+x^2}=\sqrt{1+\frac{1}{z^2}}$ $=\frac{\sqrt{1+z^2}}{z}$. $\therefore$ Integral$=-\int\frac{dz}{z^2.\frac{1}{z}.\frac{\sqrt{1+z^2}}{z}}$

$$=-\int\frac{dz}{\sqrt{1+z^2}}=-\log(z+\sqrt{1+z^2})+c$$

$$=-\log\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right)+c=-\log\frac{1+\sqrt{1+x^2}}{x}+c.$$

$$=\log\frac{x}{1+\sqrt{1+x^2}}+c$$

(*f*) $$\int\frac{dx}{\sqrt{2ax-x^2}}=\int\frac{dx}{\sqrt{a^2-(x^2-2ax+x^2)}}=\int\frac{dx}{\sqrt{a^2-(x-a)^2}}$$

$$=\sin^{-1}\left(\frac{x-a}{a}\right)+c.$$

Ex. 28. Integrate : $\int \frac{3x+1}{\sqrt{(2-3x-2x^2)}}dx$

$$\int \frac{3x+1}{\sqrt{(2-3x-2x^2)}}\,dx=-\tfrac{3}{4}\int \frac{-4x-3}{\sqrt{2-3x-2x^2}}dx-\tfrac{5}{4}\int \frac{dx}{\sqrt{2-3x-2x^2}}$$

$$=-\tfrac{3}{2}\sqrt{(2-3x-2x^2)}-\frac{5}{4\sqrt{2}}\int \frac{dx}{\sqrt{\{(\frac{5}{4})^2-(x+\frac{3}{4})^2\}}}+c_1$$

$$=-\tfrac{3}{2}\sqrt{(2-3x-2x^2)}-\frac{5}{4\sqrt{2}}\sin^{-1}\left(\frac{4x+3}{5}\right)+c.$$

Ex. 29. Integrate : $\int \frac{dx}{(x+1)\sqrt{x+2}}$

Let, $x+2=u^2$ $\therefore$ $dx=2u\,du$ and $x+1=u^2-1.$

$$\therefore \int \frac{dx}{(x+1)\sqrt{x+2}}=\int \frac{2u\,du}{(u^2-1)u}=2\int \frac{du}{u^2-1}$$

$$=\log \frac{u-1}{u+1}+c=\log \frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}+c.$$

Ex. 30. Integrate : $\int \frac{dx}{x\sqrt{x+1}}.$

Let, $x+1=t^2.$ $\therefore$ $dx=2t\,dt$ and $x=t^2-1$

$$\therefore \int \frac{dx}{x\sqrt{x+1}}=\int \frac{2t\,dt}{(t^2-1).t}=2\int \frac{dt}{t^2-1}.$$

$$=2.\tfrac{1}{2}\log \frac{t-1}{t+1}+c=\log \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}+c.$$

Ex. 31. Integrate : $\int \frac{dx}{(2-x)\sqrt{1-2x+3x^2}}$ [Punjab '60]

Let, $2-x=\frac{1}{t}.$ $\therefore$ $-dx=-\frac{1}{t^2}dt.$

or, $dx=\frac{dt}{t^2}$; $1-2x+3x^2=1-2\left(2-\frac{1}{t}\right)+3\left(2-\frac{1}{t}\right)^2$

$$=1-4+\frac{2}{t}+12-\frac{12}{t}+\frac{3}{t^2}=\frac{9t^2-10t+3}{t^2}$$

$$\therefore \int \frac{dx}{(2-x)\sqrt{1-2x+3x^2}}=\int \frac{dt}{t^2.\frac{1}{t}\ \frac{\sqrt{9t^2-10t+3}}{t}}$$

$$=\int\frac{dt}{\sqrt{9t^2-10t+3}}=\int\frac{dt}{3\sqrt{t^2-\frac{10}{9}t+\frac{1}{3}}}$$

$$=\tfrac{1}{3}\int\frac{dt}{\sqrt{(t-\frac{5}{9})^2+\left(\frac{\sqrt{2}}{9}\right)^2}}$$

$$=\tfrac{1}{3}\log\left\{(t-\tfrac{5}{9})+\sqrt{(t-\tfrac{5}{9})^2+\left(\frac{\sqrt{2}}{9}\right)^2}\right\}+c$$

$$=\tfrac{1}{3}\log\left\{\left(\frac{1}{2-x}-\frac{5}{9}\right)+\sqrt{t^2-\tfrac{10}{9}t+\tfrac{1}{3}}\right\}+c$$

$$=\tfrac{1}{3}\log\left\{\frac{5x-1}{9(2-x)}+\sqrt{\frac{1}{(2-x)^2}-\frac{10}{9(2-x)}+\tfrac{1}{3}}\right\}+c$$

$$=\tfrac{1}{3}\log\left\{\frac{5x-1}{9(2-x)}+\frac{\sqrt{1-2x+3x^2}}{3(2-x)}\right\}+c$$

Ex. 32. Integrate : (i) $\int\frac{\sqrt{x}}{x(x+1)}dx$ [H. S. 1986]

(ii) $\int\frac{(x+1)dx}{\sqrt{1-2x-x^2}}dx$ [H. S. 1987] (iii) $\int\frac{x(1-x^2)}{1+x^4}dx$ [H.S. 1988]

(iv) $\int\frac{dx}{(1+x)\sqrt{1-x^2}}$ [Joint Entrance 1979]

(v) $\int\frac{dx}{(x-3)\sqrt{2x^2-12x+17}}$ [Joint Entrance 1978]

(vi) $\int\frac{dx}{(e^x-1)^{\frac{1}{2}}}$ [H. S. 1987]

(i) $\int\frac{\sqrt{x}}{x(x+1)}dx.$

Let, $x=z^2$ $\therefore$ $dx=2zdz$

$\therefore$ Given integral$=\int\frac{\sqrt{x}}{x(x+1)}dx=\int\frac{z.2zdz}{z^2(z^2+1)}$

$$=2\int\frac{dz}{z^2+1}=2\tan^{-1}z+c=2\tan^{-1}(\sqrt{x})+c.$$

(ii) $\int\frac{(x+1)dx}{\sqrt{1-2x-x^2}}=-\frac{1}{2}\int\frac{-2x-2}{\sqrt{1-2x-x^2}}dx$

Let, $1-2x-x^2=z^2$ $\therefore$ $(-2-2x)dx=2zdz$

$\therefore$ Given integral$=-\frac{1}{2}\int\frac{2zdz}{z}=-\int dz=-z+c$

$=-\sqrt{1-2x-x^2}+c.$

(iii) $\int\frac{x(1-x^2)}{1+x^4}dx=\int\frac{xdx}{1+x^4}-\int\frac{x^3\,dx}{1+x^4}$

Now for $\int\frac{xdx}{1+x^4}$ let, $x^2=z$ $\therefore$ $2xdx=dz$

$\therefore$ $\int\frac{xdx}{1+x^4}=\frac{1}{2}\int\frac{dz}{1+z^2}=\frac{1}{2}\tan^{-1}z+c_1=\frac{1}{2}\tan^{-1}(x^2)+c_1$

$\int\frac{x^3}{1+x^4}\,dx$ for let, $1+x^4=z$

$\therefore$ $4x^3dx=dz$ or, $x^3dx=\frac{dz}{4}$

$\therefore$ Integral$=\int\frac{dz}{4z}=\frac{1}{4}\log z+c_2=\frac{1}{4}\log(1+x^4)+c_2.$

So, given integral$=\frac{1}{2}\tan^{-1}x^2-\frac{1}{4}\log(1+x^4)+c$

(iv) $\int\frac{dx}{(1+x)\sqrt{1-x^2}}$

Let, $x=\cos\theta$ $\therefore$ $dx=-\sin\theta\,d\theta$

$\therefore$ Given integral$=\int\frac{-\sin\theta\,d\theta}{(1+\cos\theta)\sqrt{1-\cos^2\theta}}$

$$=-\int\frac{\sin\theta\,d\theta}{2\cos^2\frac{\theta}{2}\sin\theta}=-\int\frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta=-\tan\frac{\theta}{2}+c$$

$$=-\sqrt{\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}+c=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}+c=-\sqrt{\frac{1-x}{1+x}}+c$$

(v) $\int\frac{dx}{(x-3)\sqrt{2x^2-12x+17}}$

Let, $x-3=\frac{1}{z}$ $\therefore$ $dx=-\frac{1}{z^2}dz.$

$x=\frac{1}{z}+3$ and $2x^2-12x+17=2\left(\frac{1}{z}+3\right)^2-12\left(\frac{1}{z}+3\right)+17$

$$=2\left(\frac{1+6z+9z^2}{z^2}\right)-\frac{12(1+3z)}{z}+17$$

$$=\frac{2+12z+18z^2-12z-36z^2+17z^2}{z^2}=\frac{2-z^2}{z^2}$$

$\therefore$ Given integral$=-\displaystyle\int\frac{dz}{z^2.\,\frac{1}{z}\sqrt{\frac{(2-z^2)}{z^2}}}=-\int\frac{dz}{\sqrt{2-z^2}}$

$$=-\sin^{-1}\left(\frac{z}{\sqrt2}\right)+c_1=\cos^{-1}\sqrt{\frac{1}{(x-3)}\cdot\frac{1}{\sqrt2}}+c$$

(vi) $\displaystyle\int\frac{dx}{(e^x-1)^{\frac{1}{2}}}$

Let, $e^x-1=z^2$ $\quad\therefore\quad e^x dx=2zdz \quad\therefore\quad dx=\dfrac{2zdz}{e^x}$

$\therefore$ Given integral$=\displaystyle\int\frac{2zdz}{e^x.(z^2)^{\frac{1}{2}}}=\int\frac{2zdz}{(z^2+1).z}=2\int\frac{dz}{z^2+1}$

$$=2\tan^{-1}z+c=2\tan^{-1}\sqrt{e^x-1}+c.$$

Ex. 33. Integrate :

(i) $\displaystyle\int\frac{x^2+\cos^2x}{x^2+1}\operatorname{cosec}^2x\,dx$ (ii) $\displaystyle\int\left(\frac{x-1}{x^2+1}\right)^2dx$

(iii) $\displaystyle\int\frac{x^3}{(1+x^2)^3}dx$ (iv) $\displaystyle\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}$ (v) $\displaystyle\int\frac{\sqrt{x^2+1}}{x^2}dx$

(i) $\displaystyle\int\frac{x^2+\cos^2x}{x^2+1}\operatorname{cosec}^2x\,dx$

$$=\int\frac{x^2+1-\sin^2x}{x^2+1}\operatorname{cosec}^2x\,dx=\int\frac{(x^2+1)\operatorname{cosec}^2x}{x^2+1}dx$$

$$-\int\frac{\sin^2x\operatorname{cosec}^2x\,dx}{x^2+1}$$

$$=\int\operatorname{cosec}^2x\,dx-\int\frac{dx}{x^2+1}=-\cot x-\tan^{-1}x+c.$$

(ii) $\displaystyle\int\left(\frac{x-1}{x^2+1}\right)^2dx=\int\frac{x^2-2x+1}{(x^2+1)^2}dx=\int\frac{x^2+1}{(x^2+1)^2}dx-\int\frac{2x\,dx}{(x^2+1)^2}$

$$=\int\frac{dx}{x^2+1}-\int\frac{2x\,dx}{(x^2+1)^2}=\mathrm{I}_1-\mathrm{I}_2\text{ (say)}$$

Now, $I_1=\int\frac{dx}{x^2+1}=\tan^{-1}x+c_1$

For I_2 let, $x^2+1=z$ $\therefore$ $2x\,dx=dz$

$\therefore$ $I_2=\int\frac{dz}{z^2}=-\frac{1}{z}+c_2=-\frac{1}{x^2+1}+c_2$

$\therefore$ Integral $=\tan^{-1}x+c_1-\left(-\frac{1}{x^2+1}+c_2\right)$

$=\tan^{-1}x+\frac{1}{x^2+1}+c.$

(iii) $\int\frac{x^3dx}{(1+x^2)^3}$

Let, $1+x^2=z$ $\therefore$ $2x\,dx=dz$ or, $x\,dx=\frac{1}{2}\,dz.$

Now; $\int\frac{x^3\,dx}{(1+x^2)^3}=\int\frac{x^2\,x\,dx.}{(1+x^2)^3}=\int\frac{(z-1)dz}{2.z^3}$

$=\frac{1}{2}\int\frac{dz}{z^2}-\frac{1}{2}\int\frac{dz}{z^3}=-\frac{1}{2z}+\frac{1}{4z^2}+c$

$=-\frac{1}{2(1+x^2)}+\frac{1}{4(1+x^2)^2}+c$

(iv) $\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Let, $x=\sin\theta$ $\therefore$ $dx=\cos\theta\,d\theta.$

$\therefore$ Given integral $=\int\frac{\cos\theta\,d\theta}{(1+\sin^2\theta)\cos\theta}d\theta=\int\frac{d\theta}{1+\sin^2\theta}$

$=\int\frac{\sec^2\theta\,d\theta}{\sec^2\theta+\tan^2\theta}$

[Multiplying both numerator and denominator by $\sec^2\theta$]

$=\int\frac{\sec^2\theta\,d\theta}{1+2\tan^2\theta}=\int\frac{dt}{1+2t^2}$ [where $\tan\theta=t$ and $\sec^2\theta\,d\theta=dt$]

$=\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}t)+c=\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan\theta)+c$

$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)+c$

$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1-x^2}}\right)+c.$

(v) $\int \frac{\sqrt{x^2+1}}{x^2}dx=\int \frac{x^2+1}{x^2\sqrt{x^2+1}}dx$

$=\int \frac{dx}{\sqrt{x^2+1}}+\int \frac{1}{x^2\sqrt{x^2+1}}=\mathrm{I}_1+\mathrm{I}_2$ (say)

Here, $\mathrm{I}_1=\int \frac{dx}{\sqrt{x^2+1}}=\log(x+\sqrt{x^2+1})+c_1$

For I_2 let, $x=\tan\theta \quad \therefore \quad dx=\sec^2\theta\, d\theta$

$\sqrt{x^2+1}=\sqrt{1+\tan^2\theta}=\sec\theta\, d\theta$

$\therefore \quad \mathrm{I}_2=\int \frac{\sec^2\theta\, d\theta}{\tan^2\theta \sec\theta}=\int \operatorname{cosec}\theta \cot\theta\, d\theta=-\operatorname{cosec}\theta+c_2$

$=-\sqrt{1+\cot^2\theta}+c=-\sqrt{1+\frac{1}{x^2}}+c_2=-\frac{\sqrt{x^2+1}}{x}+c_2$

$\therefore$ Integral$=\log(x+\sqrt{x^2+1})-\frac{\sqrt{x^2+1}}{x}+c.$

Exercise 2B

Integrate :

1. (i) $\int \frac{dx}{x^2+9}$ (ii) $\int \frac{dx}{a^2+b^2x^2}$ (iii) $\int \frac{2x\,dx}{x^4+16}$

(iv) $\int \frac{\sec^2 x\,dx}{\sec^2 x+3}$ (v) $\int \frac{e^{2x}dx}{e^{4x}+4}$ (vi) $\int \frac{\cos x\,dx}{\sin^2 x+4}$

(vii) $\int \frac{dx}{(1+x^2)\{(1+(\tan^{-1}x)^2\}}$ (viii) $\int \frac{dx}{x\{3+(\log x)^2\}}$

(ix) $\int \frac{dx}{e^x+e^{-x}}$. [C. U. '58]

2. (i) $\int \frac{dx}{x^2-2}\ (x>\sqrt{2})$ (ii) $\int \frac{dx}{1-x^2}\ (x<1)$

(iii) $\int \frac{dx}{x^2-2ax}\ (x>2a)$ (iv) $\int \frac{dx}{2ax-x^2}\ (x<2a)$

(v) $\int \frac{dx}{x\{1-(\log x)^2\}}$ (vi) $\int \frac{e^{2x}dx}{1-e^{4x}}$

(vii) $\int \frac{\sec^2\theta}{\sec^2\theta-2}d\theta$ (viii) $\int \operatorname{cosec}\theta\, d\theta$

3. (i) $\int \frac{dx}{\sqrt{x^2+9}}$ (ii) $\int \frac{dx}{\sqrt{a^2+b^2x^2}}$ (iii) $\int \frac{2x}{\sqrt{1+x^4}}dx$

(iv) $\int \frac{dx}{\sqrt{a^2-b^2x^2}}$ (v) $\int \frac{dx}{(1+x^2)\sqrt{1-(\tan^{-1}x)^2}}$

(vi) $\int \frac{\sec^2 x\, dx}{\sqrt{1-\tan^2 x}}$

4. (i) $\int \frac{dx}{x^2+x-12}$ (ii) $\int \frac{dx}{3x^2+13x+14}$ (iii) $\int \frac{dx}{x^2+4x+5}$

(iv) $\int \frac{dx}{1-x-x^2}$ (v) $\int \frac{\cos x\, dx}{\sin^2 x+2\sin x+5}$

(vi) $\int \frac{dx}{x\{(\log x)^2+\log x+1\}}$ (vii) $\int \frac{x\, dx}{x^4+4x^2+3}$

(viii) $\int \frac{e^x\, dx}{2+3e^x-2e^{2x}}$ (ix) $\int \frac{\cos x\, dx}{5\sin^2 x-12\sin x+4}$ [C. U. '67]

5. (i) $\int \frac{x+1}{x^2+4x+5}dx$ [C. U. 1926, '28]

(ii) $\int \frac{x^2}{x^2-4}dx$ [C. U. 1935] (iii) $\int \frac{2x-1}{x^2+2x+3}dx$

(iv) $\int \frac{2x-3}{1-x-x^2}dx$ (v) $\int \frac{(1-x)\, dx}{4x^2-4x-3}$

6. (i) $\int \frac{dx}{\sqrt{x^2-5x+6}}$ (ii) $\int \frac{dx}{\sqrt{2+8x-3x^2}}$

7. (i) $\int \frac{dx}{\sqrt{x^2+2x+6}}$ (ii) $\int \frac{dx}{\sqrt{(5x-x^2-6)}}$

(iii) $\int \frac{dx}{\sqrt{2x^2+3x+4}}$ [P. P. 1932] (iv) $\int \frac{dx}{\sqrt{1+x-x^2}}$

(v) $\int \frac{dx}{\sqrt{2+x-3x^2}}$ [Gorakhpur, '63]

(vi) $\int \frac{dx}{\sqrt{3x^2-x-3}}$ [Agra, '61]

(vii) $\int \frac{dx}{\sqrt{x^2+2x+5}}$ [Agra, '49]

(viii) $\int \frac{\sec^2 x dx}{\sqrt{5\tan^2 x-12\tan x+4}}$

(ix) $\int \frac{dx}{x\sqrt{(\log x)^2+2\log x+5}}$

8. (i) $\int \frac{2x+3}{\sqrt{x^2+x+1}}$ [C. U. '28 ; B. U. '45]

(ii) $\int \frac{x-2}{\sqrt{2x^2-8x+5}}$ [C. U. '26]

(iii) $\int \frac{(2x+5)dx}{\sqrt{(x^2+3x+1)}}$

(iv) $\int \frac{xdx}{\sqrt{x^2+x+1}}$ [Nagpur '52]

(v) $\int \frac{x+3}{\sqrt{x^2+2x+2}}dx$ [Poona '63]

(vi) $\int \frac{5-6x}{\sqrt{1+2x-3x^2}}dx$ (vii) $\int \frac{(2x+5)dx}{\sqrt{3x-x^2-2}}$

9. (i) $\int \frac{dx}{(2+x)\sqrt{1+x}}$

(ii) $\int \frac{dx}{(2x+1)\sqrt{3x+4}}$ (iii) $\int \frac{dx}{(x+2)\sqrt{x+3}}$ [C. U. '41]

10. (i) $\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$ [Andhra '63]

(ii) $\int \frac{dx}{(1+2x)\sqrt{1+x^2}}$ (iii) $\int \frac{dx}{x\sqrt{x^2+x+1}}$ [Poona '63]

(iv) $\int \frac{dx}{(x-a)\sqrt{x^2-a^2}}$ (v) $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

(vi) $\int \frac{dx}{\sqrt{\frac{2}{3}x^3-x^2+\frac{1}{3}}}$

11. $\int \sqrt{\frac{x-3}{x-4}}dx.$ **12.** $\int \frac{x^2-4}{x^2+4}dx.$ **13.** $\int \frac{x^2dx}{x^2+1}.$

14. $\int \frac{\sqrt{a^2-x^2}}{x^2}dx.$ **15.** $\int \sqrt{\frac{x}{a-x}}dx.$

16. $\int \frac{dx}{(1-x)\sqrt{1+x^2}}.$ **17.** $\int \frac{dx}{(x^2+1)\sqrt{x^2+4}}$

18. $\int \frac{\sqrt{x}dx}{(x+1)(x+2)}$ **19.** $\int \frac{dx}{(1+x)^{\frac{1}{2}}+(1+x)^{\frac{3}{2}}}$.

20. $\int \frac{dx}{(x^2-4)\sqrt{x^2-1}}$. **21.** $\int \frac{dx}{x\sqrt{x^4-1}}$

§ 2·12. Some special devices for integration of trigonometric functions.

(A) $\int \sin^m x \cos^n x \, dx$.

(i) If m be a positive odd integer and n be any quantity, put $\cos x=z$.

Ex. $\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$.

Let, $\cos x=z \quad \therefore \quad -\sin x \, dx=dz$ or, $\sin x \, dx=-dz$

$\therefore$ Given integral$=\int (1-\cos^2 x)\cos^2 x \sin x \, dx$

$$=-\int (1-z^2)z^2 \, dz=-\int z^2 \, dz+\int z^4 \, dz$$

$$=-\frac{z^3}{3}+\frac{z^5}{5}+c=-\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+c.$$

(ii) If n be a positive odd integer and m be any quantity, put $\sin x=z$.

Ex. $\int \cos^7 x \, dx=\int \cos^6 x \cos x \, dx$

$=\int (1-\sin^2 x)^3 \cos x \, dx=\int (1-z^2)^3 \, dz$

[Where $\sin x=z \quad \therefore \quad \cos x \, dx=dz$]

$=\int (1-3z^2+3z^4-z^6) \, dz$

$=z-z^3+\frac{3}{5}z^5-\frac{z^7}{7}+c=\sin x-\sin^3 x+\frac{3}{5}\sin^5 x-\frac{\sin^7 x}{7}+c.$

(iii) If m and n be both positive even or both positive odd integers, then express the integrand into multiple angles.

Ex. $\int \sin^4 x \cos^4 x \, dx=\frac{1}{16}\int (2\sin x \cos x)^4 \, dx$

$=\frac{1}{16}\int \sin^2 2x \, dx=\frac{1}{64}\int (2\sin^2 2x)^2 \, dx$

$=\frac{1}{64}\int (1-\cos 4x)^2 \, dx=\frac{1}{64}\int (1-2\cos 4x+\cos^2 4x) \, dx$

$=\frac{1}{64}\int dx-\frac{1}{32}\int \cos 4x \, dx+\frac{1}{128}\int (1+\cos 8x) \, dx$

$=\frac{1}{64}x-\frac{1}{128}\sin 4x+\frac{1}{128}x+\frac{1}{1024}\sin 8x+c$

$=\frac{3}{128}x-\frac{1}{128}\sin 4x+\frac{1}{1024}\sin 8x+c.$

Ex. $\int \sin^3 x \cos^3 x \, dx$

$=\frac{1}{8}\int (2\sin x \cos x)^3 \, dx=\frac{1}{8}\int \sin^3 2x \, dx$

$=\frac{1}{32}\int (3\sin 2x-\sin 6x)\, dx=-\frac{3}{64}\cos 2x+\frac{1}{192}\cos 6x+c.$

(iv) If for any values of m and n, $m+n$ be negative even integer, then put $\tan x=z$ or $\cot x=z$.

Ex. $\int \frac{\sin^2 x}{\cos^4 x} dx$

Here, $m=2$, $n=-4$ $\therefore$ $m+n=2-4=-2$, a negative even integer.

Now given integral $=\int \frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\cos^2 x} dx$

$=\int \tan^2 x \sec^2 x \, dx=\int z^2 \, dz$

[$\tan x=z$ (say) $\therefore$ $\sec^2 x \, dx=dz$]

$=\frac{z^3}{3}+c=\frac{\tan^3 x}{3}+c$

Ex. $\int \frac{\cos^{\frac{1}{2}} x}{\sin^{\frac{5}{2}} x} dx$

Here $m=-\frac{5}{2}$, $n=\frac{1}{2}$ $\therefore$ $m+n=-\frac{5}{2}+\frac{1}{2}=-2$, which is a negative even integer.

Now, $\int \frac{\cos^{\frac{1}{2}} x}{\sin^{\frac{5}{2}} x} dx=\int \cot^{\frac{1}{2}} x \operatorname{cosec}^2 x \, dx$

$=-\int z^{\frac{1}{2}} dz$ [Let $\cot x=z$ $\therefore$ $-\operatorname{cosec}^2 x \, dx=dz$]

$=-\frac{2}{3} z^{\frac{3}{2}}+c=\frac{2}{3}\cot^{\frac{3}{2}} x+c.$

(B) (i) $\int \sec^n x \, dx$ or $\int \operatorname{cosec}^n x \, dx$ [where n is an even integer]

In these two cases take $\tan x=z$ or $\cot x=z$ respectively.

(ii) If n be an even integer, for the integrals

$\int \tan^n x \, dx$ or $\int \cot^n x \, dx$ use the two relations $\tan^2 x=(\sec^2 x-1)$ or $\cot^2 x=(\operatorname{cosec}^2 x-1)$ respectively.

(iii) If n be an odd integer ($\neq 1$), for the integrals $\int \tan^n x \, dx$ and $\int \cot^n x \, dx$ put $\sec x=z$ and $\operatorname{cosec} x=z$ respectively.

(iv) If $(m+n)$ be an even integer, put $\tan x=z$ for the integration of $\int \sec^m x \operatorname{cosec}^m x\, dx$.

[**Note.** For the integration of $\int \sec^n x\, dx$ or $\int \operatorname{cosec}^n x\, dx$, when $n \neq 1$ is an odd integer, we have to use the formula of integration by parts as discussed in chapter three.]

(C) For integration of integrals of the form $\int \frac{dx}{a+b \cos x}$, $\int \frac{dx}{a+b \sin x}$, $\int \frac{dx}{a+b \cos x+c \sin x}$ first use the formulas $\cos x=\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$ and $\sin x=\frac{2 \tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$ and then put $\tan \frac{x}{2}=z$.

(D) For the integrals of the form $\int \frac{dx}{a \cos^2 x+b \sin^2 x}$ multiply both the numerator and denominator by $\sec^2 x$ and then put $\tan x=z$.

(E) For the integrals of the forms $\int \frac{dx}{a \cos x+b \sin x}$ and $\int \frac{dx}{(a \cos x+b \sin x)^2}$, express $a \cos x+b \sin x$ in the forms $r \sin (\theta+\alpha)$ or $r \cos (\theta-\alpha)$.

(F) For the integration of integrals of the form $\int \frac{a \sin x+b \cos x}{c \sin x+d \cos x} dx$, express the numerator as $l \times$(denominator)$+m \times$(derivative of the denominator).

Examples 2C

Ex. 1. Integrate :— (i) $\int \sin^3 x \cos^4 x\, dx$ [Tripura, '87]

(ii) $\int \cos^5 x\, dx$ [H. S. 1981] (iii) $\int \sin^4 x\, dx$

(iv) $\int \sqrt{\sin x} \cos^3 x\, dx$ (v) $\int \frac{dx}{\sin^{\frac{1}{2}} x \cos^{\frac{7}{2}} x}$

(vi) $\int \frac{dx}{\sqrt{(\sin^5 x \cos^7 x)}}$ (vii) $\int \sin^6 x \cos^3 x\, dx$

(viii) $\int \sin^2 x \cos^2 x\, dx$.

(i) Let $\cos x=z$. $\therefore -\sin x\, dx=dz$

So, $\int \sin^3 x \cos^4 x \, dx$

$=\int (1-\cos^2 x) \cos^4 x \, (\sin x \, dx)=-\int (1-z^2)z^4 \, dz$

$=-\int z^4 \, dz+\int z^6 \, dz=\dfrac{z^7}{7}-\dfrac{z^5}{5}+c$

$=\dfrac{\cos^7 x}{7}-\dfrac{\cos^5 x}{5}+c.$

(ii) $\int \cos^5 x \, dx=\int \cos^4 x . \cos x \, dx$

$=\int (1-\sin^2 x)^2 \, d(\sin x)=\int (1-z^2)^2 \, dz$ [where $\sin x=z$]

$=\int (1-2z^2+z^4) \, dz=z-\frac{2}{3}z^3+\frac{1}{5}z^5+c$

$=\sin x-\frac{2}{3} \sin^3 x+\frac{1}{5} \sin^5 x+c.$

(iii) $\int \sin^4 x \, dx=\frac{1}{4}\int (1-\cos 2x)^2 \, dx$

$=\frac{1}{4} \int (1-2 \cos 2x+\cos^2 2x) \, dx$

$=\frac{1}{4} \int dx-\frac{1}{2} \int \cos 2x \, dx+\frac{1}{8} \int 2 \cos^2 2x \, dx$

$=\frac{1}{4} \int dx-\frac{1}{2} \int \cos 2x \, dx+\frac{1}{8} \int (1+\cos 4x) \, dx$

$=\frac{1}{4}x-\frac{1}{4} \sin 2x+\frac{1}{8}x+\dfrac{\sin 4x}{32}+c$

$=\frac{3}{8}x-\frac{1}{4} \sin 2x+\frac{1}{32} \sin 4x+c.$

(iv) $\int \sqrt{\sin x} \cos^3 x \, dx=\int \sqrt{\sin x} \cos^2 x . \cos x \, dx$

$=\int \sin^{\frac{1}{2}} x . (1-\sin^2 x) d(\sin x)$

$=\int \sin^{\frac{1}{2}} x \, d(\sin x)-\int \sin^{\frac{5}{2}} x \, d(\sin x)$

$=\int z^{\frac{1}{2}} dz-\int z^{\frac{5}{2}} dz$ [where $\sin x=z$]

$=\frac{2}{3}z^{\frac{3}{2}}-\frac{2}{7}z^{\frac{7}{2}}+c$

$=\frac{2}{3} \sin^{\frac{3}{2}} x-\frac{2}{7} \sin^{\frac{7}{2}} x+c.$

(v) $\displaystyle\int \frac{dx}{\sin^{\frac{1}{2}} x \cos^{\frac{7}{2}} x}=\int \frac{dx}{\dfrac{\sin^{\frac{1}{2}} x}{\cos^{\frac{1}{2}} x}\cos^4 x} dx=\int \frac{\sec^4 x \, dx}{\tan^{\frac{1}{2}} x}$

$\displaystyle=\int \frac{(1+\tan^2 x) \sec^2 x}{\tan^{\frac{1}{2}} x} dx.$

Now, let $\tan x=z$ $\quad\therefore\quad \sec^2 x \, dx=dz$

So given integral $=\int \frac{1+z^2}{z^{\frac{1}{2}}}dz=\int z^{-\frac{1}{2}}dz+\int z^{\frac{3}{2}}dz$

$=2z^{\frac{1}{2}}+\frac{2}{5}z^{\frac{5}{2}}+c=2\sqrt{\tan x}+\frac{2}{5}\tan^{\frac{5}{2}}x+c.$

(vi) $\int \frac{dx}{\sqrt{(\sin^5 x\cos^7 x)}}=\int \frac{dx}{\sin^{\frac{5}{2}}x\cos^{\frac{7}{2}}x}$

$$=\int \frac{dx}{\frac{\sin^{\frac{5}{2}}x}{\cos^{\frac{5}{2}}x}.\cos^{\frac{7}{2}}x.\cos^{\frac{5}{2}}x}=\int \frac{dx}{\tan^{\frac{5}{2}}x\cos^6 x}$$

$$=\int \frac{\sec^6 x\,dx}{\tan^{\frac{5}{2}}x}=\int \frac{(1+\tan^2 x)^2\sec^2 x\,dx}{\tan^{\frac{5}{2}}x}$$

$=\int \frac{(1+t^2)^2\,dt}{t^{\frac{5}{2}}}$ [where $t=\tan x$ $\therefore$ $dt=\sec^2 x\,dx$]

$=\int \frac{1+2t^2+t^4}{t^{\frac{5}{2}}}dt=\int t^{-\frac{5}{2}}dt+2\int t^{-\frac{1}{2}}dt+\int t^{\frac{3}{2}}dt$

$=-\frac{2}{3}t^{-\frac{3}{2}}+4t^{\frac{1}{2}}+\frac{2}{5}t^{\frac{5}{2}}+c$

$=-\frac{2}{3}\frac{1}{\tan x\sqrt{\tan x}}+4\sqrt{\tan x}+\frac{2}{5}(\tan^2 x\sqrt{\tan x}+c).$

(vii) $\int \sin^6 x\cos^3 x\,dx=\int \sin^6 x\cos^2 x\cos x\,dx$

$=\int \sin^6 x(1-\sin^2 x)\,d(\sin x)=\int z^6(1-z^2)dz$

[where $\sin x=z$] $=\int z^6dz-\int z^8dz$

$=\frac{z^7}{7}-\frac{z^9}{9}+c=\frac{\sin^7 x}{7}-\frac{\sin^9 x}{9}+c.$

(viii) $\int \sin^2 x\cos^2 x\,dx=\frac{1}{4}\int \sin^2 2x\,dx$

$=\frac{1}{8}\int(1-\cos 4x)dx=\frac{1}{8}\int dx-\frac{1}{8}\int\cos 4x\,dx$

$=\frac{1}{8}x-\frac{1}{32}\sin 4x+c.$

Ex. 2. Integrate :

(*a*) $\int \sin^4 x\cos^3 x\,dx$, (*b*) $\int \sin^5 x\cos^5 x\,dx$ (*c*) $\int \sin^4 x\cos^6 x\,dx$

(*d*) $\int \frac{\cos^2 x}{\sin^6 x}dx$ (*e*) $\int \frac{dx}{\sin^{\frac{5}{2}}x\cos^{\frac{3}{2}}x}$

(*a*) $\int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \cos^2 x \cos x \, dx$

$= \int \sin^4 x \, (1 - \sin^2 x) d\,(\sin x)$

$= \int z^4 \, (1 - z^2) dz$ [where $\sin x = z$]

$= \int z^4 dz - \int z^6 dz = \frac{z^5}{5} - \frac{z^7}{7} + c = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$

(*b*) $\int \sin^5 x \cos^5 x \, dx = \frac{1}{32} \int (2 \sin x \cos x)^5 \, dx$

$= \frac{1}{32} \int \sin^5 2x \, dx = \frac{1}{64} \int \sin^4 2x \; 2 \sin 2x \, dx$

$= -\frac{1}{64} \int (1 - \cos^2 2x)^2 \, d(\cos 2x)$

$= -\frac{1}{64} \int (1 - z^2)^2 \, dz$ where $\cos 2x = z$

$= -\frac{1}{64} \int (1 - 2z^2 + z^4) \, dz = -\frac{1}{64} z + \frac{1}{96} z^3 - \frac{1}{320} z^5 + c$

$= -\frac{1}{64} \cos 2x + \frac{1}{96} \cos^3 2x - \frac{1}{320} \cos^5 2x + c.$

(*c*) $\int \sin^4 x \cos^6 x \, dx$

$= \int \sin^4 x \cos^4 x \cos^2 x \, dx = \frac{1}{32} \int 16 \sin^4 x \cos^4 x \; 2 \cos^2 x \, dx$

$= \frac{1}{32} \int \sin^4 2x (1 + \cos 2x) \, dx$

$= \frac{1}{32} \int \sin^4 2x \, dx + \frac{1}{32} \int \sin^4 2x \cos 2x \, dx$

$= \frac{1}{128} \int 4 \sin^4 2x \, dx + \frac{1}{64} \int \sin^4 2x (2 \cos 2x) \, dx$

$= \frac{1}{128} \int (2 \sin^2 2x)^2 \, dx + \frac{1}{64} \int \sin^4 2x d(\sin 2x)$

$= \frac{1}{128} \int (1 - \cos 4x)^2 \, dx + \frac{1}{64} \int \sin^4 2x d(\sin 2x)$

$= \frac{1}{128} \int dx - \frac{1}{64} \int \cos 4x \, dx + \frac{1}{128} \int \cos^2 4x \, dx$

$+ \frac{1}{64} \int \sin^4 2x d(\sin 2x)$

$= \frac{1}{128} \int dx - \frac{1}{64} \int \cos 4x \, dx + \frac{1}{256} \int (1 + \cos 8x) \, dx$

$+ \frac{1}{64} \int \sin^4 2x d(\sin 2x)$

$= \frac{1}{128} x - \frac{1}{256} \sin 4x + \frac{1}{256} x + \frac{1}{2048} \sin 8x + \frac{1}{64} \cdot \frac{1}{5} \sin^5 2x + c$

$= \frac{3}{256} x - \frac{1}{256} \sin 4x + \frac{1}{2048} \sin 8x + \frac{1}{320} \sin^5 2x + c.$

(*d*) $\int \frac{\cos^2 x}{\sin^6 x} dx = \int \cot^2 x \operatorname{cosec}^4 \, dx = \int \cot^2 x \, (\cot^2 x + 1) \operatorname{cosec}^2 x \, dx$

$= \int \cot^4 x \operatorname{cosec}^2 x \, dx + \int \cot^2 x \operatorname{cosec}^2 x \, dx$

$= -\int z^4 dz - \int z^2 dz$ [where $\cot x = z$ and $-\operatorname{cosec}^2 x \, dx = dz$]

$= -\frac{z^5}{5} - \frac{z^3}{3} + c = -\frac{\cot^5 x}{5} - \frac{\cot^3 x}{3} + c.$

(*e*) $\int \frac{dx}{\sin^{\frac{5}{2}} x \cos^{\frac{3}{2}} x} = \int \frac{dx}{\dfrac{\sin^{\frac{5}{2}} x}{\cos^{\frac{5}{2}} x} \cos^4 x \, dx}$

$$=\int\frac{\sec^4 x\,dx}{\tan^{\frac{5}{2}}x}=\int\frac{(1+\tan^2 x)d\,(\tan x)}{\tan^{\frac{5}{2}}x}$$

$$=\int\frac{(1+t^2)dt}{t^{\frac{5}{2}}}\quad[\text{ where }\tan x=t\ ]$$

$$=\int t^{-\frac{5}{2}}dt+\int t^{-\frac{1}{2}}\,dt=-\tfrac{2}{3}t^{-\frac{3}{2}}+2t^{\frac{1}{2}}+c$$

$$=-\tfrac{2}{3}\ \frac{1}{\tan x\sqrt{\tan x}}+2\sqrt{\tan x}+c=2\sqrt{\tan x}\left(1-\frac{1}{3\tan^2 x}\right)+c.$$

Ex. 3. Integrate : (i) $\int\sec^4 dx$ (ii) $\int\operatorname{cosec}^6 x\,dx$

(iii) $\int\tan^3 x\,dx$

(i) $\int\sec^4 x\,dx=\int\sec^2 x\sec^2 x\,dx$

$=\int(1+\tan^2 x)\,d\,(\tan x)=\int(1+t^2)\,dt$ [where $\tan x=t$]

$$=t+\frac{t^3}{3}+c=\tan x+\frac{\tan^3 x}{3}+c$$

(ii) $\int\operatorname{cosec}^6 x\,dx=\int\operatorname{cosec}^4 x\operatorname{cosec}^2 x\,dx$

$=-\int(1+\cot^2 x)^2\,d\,(\cot x)=-\int(1+z^2)^2\,dz$ [where $\cot x=z$]

$=-\int(1+2z^2+z^4)\,dz=-\{\int dz+2\int z^2dz+\int z^4dz\}$

$$=-\left(z+\tfrac{2}{3}z^3+\frac{z^5}{5}\right)+c=-(\cot x+\tfrac{2}{3}\cot^3 x+\tfrac{1}{5}\cot^5 x)+c.$$

(iii) $\int\tan^3 x\,dx=\int\tan x.(\sec^2 x-1)\,dx$

$=\int\tan x\sec^2 x\,dx-\int\tan x\,dx$

$=\dfrac{\tan^2 x}{2}+\log\cos x+c$ [For $\int\tan x\sec^2 x\,dx$, take $\tan x=z$

$\therefore\ \sec^2 x\,dx=dz\quad\therefore\ \int\tan x\sec^2 x\,dx$

$=\int zdz=\dfrac{z^2}{2}+c_1=\dfrac{\tan^2 x}{2}+c_1$]

Ex. 4. Integrate : (*a*) $\displaystyle\int\frac{dx}{3+5\cos x}$ [Tripura, 1986]

(*b*) $\displaystyle\int\frac{dx}{5+4\cos x}$ [C. U. '74] (*c*) $\displaystyle\int\frac{dx}{3+2\sin x+\cos x}$ [C.U. '67]

(*a*) $$\int\frac{dx}{3+5\cos x}=\int\frac{dx}{3+5\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$=\int\frac{(1+\tan^2\frac{x}{2})\,dx}{8-2\tan^2\frac{x}{2}}=\int\frac{\frac{1}{2}\sec^2\frac{x}{2}\,dx}{4-\tan^2\frac{x}{2}}$$

$$=\int\frac{dz}{4-z^2}=\frac{1}{4}\log\left|\frac{2+z}{2-z}\right|+c=\frac{1}{4}\log\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right|+c.$$

(*b*) $\displaystyle\int\frac{dx}{5+4\cos x}$

$$\int\frac{dx}{5+4\cos x}=\int\frac{dx}{5+4\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$=\int\frac{(1+\tan^2\frac{x}{2})dx}{5(1+\tan^2\frac{x}{2})+4(1-\tan^2\frac{x}{2})}=\int\frac{\sec^2\frac{x}{2}dx}{9+\tan^2\frac{x}{2}}$$

$$=\int\frac{2dz}{9+z^2}\quad(\text{ Let, }\tan\tfrac{x}{2}=z\quad\therefore\ \tfrac{1}{2}\sec^2\tfrac{x}{2}dx=dz\ )$$

$$=2\int\frac{dz}{3^2+z^2}=2.\tfrac{1}{3}\tan^{-1}\tfrac{z}{3}+c.$$

$$=\tfrac{2}{3}\tan^{-1}(\tfrac{1}{3}\tan\tfrac{x}{2})+c.$$

(*c*) $\displaystyle\int\frac{dx}{3+2\sin x+\cos x}$

$$\int\frac{dx}{3+2\sin x+\cos x}=\int\frac{dx}{3+2.\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\dfrac{(1-\tan^2\frac{x}{2})}{1+\tan^2\frac{x}{2}}}$$

$$=\int\frac{(1+\tan^2\frac{x}{2})dx}{3(1+\tan^2\frac{x}{2})+4\tan\frac{x}{2}+(1-\tan^2\frac{x}{2})}$$

$$=\int\frac{\sec^2\frac{x}{2}dx}{2(\tan^2\frac{x}{2}+2\tan\frac{x}{2}+2)}=\int\frac{2dz}{2(z^2+2z+2)}$$

$$[\text{ Let }z=\tan\tfrac{x}{2}\quad\therefore\ dz=\tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx\ ]$$

$$=\int\frac{dz}{(z+1)^2+1^2}=\tan^{-1}(z+1)+c=\tan^{-1}(\tan\tfrac{x}{2}+1)+c.$$

Ex. 5. Integrate : $\displaystyle\int\frac{dx}{3\sin x-4\cos x}$ [C. U. '66]

$$\text{Given integral}=\int\frac{dx(1+\tan^2\frac{x}{2})}{3.2\tan\frac{x}{2}-4(1-\tan^2\frac{x}{2})}$$

$$=\int\frac{2dz}{4z^2+6z-4}\quad[\ \because\ z=\tan\tfrac{x}{2}\quad\therefore\ \sec^2\tfrac{x}{2}dx=2dz\ ]$$

$$=\int\frac{dz}{2\{(z+\frac{3}{4})^2-(\frac{5}{4})^2\}}=\tfrac{1}{2}\cdot\frac{1}{2.\frac{5}{4}}\log\frac{z+\frac{3}{4}-\frac{5}{4}}{z+\frac{3}{4}+\frac{5}{4}}+c$$

$$=\tfrac{1}{5}\log\frac{2\tan\frac{x}{2}-1}{2\tan\frac{x}{2}+4}+c.$$

Alternative method :

Let, $a=r\sin\theta$, $b=r\cos\theta$

$\therefore\quad r=\sqrt{a^2+b^2}$ and $\tan\theta=\frac{a}{b}$ or, $\theta=\tan^{-1}\frac{a}{b}$

$\therefore$ Given Integral :

$$=\int\frac{dx}{a\cos x+b\sin x}=\int\frac{dx}{r(\cos x\sin\theta+\sin x\cos\theta)}$$

$$=\frac{1}{r}\int\frac{dx}{\sin(x+\theta)}=\frac{1}{r}\int\operatorname{cosec}(x+\theta)dx=\frac{1}{r}\log\tan\frac{x+\theta}{2}$$

$$=\frac{1}{\sqrt{a^2+b^2}}\log\tan\left(\frac{1}{2}x+\frac{1}{2}\tan^{-1}\frac{a}{b}\right)$$

Now putting $a=-4$ and $b=3$ we get

$$\int\frac{dx}{3\sin x-4\cos x}=\frac{1}{\sqrt{3^2+4^2}}\log\tan\left(\tfrac{1}{2}x+\tfrac{1}{2}\tan^{-1}\frac{-4}{3}\right)+c$$

$$=\tfrac{1}{5}\log\tan\left(\tfrac{x}{2}-\tfrac{1}{2}\tan^{-1}\tfrac{4}{3}\right)+c.$$

Ex. 6. Integrate : $\int\frac{\sin x}{\sin x+\cos x}dx.$ [C. U. '58]

$$\int\frac{\sin x}{\sin x+\cos x}dx=\tfrac{1}{2}\int\frac{(\sin x+\cos x)-(\cos x-\sin x)}{\sin x+\cos x}dx$$

$$=\tfrac{1}{2}\int dx-\tfrac{1}{2}\int\frac{\cos x-\sin x}{\sin x+\cos x}dx=\tfrac{1}{2}x-\tfrac{1}{2}\log(\sin x+\cos x)+c$$

$\left[\because\right.$ the second integral is of the form $\left.\int\frac{f'(x)}{f(x)}dx\right]$

Note : If an integral be of the form $\int\frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx$ then express the integrand in the form $l\times$(denominator)$+m$ (denominator)$'$ See the illustrations.

Ex. 7. Integrate : $\int\frac{2\sin x+3\cos x}{3\sin x+4\cos x}dx.$

Let $2\sin x+3\cos x=l(3\sin x+4\cos x)+m(3\cos x-4\sin x)$
$=(3l-4m)\sin x+(4l+3m)\cos x$

Equating the coefficients of $\sin x$ and $\cos x$ from both sides we get,

$$\left.\begin{array}{l}2=3l-4m\\3=4l+3m\end{array}\right\}. \quad \text{Solving, } l=\tfrac{18}{25},\ m=\tfrac{1}{25}$$

$\therefore$ Required integral

$$=\int\frac{\frac{18}{25}(3\sin x+4\cos x)+\frac{1}{25}(3\cos x-4\sin x)}{3\sin x+4\cos x}dx$$

$$=\int\tfrac{18}{25}dx+\tfrac{1}{25}\int\frac{3\cos x-4\sin x}{3\sin x+4\cos x}\,dx$$

$$=\tfrac{18}{25}x+\tfrac{1}{25}\log(3\sin x+4\cos x)+c$$

Ex. 8. Integrate : (i) $\displaystyle\int\frac{\cos 2x}{\cos x}\,dx$, (ii) $\displaystyle\int\frac{\cos x}{\cos 2x}\,dx$.

(i) $$\int\frac{\cos 2x}{\cos x}dx=\int\frac{2\cos^2 x-1}{\cos x}\,dx=2\int\cos x\,dx-\int\sec x\,dx$$

$$=2\sin x-\log(\sec x+\tan x)+c.$$

(ii) $$\int\frac{\cos x}{\cos 2x}dx=\int\frac{\cos x}{1-2\sin^2 x}dx=\frac{1}{\sqrt2}\int\frac{\sqrt2\cos x\,dx}{1-(\sqrt2\sin x)^2}$$

$$=\frac{1}{\sqrt2}\int\frac{dz}{1-z^2}\quad[\text{Taking }\sqrt2\sin x=z,\ \sqrt2\cos x\,dx=dz\,]$$

$$=\frac{1}{\sqrt2}\cdot\tfrac12.\log\frac{1+z}{1-z}=\frac{1}{2\sqrt2}\log\frac{1+\sqrt2\sin x}{1-\sqrt2\sin x}$$

Ex. 9. Integrate : $\displaystyle\int\frac{dx}{(a\cos x+b\sin x)^2}$

$$\int\frac{dx}{(a\cos x+b\sin x)^2}=\int\frac{dx}{\cos^2 x(a+b\tan x)^2}$$

$$=\int\frac{\sec^2 x\,dx}{(a+b\tan x)^2}=\int\frac{\frac{1}{b}dz}{z^2}.\qquad\left[\begin{array}{l}a+b\tan x=z\text{ (say)}\\ \therefore\ b\sec^2 x\,dx=dz\end{array}\right]$$

$$=-\frac{1}{b}\cdot\frac{1}{z}+c=-\frac{1}{b}\cdot\frac{1}{a+b\tan x}+c.$$

Alternative method : Let $a=r\cos\theta$, $b=r\sin\theta$

$\therefore\ a\cos x+b\sin x=r(\cos x.\cos\theta+\sin x.\sin\theta)=r\cos(x-\theta)$

where $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\dfrac{b}{a}$

$$\therefore \text{ Required integral}=\int\frac{dx}{r^2\cos^2(x-\theta)}=\frac{1}{r^2}\int\sec^2(x-\theta)\,dx$$

$$=\frac{1}{a^2+b^2}\tan\left(x-\tan^{-1}\frac{b}{a}\right)+c'.$$

Ex. 10. Integrate : (i) $\int\sin^6x\cos^3x\,dx$. [C. U. '68]

(ii) $\int\sin^2x\cos^2x\,dx$ (iii) $\int\dfrac{\sin^2x}{\cos^6x}\,dx$.

(i) $\int\sin^6x\cos^3x\,dx=\int\sin^6x\,.(1-\sin^2x)\cos x\,dx$

[Let $\sin x=z$, $\therefore$ $\cos x\,dx=dz$]

$=\int z^6(1-z^2)\,dz=\int z^6dz-\int z^8dz$

$=\dfrac{z^7}{7}-\dfrac{z^9}{9}+c=\frac{1}{7}\sin^7x-\frac{1}{9}\sin^9x+c$

(ii) $\int\sin^2x\cos^2x\,dx=\int\frac{1}{4}\,.\,4\sin^2x\,.\cos^2x\,dx$

$=\frac{1}{4}\int\sin^2 2x\,dx=\frac{1}{8}\int 2\sin^2 2x\,dx$

$=\frac{1}{8}\int(1-\cos 4x)\,dx=\frac{1}{8}\left(x-\dfrac{\sin 4x}{4}\right)+c.$

(iii) $\int\dfrac{\sin^2x}{\cos^6x}\,dx=\int\tan^2x.\,(1+\tan^2x)\sec^2x\,dx,$

[Let $\tan x=z$ $\therefore$ $\sec^2x\,dx=dz$]

$=\int z^2(1+z^2)\,dz=\dfrac{z^3}{3}+\dfrac{z^5}{5}+c=\frac{1}{3}\tan^3x+\frac{1}{5}\tan^5x+c.$

Ex. 11. Integrate : $\int\sqrt{1+\sec x}\,dx$. [C. U. '62]

$\int\sqrt{1+\sec x}\,dx=\int\dfrac{\sqrt{1+\cos x}}{\sqrt{\cos x}}dx=\int\dfrac{\sqrt{2}\cos\frac{x}{2}}{\sqrt{1-2\sin^2\frac{x}{2}}}dx$

$=\int\dfrac{\sqrt{2}.\sqrt{2}dz}{\sqrt{1-z^2}}$ [$\sqrt{2}\sin\frac{x}{2}=z$ (say) $\therefore$ $\dfrac{1}{\sqrt{2}}\cos\frac{x}{2}\,dx=dz$]

$=2\sin^{-1}z+c=2\sin^{-1}(\sqrt{2}\sin\frac{x}{2})+c.$

Ex. 12. Integrate : $\int\dfrac{\sin x}{\sqrt{1+\sin x}}dx$ [C. U.]

$\int\dfrac{\sin x}{\sqrt{1+\sin x}}dx=\int\dfrac{1+\sin x-1}{\sqrt{1+\sin x}}dx$

$$=\int \sqrt{1+\sin x}\, dx-\int \frac{1}{\sqrt{1+\sin x}} dx$$

$$=\int \left(\cos \tfrac{x}{2}+\sin \tfrac{x}{2}\right) dx-\int \frac{dx}{\cos \frac{x}{2}+\sin \frac{x}{2}}$$

for, $\left(\cos \frac{x}{2}+\sin \frac{x}{2}\right)^2=1+\sin x$

$$=2 \sin \tfrac{x}{2}-2 \cos \tfrac{x}{2}-\int \frac{dx}{\sqrt{2} \sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$

$$=2 \sin \tfrac{x}{2}-2 \cos \tfrac{x}{2}-\frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(\frac{x}{2}+\frac{\pi}{4}\right) dx$$

$$=2 \sin \tfrac{x}{2}-2 \cos \tfrac{x}{2}-\sqrt{2} \log \tan\left(\frac{x}{4}+\frac{\pi}{8}\right)+c.$$

Ex. 13. Integrate :

(i) $\int (\sqrt{\tan x}+\sqrt{\cot x})\, dx$ (ii) $\int (\sqrt{\cot x}-\sqrt{\tan x})\, dx$

(iii) $\int \sqrt{\tan x}\, dx$ (iv) $\int \sqrt{\cot x}\, dx.$

(i) Let $\sin x-\cos x=z$

$\therefore \quad (\cos x+\sin x)\, dx=dz$

Now, $\int (\sqrt{\tan x}+\sqrt{\cot x})\, dx$

$$=\int\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx=\int \frac{\sin x+\cos x}{\sqrt{\cos x \sin x}} dx$$

$$=\int \frac{\sin x+\cos x}{\sqrt{\dfrac{1-(\sin x-\cos x)^2}{2}}} dx=\int \frac{\sqrt{2}\, dz}{\sqrt{1-z^2}}$$

$$=\sqrt{2} \sin^{-1} z+c=\sqrt{2} \sin^{-1} (\sin x-\cos x)+c.$$

(ii) Let, $\sin x+\cos x=z$

$\therefore \quad (\cos x-\sin x)\, dx=dz$

Now, $\int (\sqrt{\cot x}-\sqrt{\tan x})\, dx$

$$=\int\left(\sqrt{\frac{\cot x}{\sin x}}-\sqrt{\frac{\sin x}{\cos x}}\right)dx=\int \frac{(\cos x-\sin x)\, dx}{\sqrt{\sin x \cos x}}$$

$$=\int \frac{\sqrt{2}(\cos x-\sin x)\, dx}{(\sqrt{(\sin x+\cos x)^2-1}}=\int \frac{\sqrt{2} dz}{\sqrt{z^2-1}}$$

$$=\sqrt{2} \log (z+\sqrt{z^2-1})+c'$$

$$= \sqrt{2}\log\{\sin x+\cos x+\sqrt{(\sin x+\cos x)^2-1}\}+c'$$

$$= \sqrt{2}\log\{\sin x+\cos x+\sqrt{2\sin x\cos x}\}+c'$$

$$= \sqrt{2}\log\{\sin x+\cos x+\sqrt{\sin 2x}\}+c'$$

(iii) Form (i)—(ii) [above]

$$\int(\sqrt{\cot x}+\sqrt{\tan x})\,dx-\int(\sqrt{\cot x}-\sqrt{\tan x})\,dx$$

$$= \sqrt{2}\sin^{-1}(\sin x-\cos x)+c-\sqrt{2}\log\{\sin x+\cos x+\sqrt{\sin 2x}\}-c'$$

or, $2\int\sqrt{\tan x}\,dx=\sqrt{2}\,[\sin^{-1}(\sin x-\cos x)-\log\{\sin x+\cos x+\sqrt{\sin 2x}\}+k]$

$$\therefore\ \int\sqrt{\tan x}\,dx=\frac{1}{\sqrt{2}}[\sin^{-1}(\sin x-\cos x)-\log\{\sin x+\cos x+\sqrt{\sin 2x}\}+k]$$

(iv) Similarly adding (i) and (ii)

above, $\int\sqrt{\cot x}\,dx=\frac{1}{\sqrt{2}}\,[\sin^{-1}(\sin x-\cos x)+\log\{\sin x+\cos x+\sqrt{\sin 2x}\}+k']$

Ex. 14. Integrate :

(i) $\int\frac{\sin^2 x}{\cos^4 x}\,dx.$ [H. S. '82] (ii) $\int\frac{dx}{\cos 2x+3\sin^2 x}.$ [H.S. '84]

(iii) $\int\frac{\sin 2x\,dx}{\sin^4 x+\cos^4 x}$ [H. S. '86] (iv) $\int\frac{dx}{\sin^4 x+\cos^4 x}$

(i) $\int\frac{\sin^2 x}{\cos^4 x}dx=\int\tan^2 x\sec^2 x\,dx=\int z^2\,dz=\frac{z^3}{3}+c$

[where $\tan x=z$ $\therefore$ $\sec^2 x\,dx=dz$]$=\frac{\tan^3 x}{3}+c.$

(ii) $\int\frac{dx}{\cos 2x+3\sin^2 x}=\int\frac{dx}{\cos^2 x-\sin^2 x+3\sin^2 x}$

$$=\int\frac{dx}{\cos^2 x+2\sin^2 x}=\int\frac{\sec^2 x\,dx}{1+2\tan^2 x}=\int\frac{dz}{1+2z^2}$$

[where $\tan x=z$]

$=\frac{1}{\sqrt{2}}\int\frac{dt}{1+t^2}$ [where $\sqrt{2}\,z=t$ $\therefore$ $\sqrt{2}\,dz=dt$]

$$=\frac{1}{\sqrt{2}}\tan^{-1}(t)+c=\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+c.$$

(iii) $\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}dx=\int \frac{2\sin x\cos x}{\sin^4 x+\cos^4 x}dx=\int \frac{2\tan x\sec^2 x}{\tan^4 x+1}dx$

[Dividing both numerator and denominator by $\cos^4 x$]

$=\int \frac{2t\,dt}{t^4+1}$ [where $\tan x=t \quad \therefore \quad \sec^2 x\,dx=dt$]

$=\int \frac{dz}{z^2+1}$ [where $t^2=z \quad \therefore \quad 2t\,dt=dz$]

$=\tan^{-1}z+c=\tan^{-1}(t^2)+c=\tan^{-1}(\tan^2 x)+c.$

(iv) $\int \frac{dx}{\sin^4 x+\cos^4 x}$

$=4\int \frac{dx}{(2\sin^2 x)^2+(2\cos^2 x)^2}=4\int \frac{dx}{(1-\cos 2x)^2+(1+\cos 2x)^2}$

$=4\int \frac{dx}{2(1+\cos^2 2x)}=2\int \frac{\sec^2 2x}{\sec^2 2x+1}dx=\int \frac{2\sec^2 2x}{2+\tan^2 2x}dx$

Now let $\tan 2x=z \quad \therefore \quad 2\sec^2 2x\,dx=dz$

So given integral$=\int \frac{dz}{2+z^2}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}+c$

$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan 2x}{\sqrt{2}}\right)+c.$

Ex. 15. Integrate : (i) $\int \frac{\sin x\,dx}{\sin 3x}$ (ii) $\int \frac{\sin x\,dx}{\cos 2x}$

(iii) $\int \frac{dx}{1-\sin^4 x}$ (iv) $\int \frac{dx}{\sin^4 x\cos^2 x}$ (v) $\int \frac{\cot^2 x+1}{\cot^2 x-1}dx$

(vi) $\int \frac{dx}{1+\cos\alpha\cos x}$ (vii) $\int \sqrt{\frac{\operatorname{cosec} x-\cot x}{\operatorname{cosec} x+\cot x}}\,\frac{\sec x}{\sqrt{1+2\sec x}}dx$

(i) $\int \frac{\sin x\,dx}{\sin 3x}=\int \frac{\sin x}{3\sin x-4\sin^3 x}dx=\int \frac{dx}{3-4\sin^2 x}$

$=\int \frac{\sec^2 x\,dx}{3\sec^2 x-4\tan^2 x}=\int \frac{\sec^2 x\,dx}{3-\tan^2 x}$

$=\int \frac{dt}{3-t^2}$ [where $\tan x=t \quad \therefore \quad \sec^2 x\,dx=dt$]

$=\frac{1}{2\sqrt{3}}\log\frac{\sqrt{3}+t}{\sqrt{3}-t}+c=\frac{1}{2\sqrt{3}}\log\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}+c.$

(ii) $\int \frac{\sin x\, dx}{\cos 2x} = \int \frac{\sin x\, dx}{2\cos^2 x - 1}$

Now let, $\sqrt{2} \cos x = z \quad \therefore \quad -\sqrt{2} \sin x\, dx = dz$

or, $\sin x\, dx = -\frac{1}{\sqrt{2}} dz$

$\therefore$ Given integral $= -\frac{1}{\sqrt{2}} \int \frac{dz}{z^2-1} = \frac{1}{\sqrt{2}} \int \frac{dz}{1-z^2}$

$$= \frac{1}{2\sqrt{2}} \log \frac{1+z}{1-z} + c = \frac{1}{2\sqrt{2}} \log \frac{(\sqrt{2} \cos x + 1)}{(-\sqrt{2} \cos x + 1)} + c$$

(iii) $\int \frac{dx}{1-\sin^4 x} = \int \frac{dx}{(1-\sin^2 x)(1+\sin^2 x)} = \int \frac{dx}{\cos^2 x\, (1+\sin^2 x)}$

$$= \int \frac{\sec^2 x \sec^2 x\, dx}{\sec^2 x + \tan^2 x} = \int \frac{(1+\tan^2 x) \sec^2 x\, dx}{1 + 2\tan^2 x}$$

Now, let $\tan x = z \quad \therefore \quad \sec^2 x\, dx = dz$

$\therefore$ Given integral $= \int \frac{1+z^2}{1+2z^2} dz = \frac{1}{2} \int \frac{2+2z^2}{1+2z^2} dz$

$$= \frac{1}{2} \int \frac{1+2z^2}{1+2z^2} dz + \frac{1}{2} \int \frac{dz}{1+2z^2}$$

$$= \frac{1}{2} \int dz + \frac{1}{2\sqrt{2}} \int \frac{dt}{1+t^2} \qquad \left[\sqrt{2}z = t \text{ (say) } \quad \therefore \quad \sqrt{2} dz = dt \right]$$

$$= \frac{1}{2} z + \frac{1}{2\sqrt{2}} \tan^{-1}(t) + c$$

$$= \frac{1}{2} z + \frac{1}{2\sqrt{2}} \tan^{-1}(\sqrt{2} z) + c$$

$$= \frac{1}{2} \tan x + \frac{1}{2\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + c.$$

(iv) $\int \frac{dx}{\sin^4 x \cos^2 x} = \int \frac{\sec^4 x \sec^2 x\, dx}{\tan^4 x} = \int \frac{(1+\tan^2 x)^2 \sec^2 x\, dx}{\tan^4 x}$

$$= \int \frac{(1+t^2)^2 dt}{t^4} \qquad \left[\text{where } \tan x = t\,; \quad \therefore \quad \sec^2 x\, dx = dt \right]$$

$$= \int \frac{1+2t^2+t^4}{t^4} dt = \int t^{-4} dt + 2 \int t^{-2} dt + \int dt$$

$$= -\frac{1}{3t^3} - \frac{2}{t} + t + c = -\frac{1}{3} \cot^3 x - 2 \cot x + \tan x + c.$$

(v) $$\int\frac{\cot^2 x+1}{\cot^2 x-1}dx=\int\frac{\dfrac{1}{\tan^2 x}+1}{\dfrac{1}{\tan^2 x}-1}dx=\int\frac{\sec^2 x\ dx}{1-\tan^2 x}$$

$$=\int\frac{dt}{1-t^2}\quad\left[\text{Let, }\tan x=t\quad\therefore\quad \sec^2 x\ dx=dt\right]$$

$$=\tfrac{1}{2}\log\frac{1+t}{1-t}+c=\tfrac{1}{2}\log\frac{1+\tan x}{1-\tan x}+c.$$

(vi) $$\int\frac{dx}{1+\cos\alpha\cos x}=\int\frac{dx}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+\cos\alpha\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$=\int\frac{\sec^2\frac{x}{2}\ dx}{1+\tan^2\frac{x}{2}+\cos\alpha\left(1-\tan^2\frac{x}{2}\right)}=\int\frac{\sec^2\frac{x}{2}\ dx}{(1+\cos\alpha)+(1-\cos\alpha)\tan^2\frac{x}{2}}$$

Now let, $\tan\frac{x}{2}=z$ $\therefore$ $\frac{1}{2}\sec^2\frac{x}{2}\ dx=dz$ or, $\sec^2\frac{x}{2}\ dx=2dz.$

$\therefore$ Given integral $$=\int\frac{2dz}{(1+\cos\alpha)+(1-\cos\alpha)z^2}$$

$$=\frac{1}{1-\cos\alpha}\int\frac{2dz}{\dfrac{1+\cos\alpha}{1-\cos\alpha}+z^2}$$

$$=\frac{2}{1-\cos\alpha}\cdot\frac{1}{\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}}\tan^{-1}\left\{\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}z\right\}+c$$

$$=\frac{2}{\sqrt{1-\cos^2\alpha}}\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{x}{2}\right)+c$$

$$=2\operatorname{cosec}\alpha\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{x}{2}\right)+c.$$

(vii) $$\sqrt{\frac{\operatorname{cosec}x-\cot x}{\operatorname{cosec}x+\cot x}}=\sqrt{\frac{(\operatorname{cosec}^2 x-\cot^2 x)}{(\operatorname{cosec}x+\cot x)^2}}=\frac{1}{\operatorname{cosec}x+\cot x}$$

$$=\frac{1}{\dfrac{1}{\sin x}+\dfrac{\cos x}{\sin x}}=\frac{\sin x}{1+\cos x}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}=\tan\tfrac{x}{2}.$$

$$\frac{\sec x}{\sqrt{(1+2\sec x)}}=\frac{\dfrac{1}{\cos x}}{\sqrt{\left(1+\dfrac{2}{\cos x}\right)}}=\frac{\dfrac{1}{\sqrt{\cos x}}}{\sqrt{(2+\cos x)}}$$

$$=\frac{1}{\sqrt{\cos x}\sqrt{(2+\cos x)}}$$

$\therefore$ Given Integral $=\int \frac{\tan \frac{x}{2}\, dx}{\sqrt{(2\cos^2 \frac{x}{2}-1)(2\cos^2 \frac{x}{2}+1)}}$

$$=\int \frac{\tan \frac{x}{2} \sec^2 \frac{x}{2}\, dx}{\sqrt{(4-\sec^4 \frac{x}{2})}}$$

Let, $\sec^2 \frac{x}{2}=z \quad \therefore \quad 2 \sec^2 \frac{z}{2} \tan \frac{x}{2}.\frac{1}{2}\, dx=dz$
or, $\sec^2 \frac{x}{2} \tan \frac{x}{2}\, dx=dz.$

$\therefore$ Given integral $=\int \frac{dz}{\sqrt{(4-z^2)}}=\sin^{-1} \frac{z}{2}+c$

$=\sin^{-1} (\frac{1}{2} \sec^2 \frac{x}{2})+c.$

Ex. 16. Evaluate : $\int \left[\frac{(\cos 2x)^{\frac{1}{2}}}{\sin x}\right] dx$ [I. I. T. 1987]

$\int \left[\frac{(\cos 2x)^{\frac{1}{2}}}{\sin x}\right] dx=\int \frac{1}{\sin x}\sqrt{\frac{1-\tan^2 x}{1+\tan^2 x}}\,dx=\int \frac{1}{\sin x \sec x} \sqrt{1-\tan^2 x}\, dx$

$=\int \frac{\sqrt{1-\tan^2 x}}{\tan x} dx.$

Now let $\tan x=\sin \theta \quad \therefore \quad \sec^2 x\, dx=\cos \theta\, d\theta$

or, $dx=\frac{\cos \theta\, d\theta}{\sec^2 x}=\frac{\cos \theta\, d\theta}{1+\tan^2 x}=\frac{\cos \theta\, d\theta}{1+\sin^2 \theta}$

$\therefore$ Given integral $=\int \frac{\sqrt{1-\sin^2 \theta}}{\sin \theta} \frac{\cos \theta}{1+\sin^2 \theta} d\theta$

$=\int \frac{1-\sin^2 \theta}{\sin \theta\, (1+\sin^2 \theta)} d\theta=\int \frac{1+\sin^2 \theta-2 \sin^2 \theta}{\sin \theta\, (1+\sin^2 \theta)} d\theta$

$[\because \ \cos \theta=\sqrt{1-\sin^2 \theta}]$

$=\int \frac{d\theta}{\sin \theta}-2\int \frac{\sin \theta}{1+\sin^2 \theta} d\theta=\mathrm{I}_1-2\mathrm{I}_2.$

$\mathrm{I}_2=\int \frac{\sin \theta}{1+\sin^2 \theta} d\theta=-\int \frac{d\,(\cos \theta)}{2-\cos^2 \theta}=-\int \frac{dz}{2-z^2}$ [where $\cos \theta=z$]

$=-\frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}+z}{\sqrt{2}-z}+c_1=-\frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}+\cos \theta}{\sqrt{2}-\cos \theta}+c_1$

$=-\frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}+\sqrt{1-\sin^2 \theta}}{\sqrt{2}-\sqrt{1-\sin^2 \theta}}+c_1$

$=-\frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}+c_1$

$=-\frac{1}{2\sqrt{2}} \log \frac{\sqrt{2} \cos x+\sqrt{\cos 2x}}{\sqrt{2} \cos x-\sqrt{\cos 2x}}+c_1$

Again $I_1=\int\frac{d\theta}{\sin\theta}=\int \operatorname{cosec}\theta\, d\theta=\log(\operatorname{cosec}\theta-\cot\theta)+c_2$

$$=\log\left(\frac{1}{\sin\theta}-\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right)+c_2=\log\left(\frac{1}{\tan x}-\frac{\sqrt{1-\tan^2 x}}{\tan x}\right)+c_2$$

$$=\log\frac{1-\sqrt{1-\tan^2 x}}{\tan x}+c_2=\log\left(\frac{\cos x-\sqrt{\cos 2x}}{\sin x}\right)+c_2$$

$\therefore$ Given integral $=\log\dfrac{\cos x-\sqrt{\cos 2x}}{\sin x}$

$$+\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}\cos x+\sqrt{\cos 2x}}{\sqrt{2}\cos x-\sqrt{\cos 2x}}+c.$$

Exercise 2C

Integrate :—

1. $\int\frac{dx}{4+5\cos x}$ 2. $\int\frac{dx}{5+4\sin x}$ 3. $\int\frac{dx}{4-5\sin x}$

4. $\int\frac{dx}{3\sin x+4\cos x}$ 5. $\int\frac{dx}{2+\sin x+\cos x}$

6. $\int\sin^6 x\cos^3 x\,dx$ 7. $\int\sin^4 x\cos^2 x\,dx$ 8. $\int\frac{\sin^6 x}{\cos^4 x}dx$

9. $\int\frac{\sec x}{a+b\tan x}dx$ 10. $\int\frac{dx}{3+2\sin x}$ [C. U. '65]

11. $\int\frac{dx}{3+2\cos x}$ 12. $\int\frac{dx}{4-5\sin^2 x}$ 13. (a) $\int\frac{dx}{4-5\cos^2 x}$

(b) $\int\frac{dx}{1+\cos^2 x}$ 14. $\int\frac{dx}{4\cos^2 x+3\sin^2 x}$ 15. $\int\frac{dx}{(4\sin x-3\cos x)^2}$

16. $\int\frac{\cos x}{\cos x+\sin x}dx$ 17. $\int\frac{\cos x+2\sin x}{3\cos x+4\sin x}$

18. $\int\frac{dx}{13+3\cos x+4\sin x}$ 19. $\int\frac{\sin 2x\,dx}{\sin 5x\sin 3x}$

20. $\int\frac{dx}{\sin(x-a)\sin(x-b)}$ 21. $\int\frac{dx}{4-5\sin^2 x}$ 22. $\int\frac{dx}{1+\cos^2 x}$

23. $\int\frac{dx}{\cos 3x-\cos x}$ 24. $\int\frac{\sin 2x}{(\sin x+\cos x)^2}dx$ 25. $\int\frac{dx}{a+b\tan x}$

26. $\int\frac{dx}{1+\tan x}$ 27. $\int\frac{\cos x\,dx}{5-3\cos x}$ 28. $\int\frac{\tan x}{\sqrt{a+b\tan^2 x}}dx.$

CHAPTER THREE

Integration by Parts

§ 3·1. In this chapter we shall discuss integration of the product of two functions. Generally, these integrations are performed by the method of integration by parts.

Theorem. If u and v be two differentiable functions of the same variable x for all values of x, then

$$\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx.$$

Proof : $\dfrac{d}{dx}(uv)=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$

Hence from the definition of integral,

$$uv=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)dx$$

$$=\int u\frac{dv}{dx}dx+\int v\frac{du}{dx}\,dx.$$

or, $\displaystyle\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}dx.$

In the above theorem both u and $\dfrac{dv}{dx}$ are functions of x.

Let $u=f(x)$ and $\dfrac{dv}{dx}=\phi(x)$.

as $\dfrac{dv}{dx}=\phi(x)$, $\therefore$ $dv=\phi(x)\,dx.$

or, $\int dv=\int\phi(x)\,dx$, or, $v=\int\phi(x)\,dx$

and $\dfrac{du}{dx}=\dfrac{d}{dx}\{f(x)\}$

Hence from the above theorem one can write,

$$\int f(x)\,\phi(x)\,dx=f(x)\int\phi(x)\,dx-\int\left[\left\{\frac{d}{dx}f(x)\right\}\int\phi(x)\,dx\right]dx.$$

i.e., the integral of the product of two functions
=(first function) × (integral of the second) − integral
of {the differential coefficient of the first function
× the integral of the second function}.

This formula is called the formula for **integration by parts.**

Ex. 1. $\int x \sin x\, dx$

$$=\{x \int \sin x\, dx\} - \int \left[\left\{ \frac{d}{dx}(x) \right\} \int \sin x\, dx \right] dx$$

$$= -x \cos x - \int \{(1)(-\cos x)\}\, dx$$

$$= -x \cos x + \int \cos x\, dx = -x \cos x + \sin x + c.$$

Ex. 2. $\int x^2 \cos x\, dx = x^2 \int \cos x\, dx$

$$- \int \left[\left\{ \frac{d}{dx}(x^2) \right\} \int \cos x\, dx \right] dx$$

$$= x^2 \sin x - \int (2x) \sin x\, dx = x^2 \sin x - 2 \int x \sin x\, dx$$

$$= x^2 \sin x - 2(-x \cos x + \sin x + c) \qquad [\text{ from Ex. 1 }]$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c'.$$

Note that : One can take any of $f(x)$ and $\phi(x)$ as the first function ; but one should first determine the function which is to be taken as the first function, so that integration can be easily performed. In example 1, x has been taken as the first function and $\sin x$ as the second. As $\int x\, dx$ and $\int \sin x\, dx$ are both of the standard forms, so each of them can be easily determined. Let us now examine the situation if $\sin x$ is taken as the first function. In that case,

$$\int x \sin x\, dx = \sin x \int x\, dx - \int \left\{ \left(\frac{d}{dx} (\sin x) \ x\, dx \right) \right\} dx$$

$$= \frac{x^2}{2} \sin x - \int \frac{x^2}{2} \cos x\, dx \cdots\cdots (\alpha)$$

In the given integral, the integrand is the product of x and a trigonometric function. By taking $\sin x$ as the first function, we get in (α), $\int \frac{x^2}{2} \cos x\, dx$ and here also the integrand is the product of a trigonometric function and a power of x. In $\int \frac{x^2}{2} \cos x\, dx$, the power of x has increased and the integration will be lengthened. If in $\int \frac{x^2}{2} \cos x\, dx$, $\cos x$ is taken as the first function, you will find that integration of $\int \frac{x^3}{6} \sin x\, dx$ will be necessary and integration cannot be completed.

Hence if in integrals of the type $\int x \sin x\, dx$, trigonometric functions are taken as first functions, integration will never be completed. So, you find that the success of the integration process depends upon the choice of the first function. There is no hard and fast rule for the choice of the first function. But a general rule is that the integral which cannot be evaluted easily is to be taken as the first function. In case of $\int x \sin x\, dx$, evaluations of both $\int x\, dx$ and $\int \sin x\, dx$ are easy. But if $\sin x$ is taken as the first function, integration cannot be completed. So, this rule is to be followed, when integration of one function is difficult. We give below a list of rules for the choice of the first function. There are exceptions to these rules. But in the primary stage they will be useful.

Rules for choice of the first function :

If the integrand is a product of

(1) an algebraic and a trigonometric function, take the algebraic function as the first function.

(2) an algebraic and an exponential function select the algebraic function as the first function.

(3) an algebraic and a logarithmic function, select the logarithmic function as the first function.

(4) an algebraic and an inverse circular function, take the inverse circular function as the first function.

(5) a trigonometric and an exponential function, take any of the functions as the first function.

(6) Sometimes in determination of integrals of the form $\int f(x)\, dx$ the integrand is expressed as $1 . f(x)$ and in those cases $f(x)$ is to be selected as the first function.

These rules for choice of the first function can be remembered by the following artifice :

Remember the word "LIATE"

L stands for Logarithmic function.

I stands for inverse circular function.

A stands for Algebraic function.

T stands for Trigonometric function.

E stands for Exponential function.

In the product of two functions, the letter which comes earlier in the word LIATE should be taken as the first function.

Examples. In $\int x^2 \sin x\, dx$

x^2 is A and $\sin x$ is T. In 'LIATE' A comes earlier than T. So A i.e., x^2 is the first function.

In $\int \log x\, dx = \int \log x .\, 1\, dx$, L i.e., $\log x$ comes earlier than A i.e., 1 which is algebraic in 'LIATE'. So, L i.e., $\log x$ is to be chosen as the first function.

Ex. 3. $\int x \sec^2 x\, dx = x \int \sec^2 x\, dx - \int \left\{ \frac{d}{dx}(x) \int \sec^2 x\, dx \right\} dx$

$= x \tan x - \int \tan x\, dx = x \tan x - \log (\sec x) + c.$

Ex. 4. $\int x e^{ax}\, dx = x . \int e^{ax}\, dx - \int \left\{ \frac{d}{dx}(x) \int e^{ax} dx \right\} dx$

$= x . \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} dx = \frac{x e^{ax}}{a} - \frac{e^{ax}}{a^2} = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$

§ 3·2. Integration of logarithmic functions.

Ex. 1. $\int \log x\, dx = \int 1 .\, \log x\, dx$

$= \log x \int 1 . dx - \int \left\{ \frac{d}{dx}(\log x) \int 1 . dx \right\} dx = x \log x - \int \frac{1}{x} .\, x\, dx$

$= x \log x - \int dx = x \log x - x = x(\log x - 1).$

Cor. $\int \log x^n dx = \int n \log x\, dx = n \int \log x\, dx = nx\,(\log x - 1).$

Ex. 2. $\int (\log x)^2\, dx = \int 1 .\, (\log x)^2\, dx$

$= (\log x)^2 \int 1 . dx - \int \left\{ \frac{d}{dx} (\log x)^2 \int 1 .\, dx \right\} dx$

$= x \{\log x\}^2 - \int \frac{2 \log x}{x} . x .\, dx = x\,(\log x)^2 - 2 \int \log x\, dx$

$= x\,(\log x)^2 - 2x\,(\log x - 1)$ [from Ex. 1]

$= x\,\{(\log x)^2 - 2 \log x + 2\},$

§ 3·3. Integration of inverse circular functions.

Ex. 1. $\int \sin^{-1} x\, dx = \int 1 .\, \sin^{-1} x\, dx$

$= \sin^{-1} x . \int 1 .\, dx - \int \left\{ \frac{d}{dx} (\sin^{-1} x) \int 1 .\, dx \right\} dx$

$= x \sin^{-1} x - \int \frac{1}{\sqrt{1 - x^2}} . x . dx$

Now, to determine $\int \frac{x\, dx}{\sqrt{1 - x^2}}$, let $1 - x^2 = t^2$

$\therefore \quad -2x\, dx = 2t\, dt$ or $x\, dx = -t\, dt$ and $\sqrt{1 - x^2} = \sqrt{t^2} = t$

$$\therefore\quad \int \frac{x\,dx}{\sqrt{1-x^2}} = -\int \frac{t\,dt}{t} = -\int dt = -t = -\sqrt{1-x^2}$$

$\therefore$ required integral $= x \sin^{-1} x + \sqrt{1-x^2}$.

Cor. $\int \cos^{-1} x\ dx = \int \left(\frac{\pi}{2} - \sin^{-1} x\right) dx$

$$= \int \frac{\pi}{2}\ dx - \int \sin^{-1} x\ dx = \frac{\pi}{2}\ x - x \sin^{-1} x - \sqrt{1-x^2}$$

$$= \frac{\pi}{2} x - x\left(\frac{\pi}{2} - \cos^{-1} x\right) - \sqrt{1-x^2} = x \cos^{-1} x - \sqrt{1-x^2}$$

Ex. 2. $\int \tan^{-1} x\ dx = \int 1.\ \tan^{-1} x\ dx$

$$= \tan^{-1} x. \int 1.\ dx - \int \left\{\frac{d}{dx}\ (\tan^{-1} x) \int 1.\ dx\right\} dx$$

$$= x \tan^{-1} x - \int \frac{1}{1+x^2}\ x\ dx = x \tan^{-1} x - \tfrac{1}{2} \log\ (1+x^2)$$

Cor. $\int \cot^{-1} x\ dx$

$$= \int \left(\frac{\pi}{2} - \tan^{-1} x\right) dx = \int \frac{\pi}{2}\ dx - \int \tan^{-1} x\ dx$$

$$= \frac{\pi}{2}\ x - \{x \tan^{-1} x - \tfrac{1}{2} \log\ (1+x^2)\}$$

$$= \frac{\pi}{2} x - x\left(\frac{\pi}{2} - \cot^{-1} x\right) + \tfrac{1}{2} \log\ (1+x^2) = x \cot^{-1} x + \tfrac{1}{2} \log\ (1+x^2)$$

Ex. 3. $\int \sec^{-1} x.dx = \int 1.\sec^{-1} x.dx$

$$= \sec^{-1} x. \int 1.\ dx - \int \left\{\frac{d}{dx}\ (\sec^{-1} x). \int 1.dx\right\} dx$$

$$= x \sec^{-1} x - \int \left\{\frac{d}{dx}\ (\sec^{-1} x) \int 1.dx\right\} dx$$

$$= x \sec^{-1} x - \int \frac{1}{x\sqrt{x^2-1}}\ x\ dx = x \sec^{-1} x - \log\ (x + \sqrt{x^2-1}).$$

Cor. $\int \operatorname{cosec}^{-1} x\ dx = x \operatorname{cosec}^{-1} x + \log\ (x + \sqrt{x^2-1})$.

§ 3·4. Standard forms :

$\int e^{ax} \cos bx\ dx$ and $\int e^{ax} \sin bx\ dx$.

Let, $\int e^{ax} \cos bx\ dx = \mathrm{I}_1$ and $\int e^{ax} \sin bx\ dx = \mathrm{I}_2$

$\therefore\quad \mathrm{I}_1 = \int e^{ax} \cos bx\ dx.$

$$= e^{ax} \int \cos bx\ dx - \int \left\{\frac{d}{dx}\ (e^{ax}) \int \cos bx\ dx\right\} dx.$$

$$=e^{ax}\frac{\sin bx}{b}-\int ae^{ax}\frac{\sin bx}{b}dx$$

$$=\frac{e^{ax}\sin bx}{b}-\frac{a}{b}\int e^{ax}\sin bx\,dx \qquad \cdots(1)$$

$$=\frac{e^{ax}}{b}\sin bx-\frac{a}{b}I_2 \quad \cdots(2)$$

Similarly, $I_2=-\frac{e^{ax}\cos bx}{b}+\int ae^{ax}\frac{\cos bx}{b}dx$

$$=-\frac{e^{ax}\cos bx}{b}+\frac{a}{b}\int e^{ax}\cos bx\,dx \qquad \cdots(3)$$

$$=\frac{a}{b}I_1-\frac{e^{ax}\cos bx}{b} \quad \cdots(4)$$

From (2) by transposition we get,

$$bI_1+aI_2=e^{ax}\sin bx \qquad \cdots(5)$$

and from (4) by transposition we get,

$$aI_1-bI_2=e^{ax}\cos bx \qquad \cdots(6)$$

From $(5)\times b+(6)\times a$ we get,

$$(a^2+b^2)I_1=e^{ax}(a\cos bx+b\sin bx)$$

or, $I_1=\frac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}$ $\cdots(7)$

Again, from$(5)\times a-(6)\times b$ we get,

$$(a^2+b^2)I_2=e^{ax}(a\sin bx-b\cos bx)$$

or, $I_2=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$ $\cdots(8)$

Alternative Method.

In the first method I_1 and I_2 have been determined simultaneously. They can also be determined separately.

$$I_1=\int e^{ax}\cos bx\,dx=\frac{e^{ax}\sin bx}{b}-\frac{a}{b}\int e^{ax}\sin bx\,dx$$

$$=\frac{e^{ax}\sin bx}{b}+\frac{a}{b^2}e^{ax}\cos bx-\frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

$$=\frac{e^{ax}(b\sin bx+a\cos bx)}{b^2}-\frac{a^2}{b^2}I_1$$

or, $\left(1+\frac{a^2}{b^2}\right)I_1=\frac{e^{ax}(a\cos bx+b\sin bx)}{b^2}$

or, $\mathrm{I}_1=\dfrac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}$

Similarly, $\mathrm{I}_2=\dfrac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$

Now, let $a=r\cos\theta$ and $b=r\sin\theta$.

$\therefore\ a^2+b^2=r^2$ or, $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\dfrac{b}{a}$

$$\therefore\ \mathrm{I}_1=\frac{e^{ax}(r\cos\theta\cos bx+r\sin\theta\sin bx)}{r^2}$$

$$=\frac{e^{ax}}{r}\cos(bx-\theta)=\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)$$

Similarly, $\mathrm{I}_2=\dfrac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\dfrac{b}{a}\right)$

$$\therefore\ \int e^{ax}\cos bx\,dx=\frac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}$$

$$=\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)$$

and $\int e^{ax}\sin bx\,dx=\dfrac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$

$$=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)$$

Remember these two forms as standard ones.

§ 3·5. $\int e^x\{f(x)+f'(x)\}\,dx$.

$\int e^x\{f(x)+f'(x)\}\,dx=\int e^x f(x)\,dx+\int e^x f'(x)\,dx.$

$=f(x)\,e^x-\int f'(x)\,e^x\,dx+\int e^x f'(x)\,dx$ [Integrating $\int e^x f(x)\,dx$ by parts.]

$=f(x)\,e^x$.

§ 3·6. Standard forms.

(i) $\int\sqrt{x^2+a^2}\,dx$ (ii) $\int\sqrt{x^2-a^2}\,dx$ (iii) $\int\sqrt{a^2-x^2}\,dx$

(i) $\int\sqrt{x^2+a^2}\,dx=\int 1.\sqrt{x^2+a^2}\,dx$

$$=\sqrt{x^2+a^2}.x-\int\frac{2x}{2\sqrt{x^2+a^2}}.\,x\,dx$$

$$=x\sqrt{x^2+a^2}-\int\frac{x^2}{\sqrt{x^2+a^2}}\,dx$$

$$=x\sqrt{x^2+a^2}-\int\frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx=x\sqrt{x^2+a^2}-\int\sqrt{x^2+a^2}\,dx$$

$$+a^2\int\frac{dx}{\sqrt{x^2+a^2}}$$

or, $2\int\sqrt{x^2+a^2}\,dx=x\sqrt{x^2+a^2}+a^2\log(x+\sqrt{x^2+a^2})$

(by transposition)

$$\therefore\ \int\sqrt{x^2+a^2}\,dx=\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\log(x+\sqrt{x^2+a^2}).$$

(ii) $\int\sqrt{x^2-a^2}\,dx=\int 1.\sqrt{x^2-a^2}\,dx$

$$=x\sqrt{x^2-a^2}-\int\frac{2x}{2\sqrt{x^2-a^2}}\,x\,dx$$

$$=x\sqrt{x^2-a^2}-\int\frac{x^2}{\sqrt{x^2-a^2}}dx \qquad \cdots(1)$$

Again, $\int\sqrt{x^2-a^2}\,dx=\int\frac{x^2-a^2}{\sqrt{x^2-a^2}}\,dx$

$$=\int\frac{x^2}{\sqrt{x^2-a^2}}dx-\int\frac{a^2}{\sqrt{x^2-a^2}}dx \qquad \cdots(2)$$

Adding, (1) and (2) we obtain,

$$2\int\sqrt{x^2-a^2}\,dx=x\sqrt{x^2-a^2}-a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$=x\sqrt{x^2-a^2}-a^2\log(x+\sqrt{x^2-a^2})$$

$$\therefore\ \int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log(x+\sqrt{x^2-a^2}).$$

(iii) $\int\sqrt{a^2-x^2}\,dx=\int 1.\sqrt{a^2-x^2}\,dx$

$$=x\sqrt{a^2-x^2}-\int\frac{-2x}{2\sqrt{a^2-x^2}}\,x\,dx$$

$$=x\sqrt{a^2-x^2}+\int\frac{x^2}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

on transposition,

$$2\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}$$

$$\therefore\quad\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Note : Putting $x=a\sin\theta$ you can easily determine this integral.

Examples 3A

Example 1. Integrate : (i) $\int xe^{mx}\,dx$ [Tripura, 1978]

(ii) $\int xe^{x}\,dx$ [Tripura, '82] (iii) $\int e^{\cos x}\sin 2x\,dx.$

[Tripura, 1978]

(i) $\int xe^{mx}dx=x.\int e^{mx}dx-\int\left\{\frac{d}{dx}(x)\int e^{mx}dx\right\}dx.$

$$=x.\frac{e^{mx}}{m}-\int 1.\frac{e^{mx}}{m}dx=x.\frac{e^{mx}}{m}-\frac{1}{m^2}e^{mx}+c$$

$$=\frac{e^{mx}}{m}\left(x-\frac{1}{m}\right)+c.$$

(ii) $\int xe^{x}dx=x\int e^{x}dx-\int\left\{\frac{d}{dx}(x)\int e^{x}dx\right\}dx$

$$=x.e^{x}-\int 1.e^{x}\,dx=xe^{x}-e^{x}dx+c=e^{x}(x-1)+c.$$

(iii) $\int e^{\cos x}\sin 2x\,dx=\int e^{\cos x}.\,2\sin x\cos x\,dx$

$=2\int e^{z}.\,z\,(-dz)$ [where $\cos x=z$ and so $-\sin x\,dx=dz$]

$=-2\int e^{z}.zdz=-2e^{z}(z-1)+c$ [see (ii) above]

$=2e^{\cos x}(1-\cos x)+c.$

Ex. 2. Integrate : (i) $\int\log x^2\,dx.$ [H. S. 1979]

(ii) $\int(x+1)^2\log x\,dx$ [H. S. 1981]

(iii) $\int x^4(\log_e^x)^2\,dx$ [H. S. 1985]

(iv) $\int x^2\log x\,dx$ [Tripura, 1979, '81]

(v) $\int x^3(\log x)^2\,dx$ [Tripura, '86]

(i) $\int\log x^2\,dx=2\int\log x\,dx$

$$=2\left[\log x.\int 1dx-\left\{\int\frac{d}{dx}(\log x)\int 1dx\right\}dx\right]$$

$$=2\left[\log x.x-\int\frac{1}{x}.x\,dx\right]=2\,[x\log x-x]+c$$

$$=2x(\log x-1)+c.$$

(ii) $\int (x+1)^2 \log x \, dx$

$$=\log x \int (x+1)^2 \, dx - \int \left\{ \frac{d}{dx} (\log x) \int (x+1)^2 \, dx \right\} dx$$

$$=\log x \, \frac{(x+1)^3}{3} - \int \frac{1}{x} \frac{(x+1)^3}{3} \, dx$$

$$=\frac{(x+1)^3}{3} \log x - \tfrac{1}{3} \int \left(x^2+3x+3+\frac{1}{x} \right) dx$$

$$=\frac{(x+1)^3}{3} \log x - \left(\frac{x^3}{9} + \frac{x^2}{2} + x + \tfrac{1}{3} \log x \right) + c.$$

(iii) $\int x^4 (\log_e^x)^2 \, dx$

$$=(\log_e^x)^2 \int x^4 \, dx - \int \left\{ \frac{d}{dx} (\log_e^x)^2 \int x^4 \, dx \right\} dx$$

$$=(\log_e^x)^2 . \frac{x^5}{5} - \int \left\{ 2 \log_e^x . \frac{1}{x} . \frac{x^5}{5} \right\} dx$$

$$=\frac{x^5}{5} (\log_e^x)^2 - \tfrac{2}{5} \int \log_e^x \, x^4 \, dx$$

$$=\frac{x^5}{5} (\log_e^x)^2 - \tfrac{2}{5} \left[\log_e^x \int x^4 \, dx - \int \left\{ \frac{d}{dx} (\log_e^x) \int x^4 \, dx \right\} dx \right]$$

$$=\frac{x^5}{5} (\log_e^x)^2 - \tfrac{2}{5} \left[(\log_e^x) \frac{x^5}{5} - \int \frac{1}{x} \frac{x^5}{5} dx \right]$$

$$=\frac{x^5}{5} (\log_e^x)^2 - \frac{2}{25} x^5 (\log_e^x) + \tfrac{2}{25} \int x^4 \, dx$$

$$=\frac{x^5}{5} (\log_e^x)^2 - \tfrac{2}{25} x^5 (\log_e^x) + \tfrac{2}{125} x^5 + c$$

$$=\frac{x^5}{125} \{ 25 (\log_e^x)^2 - 10 (\log_e^x) + 2 \} + c.$$

(iv) $\int x^2 \log x \, dx = \log x . \int x^2 \, dx - \int \left\{ \frac{d}{dx} (\log x) \int x^2 \, dx \right\} dx$

$$=\log x . \frac{x^3}{3} - \int \frac{1}{x} . \frac{x^3}{3} \, dx$$

$$=\frac{x^3}{3} \log x - \tfrac{1}{3} \int x^2 \, dx = \frac{x^3}{3} \log x - \frac{x^3}{9} + c$$

$$=\frac{x^3}{9} (3 \log x - 1) + c.$$

(v) $\int x^3 (\log x)^2 \, dx$

$$=(\log x)^2 \int x^3\, dx-\int\left\{\frac{d}{dx}(\log x)^2 \int x^3\, dx\right\}dx$$

$$=(\log x)^2\frac{x^4}{4}-\int\left(2\log x.\frac{1}{x}.\frac{x^4}{4}\right)dx$$

$$=\frac{x^4}{4}(\log x)^2-\tfrac{1}{2}\left[\log x\int x^3\, dx-\int\left\{\frac{d}{dx}(\log x)\int x^3\, dx\right]dx\right.$$

$$=\frac{x^4}{4}(\log x)^2-\tfrac{1}{2}(\log x)\frac{x^4}{4}+\tfrac{1}{2}\int\frac{1}{x}.\frac{x^4}{4}\,dx$$

$$=\frac{x^4}{4}(\log x)^2-\tfrac{1}{8}x^4\log x+\tfrac{1}{8}\int x^3\, dx$$

$$=\frac{x^4}{4}(\log x)^2-\tfrac{1}{8}x^4\log x+\tfrac{1}{32}x^4+c$$

$$=\tfrac{1}{32}x^4\{8(\log x)^2-4\log x+1\}+c.$$

Ex. 3. Integrate :

(i) $\int x^2 \sin x\, dx$ [H· S. '79 ; Tripura, '85]

(ii) $\int x^3 \sin x\, dx$ [Tripura, 1984]

(iii) $\int x \sec^2 x\, dx$ [Tripura, 1980]

(iv) $\int x \sin^2 x\, dx$ [Tripura, 1985]

(v) $\int x \cos^2 2x\, dx$ [Joint Entrance, 1978]

(vi) $\int x^2 \cos^2 x\, dx.$

(i) $\int x^2 \sin x\, dx=x^2\int \sin x\, dx-\int\left\{\frac{d}{dx}(x^2)\int \sin x\, dx\right\}dx$

$$=x^2(-\cos x)-\int 2x(-\cos x)\, dx$$

$$=-x^2\cos x+2\int x\cos x\, dx$$

$$=-x^2\cos x+2\left[x\int\cos x\, dx-\int\left\{\frac{d}{dx}(x)\int\cos x\, dx\right\}dx\right]$$

$$=-x^2\cos x+2[x\sin x-\int\sin x\, dx]$$

$$=-x^2\cos x+2[x\sin x+\cos x]+c$$

$$=-x^2\cos x+2x\sin x+2\cos x+c.$$

(ii) $\int x^3 \sin x\, dx$

$$=x^3\int\sin x\, dx-\int\left\{\frac{d}{dx}(x^3)\int\sin x\, dx\right\}dx$$

$$=x^3(-\cos x)-\int 3x^2(-\cos x)\, dx$$

$$=-x^3\cos x+3\int x^2\cos x\, dx$$

$$=-x^3\cos x+3\left[x^2\int\cos x\, dx-\int\left\{\frac{d}{dx}(x^2)\int\cos x\, dx\right\}dx\right]$$

$\therefore$ Given integral$=\int z.\sin z\,dz$

$$=z\int \sin z\,dz-\int\left\{\frac{d}{dz}(z)\int \sin z\,dz\right\}dz=-z\cos z-\int 1.(-\cos z)dz$$

$$=-z\cos z+\int \cos z\,dz=-z\cos z+\sin z+c$$

$$=-\sqrt{1-x^2}\sin^{-1}x+x+c$$

$$[\because \sin z=x \quad \therefore \cos z=\sqrt{1-\sin^2 z}=\sqrt{1-x^2}]$$

(iii) $\int(\sin^{-1}x)^2dx=\int 1.(\sin^{-1}x)^2dx$

$$=(\sin^{-1}x)^2\int 1.dx-\int\left(\frac{d}{dx}(\sin^{-1}x)^2\int 1.dx\right)dx$$

$$=x(\sin^{-1}x)^2-\int\frac{(2\sin^{-1}x)}{\sqrt{1-x^2}}x\,dx$$

$$=x(\sin^{-1}x)^2-2\left[\sin^{-1}x\int\frac{x\,dx}{\sqrt{1-x^2}}\right.$$

$$\left.-\int\left(\frac{d}{dx}(\sin^{-1}x)\int\frac{x}{\sqrt{1-x^2}}dx\right)dx\right]$$

$$=x(\sin^{-1}x)^2+2\left[\sin^{-1}x.\sqrt{1-x^2}+2\int\frac{1}{\sqrt{1-x^2}}(-\sqrt{1-x^2})dx\right]$$

$$=x(\sin^{-1}x)^2+2\sqrt{1-x^2}\sin^{-1}x-\int 2.dx$$

$$=x(\sin^{-1}x)^2+2\sqrt{1-x^2}\sin^{-1}x-2x$$

(iv) $\int\cos^{-1}\sqrt{x}\,dx=\int 1.\cos^{-1}\sqrt{x}\,dx.$

$$=\cos^{-1}\sqrt{x}\int 1\,dx-\int\left\{\frac{d}{dx}(\cos^{-1}\sqrt{x})\int 1.dx\right\}dx$$

$$=x\cos^{-1}\sqrt{x}-\int-\frac{1}{2\sqrt{1-x}\sqrt{x}}.x\,dx$$

$$=x\cos^{-1}\sqrt{x}+\tfrac{1}{2}\int\frac{\sqrt{x}\,dx}{\sqrt{1-x}}.$$

Now, for $\int\frac{\sqrt{x}}{\sqrt{1-x}}dx$ let $x=\sin^2\theta$

$$\therefore \int\frac{\sqrt{x}\,dx}{\sqrt{1-x}}=\int\frac{\sin\theta\,2\sin\theta\cos\theta\,d\theta}{\cos\theta}=\int 2\sin^2\theta\,d\theta$$

$$=\int(1-\cos 2\theta)d\theta=\theta-\frac{\sin 2\theta}{2}=\sin^{-1}\sqrt{x}-\frac{2\sqrt{x}\sqrt{1-x}}{2}$$

$$=\sin^{-1}\sqrt{x}-\sqrt{x-x^2}$$

∴ Given integral $= x \cos^{-1} \sqrt{x} + \dfrac{\sin^{-1} \sqrt{x}}{2} - \dfrac{\sqrt{x-x^2}}{2}$

$= x \cos^{-1} \sqrt{x} + \frac{1}{2}\left(\dfrac{\pi}{2} - \cos^{-1} \sqrt{x}\right) - \dfrac{\sqrt{x-x^2}}{2}$

$= (x-\frac{1}{2})\cos^{-1} \sqrt{x} - \dfrac{\sqrt{x-x^2}}{2} + \dfrac{\pi}{4}$

∴ $\int \cos^{-1} \sqrt{x}\, dx = (x-\frac{1}{2})\cos^{-1} \sqrt{x} - \dfrac{\sqrt{x-x^2}}{2} + c.$

(v) $\int \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)dx = \int \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right). \sec^2\theta\, d\theta$

[Let $x = \tan\theta$; $dx = \sec^2\theta\, d\theta$]

$= \int \sin^{-1}(\sin 2\theta). \sec^2\theta\, d\theta = 2\int \theta \sec^2\theta\, d\theta$

$= 2[\theta\tan\theta - \int 1. \tan\theta\, d\theta] = 2(\theta\tan\theta - \log \sec. \theta) + c$

$= 2(x\tan^{-1}x - \log\sqrt{1+x^2}) + c.$

Ex. 5. (*a*) Integrate :

(i) $\int e^x\left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)dx$ [H. S. '79 ; Joint Entrance, 1983]

(ii) $\int \dfrac{e^x}{x}(x\log x + 1)dx$ [H. S. 1982]

(iii) $\int \dfrac{xe^x}{(1+x)^2}dx$ [Joint Entrance, 1980, 1987 ; Tripura, 1986]

(iv) $\int e^{3x}\dfrac{x}{(3x+2)^3}dx$ [Joint Entrance, 1982]

(v) $\int e^x(\tan x + \sec^2 x)dx.$

(vi) $\int e^x(\tan x - \log\cos x)dx$ [State Council W.B., 1986]

(vii) $\int \dfrac{(x-1)e^x}{(x+1)^3}$ [I. I. T., 1983 ; State Council W. B., 1987]

(*b*) If $c = \int e^x \cos x\, dx$ and $s = \int e^x \sin x\, dx$,

then prove that $c + s = e^x \sin x$. [H. S. 1978]

(*a*) (i) $\int e^x\left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)dx = \int e^x. \dfrac{1}{x} - \int e^x \dfrac{1}{x^2}dx$

$= \dfrac{1}{x}\int e^x dx - \int\left\{\dfrac{d}{dx}\left(\dfrac{1}{x}\right)\int e^x dx\right\}dx - \int e^x. \dfrac{1}{x^2}dx$

$$=\frac{1}{x}.e^x+c-\int\left(-\frac{1}{x^2}\right)e^x dx-\int\frac{e^x}{x^2}dx$$

$$=\frac{e^x}{x}+c+\int\frac{e^x}{x^2}dx-\int\frac{e^x}{x^2}dx=\frac{e^x}{x}+c.$$

(ii) $\int\frac{e^x}{x}(x\log x+1)dx=\int e^x\log x+\int\frac{e^x}{x}dx$

$$=\log x\int e^x dx-\int\left\{\frac{d}{dx}(\log x)\int e^x dx\right\}dx+\int\frac{e^x}{x}dx$$

$$=e^x.\log x+c-\int\frac{1}{x}.e^x dx+\int\frac{e^x}{x}dx$$

$$=e^x.\log x+c.$$

(iii) $\int\frac{x\,e^x}{(1+x)^2}dx=\int\frac{x+1-1}{(1+x)^2}e^x\,dx$

$$=\int\left\{\frac{1}{1+x}-\frac{1}{(1+x)^2}\right\}e^x\,dx$$

$$=\int\frac{1}{1+x}e^x\,dx-\int\frac{1}{(1+x)^2}e^x\,dx$$

$$=\frac{1}{1+x}\int e^x\,dx-\int\left\{\frac{d}{dx}\left(\frac{1}{1+x}\right)\int e^x\,dx\right\}dx-\int\frac{1}{(1+x)^2}e^x\,dx$$

$$=\frac{1}{1+x}e^x+c-\int\left\{\frac{d}{dx}\left(\frac{1}{1+x}\right)\int e^x\,dx\right.-\int\frac{1}{(1+x)^2}e^x\,dx$$

$$=\frac{1}{1+x}e^x+c-\int-\frac{1}{(1+x)^2}e^x\,dx-\int\frac{e^x}{(1+x)^2}e^x\,dx$$

$$=\frac{e^x}{1+x}+c+\int\frac{e^x}{(1+x)^2}dx-\int\frac{e^x}{(1+x)}dx=\frac{e^x}{1+x}+c.$$

(iv) $\int e^{3x}\frac{x}{(3x+2)^3}dx=\int e^{3x}\frac{3x+2-2}{3(3x+2)^3}dx$

$$=\int e^{3x}.\left\{\frac{1}{3(3x+2)^2}-\frac{2}{3(3x+2)^3}\right\}dx=\int\frac{e^{3x}}{3(3x+2)^2}dx-\int\frac{e^{3x}.\,2dx}{3(3x+2)^3}$$

$$=\frac{1}{3(3x+2)^2}\int e^{3x}\,dx-\int\left[\frac{d}{dx}\frac{1}{3(3x+2)^2}\int e^{3x}\,dx\right]dx-\int\frac{e^{3x}.\,2dx}{3(3x+2)^2}$$

$$=\frac{1}{9(3x+2)^2}e^{3x}+c+\int\frac{2e^{3x}}{3(3x+2)^3}dx-\int\frac{e^{3x}.\,2}{3(3x+2)^3}dx$$

$$=\frac{1}{9(3x+2)^2}e^{3x}+c.$$

(v) $\int e^x(\tan x+\sec^2 x)dx=\int e^x \tan x\,dx+\int e^x \sec^2 x\,dx$

$=\tan x.\int e^x\,dx-\int\left\{\frac{d}{dx}(\tan x)\int e^x\,dx\right\}dx+\int e^x \sec^2 x\,dx$

$=\tan x\,.\,e^x+c-\int \sec^2 x\,.\,e^x\,dx+\int e^x \sec^2 x\,dx$

$=e^x \tan x+c.$

(vi) $\int e^x(\tan x-\log\cos x)dx$

$=\int e^x \tan x\,dx-\int e^x \log\cos x\,dx$

$=\int e^x \tan x\,dx-\Big[\log\cos x\int e^x\,dx$

$\qquad\qquad-\int\left\{\frac{d}{dx}(\log\cos x)\int e^x\,dx\right\}dx\Big]$

$=\int e^x \tan x\,dx-[\log\cos x.\,e^x+c-\int(-\tan x).\,e^x\,dx$

$=\int e^x \tan x\,dx-e^x\log\cos x+c-\int e^x \tan x\,dx$

$=-e^x\log\cos x+c.$

(vii) $\int\frac{x-1}{(x+1)^3}e^x\,dx=\int e^x\frac{x+1-2}{(x+1)^3}dx$

$=\int e^x\frac{x+1}{(x+1)^3}dx-\int\frac{2e^x\,dx}{(x+1)^3}=\int e^x\frac{1}{(x+1)^2}dx-\int\frac{2e^x}{(x+1)^3}dx$

$=\frac{1}{(x+1)^2}\int e^x\,dx-\int\left\{\frac{d}{dx}\frac{1}{(x+1)^2}\int e^x\,dx\right\}-\int\frac{2e^x\,dx}{(x+1)^3}$

$=\frac{1}{(x+1)^2}e^x+c+\int\frac{2}{(x+1)^3}\,e^x\,dx-\int\frac{2e^x\,dx}{(x+1)^3}=\frac{e^x}{(x+1)^2}+c.$

(*b*) $c+s=\int e^x\cos x\,dx+\int e^x\sin x\,dx$

$=\int e^x\cos x\,dx+\sin x\int e^x\,dx-\int\left\{\frac{d}{dx}(\sin x)\int e^x\,dx\right\}dx$

$=\int e^x\cos x\,dx+e^x\sin x+c-\int e^x\cos x.\,dx$

$=e^x\sin x+c.$

Ex. 6. Integrate :

(*a*) $\int e^x\cos x\,dx$ [Tripura, 1983] (*b*) $\int e^x\sin x\,dx$

(*a*) Let $I=\int e^x\cos x\,dx$

$=\cos x.\int e^x\,dx-\int\left\{\frac{d}{dx}(\cos x)\int e^x\,dx\right\}dx$

$=\cos x.\,e^x-\int(-\sin x)e^x\,dx=\cos x.\,e^x+\int\sin x.\,e^x\,dx$

$=e^x\cos x+\sin x\int e^x\,dx-\int\left\{\frac{d}{dx}(\sin x)\int e^x\,dx\right\}dx$

$=e^x\cos x+\sin x\,e^x+c'-\int e^x\cos x\,dx$

$=e^x(\cos x+\sin x)+c'-I$

$\therefore \quad 2I = e^x (\cos x + \sin x) + c'$

or, $I = \dfrac{e^x}{2} (\cos x + \sin x) + c$

(b) $\int e^x \sin x \, dx = \dfrac{e^x}{\sqrt{2}} \sin (x - \tan^{-1} 1) + c$

[see § 3·4 ; here $a = b = 1$]

$$= \frac{e^x}{\sqrt{2}} \sin \left(x - \frac{\pi}{4} \right) + c = \frac{e^x}{\sqrt{2}} \left\{ \sin x \cos \frac{\pi}{4} - \cos x \sin \frac{\pi}{4} \right\} + c$$

$$= \frac{e^x}{2} (\sin x - \cos x) + c$$

Ex. 7. $\int e^x \cos^2 x \, dx = \frac{1}{2} \int e^x (1 + \cos 2x) \, dx.$

$$= \tfrac{1}{2} \int e^x \, dx + \tfrac{1}{2} \int e^x \cos 2x \, dx.$$

$$= \tfrac{1}{2} e^x + \tfrac{1}{2} \frac{e^x (\cos 2x + 2 \sin 2x)}{1+4} + c$$

$$= \tfrac{1}{2} e^x \{ 1 + \tfrac{1}{5} (\cos 2x + 2 \sin 2x) \} + c.$$

Ex. 8. $\int \sqrt{x^2 + 3} \, dx = \dfrac{x \sqrt{x^2+3}}{2} + \frac{3}{2} \log (x + \sqrt{x^2 + 3}).$

Ex. 9. $\int \sqrt{x^2 - 16} \, dx = \dfrac{x \sqrt{x^2 - 16}}{2} - 8 \log (x + \sqrt{x^2 - 16}).$

Ex. 10. $\int \sqrt{a^2 - b^2 x^2} \, dx = \frac{1}{b} \int \sqrt{a^2 - t^2} \, dt$

[Let $bx = t \quad \therefore \quad b \, dx = dt$]

$$= \frac{1}{b} \left\{ \frac{t \sqrt{a^2 - t^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{t}{a} \right\}$$

$$= \frac{1}{b} \left\{ \frac{bx \sqrt{a^2 - b^2 x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{bx}{a} \right\}.$$

Ex. 11. $\int \sec^3 x \, dx.$ Let $\tan x = t.$ $\therefore$ $\sec^2 x \, dx = dt$ and $\sec x = \sqrt{1 + \tan^2 x} = \sqrt{1 + t^2}$

$\therefore \quad \int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx = \int \sqrt{1 + t^2} \, dt$

$$= \frac{t \sqrt{1+t^2}}{2} + \tfrac{1}{2} \log (t + \sqrt{1 + t^2})$$

$$= \frac{\tan x \sec x}{2} + \tfrac{1}{2} \log (\sec x + \tan x).$$

Ex. 12. $\int (x - 1) \sqrt{x^2 + x + 1} \, dx$

Here $\dfrac{d}{dx} (x^2 + x + 1) = 2x + 1$

Now, $(x-1)\sqrt{x^2+x+1}$

$$=\tfrac{1}{2}(2x+1)\sqrt{x^2+x+1}-\tfrac{3}{2}\sqrt{x^2+x+1}$$

∴ Given integral $=\frac{1}{2}\int(2x+1)\sqrt{x^2+x+1}\,dx$

$$-\tfrac{3}{2}\int\sqrt{x^2+x+1}\,dx$$

$$=\tfrac{1}{2}\cdot\tfrac{2}{3}(x^2+x+1)^{\frac{3}{2}}-\tfrac{3}{2}\int\sqrt{(x+\tfrac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$=\tfrac{1}{3}(x^2+x+1)^{\frac{3}{2}}-\frac{3(x+\frac{1}{2})\sqrt{x^2+x+1}}{4}$$

$$-\tfrac{3}{2}\cdot\tfrac{3}{8}\log\left(x+\tfrac{1}{2}+\sqrt{x^2+x+1}\right)+c$$

$$=\tfrac{1}{3}(x^2+x+1)^{\frac{3}{2}}-\tfrac{3}{8}(2x+1)\sqrt{x^2+x+1}$$

$$-\tfrac{9}{16}\log\left(x+\tfrac{1}{2}+\sqrt{x^2+x+1}\right)+c$$

Note. To reduce $(x-1)\sqrt{x^2+x+1}$ in the form
$\frac{1}{2}(2x+1)\sqrt{x^2+x+1}-\frac{3}{2}\sqrt{x^2+x+1}$
let $x-1=\alpha(2x+1)+\beta$. ∴ $\alpha=\frac{1}{2}$ and $\alpha+\beta=-1$ or, $\beta=-\frac{3}{2}$.

Ex. 13. $\int\sqrt{(x-\alpha)(\beta-x)}\,dx$. $[\alpha<x<\beta]$

$$=\int\sqrt{-x^2+(\alpha+\beta)x-\alpha\beta}\,dx$$

$$=\int\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2-\left(x-\frac{\alpha+\beta}{2}\right)^2}\,dx$$

$$=\frac{\left(x-\frac{\alpha+\beta}{2}\right)\sqrt{(x-\alpha)(\beta-x)}}{2}+\frac{\left(\frac{\beta-\alpha}{2}\right)^2}{2}\sin^{-1}\frac{x-\frac{\alpha+\beta}{2}}{\frac{\beta-\alpha}{2}}+c$$

$$=\tfrac{1}{4}\left\{(2x-\alpha-\beta)\sqrt{(x-\alpha)(\beta-x)}+\frac{(\beta-\alpha)^2}{8}\sin^{-1}\frac{2x-\alpha-\beta}{\beta-\alpha}\right\}+c$$

Alternative method : Let $x=\alpha\cos^2\theta+\beta\sin^2\theta$

∴ $dx=(\beta-\alpha)\sin 2\theta\,d\theta$

$x-\alpha=\alpha\cos^2\theta+\beta\sin^2\theta-\alpha=(\beta-\alpha)\sin^2\theta.$

$\beta-x=\beta-\alpha\cos^2\theta-\beta\sin^2\theta=(\beta-\alpha)\cos^2\theta.$

∴ Given integral $=\int\sqrt{(\beta-\alpha)\sin^2\theta\,(\beta-\alpha)\cos^2\theta}\,(\beta-\alpha)\sin 2\theta\,d\theta$

$$=\int\tfrac{1}{4}(\beta-\alpha)^2\,2\sin^2 2\theta\,d\theta$$

$$=\tfrac{1}{4}(\beta-\alpha)^2\int(1-\cos 4\theta)\,d\theta$$

$$=\tfrac{1}{4}(\beta-\alpha)^2\,\theta-\tfrac{1}{16}(\beta-\alpha)^2\sin 4\theta.$$

Now, $x-\alpha=(\beta-\alpha)\sin^2\theta$, ∴ $\sin\theta=\sqrt{\dfrac{x-\alpha}{\beta-\alpha}}$, Again, $\sin 4\theta = 4\sin\theta\cos\theta\cos 2\theta$

$$= 4\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta)$$

$$= 4\sqrt{\frac{x-\alpha}{\beta-\alpha}}\sqrt{1-\frac{x-\alpha}{\beta-\alpha}}\left\{\frac{\beta-x}{\beta-\alpha}-\frac{x-\alpha}{\beta-\alpha}\right\}$$

$$= 4\sqrt{\frac{x-\alpha}{\beta-\alpha}}\sqrt{\frac{\beta-x}{\beta-\alpha}}\cdot\frac{\alpha+\beta-2x}{\beta-\alpha}$$

$$= \frac{4}{(\beta-\alpha)^2}\sqrt{(x-\alpha)(\beta-x)}\,(\alpha+\beta-2x)$$

$\therefore$ Given integral $= \frac{1}{4}(\beta-\alpha)^2 \sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}$

$$-\tfrac{1}{4}\sqrt{(x-\alpha)(\beta-x)}(\alpha+\beta-2x)+c.$$

Miscellaneous Examples 3B

Example 1. Integrate :

(a) $\int e^{-x^2} x^3\,dx$. [H. S. 1984] (b) $\int \frac{x\,dx}{1+\sin x}dx.$

[Tripura, 1987]

(c) $\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx$ [I. I. T. 1986]

(a) Let $-x^2 = z$ $\therefore$ $-2x\,dx = dz$ or, $-x\,dx = \frac{dz}{2}$

$\therefore$ $\int e^{-x^2}.\,x^3\,dx = \int e^{-x^2}.(-x^2).(-x\,dx)$

$= \int e^z z\left(\frac{dz}{2}\right) = \frac{1}{2}\int e^z.z\,dz$

$= \frac{1}{2}\left[z\int e^z\,dz - \int\left[\frac{d}{dz}(z)\int e^z\,dz\right]\right]$

$= \frac{1}{2}\{z.e^z - \int e^z\,dz\} + c$

$= \frac{1}{2}e^z(z-1) + c = -\frac{1}{2}e^{-x^2}(x^2+1) + c.$

(b) $\int \frac{x\,dx}{1+\sin x} = x.\int\frac{dx}{1+\sin x} - \int\left\{\frac{d}{dx}(x)\int\frac{dx}{1+\sin x}\right\}dx$

$= x\int\frac{1-\sin x}{\cos^2 x}dx - \int\int\frac{dx}{1+\sin x}dx$

$= x\int(\sec^2 x - \sec x\tan x)\,dx - \int\int(\sec^2 x - \sec x\tan x)\,dx\,dx$

$$=x\ (\tan x-\sec x)-\int (\tan x-\sec x)\ dx$$
$$=x\ (\tan x-\sec x)+\log\cos x+\log\ (\sec x+\tan x)+c$$
$$=x\ (\tan x-\sec x)+\log\ (\sec x\cos x+\cos x\tan x)+c$$
$$=x\ (\tan x-\sec x)+\log\ (1+\sin x)+c$$

(c) $$\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx=\int\frac{\frac{\pi}{2}-2\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$\left[\because\ \sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}\right]$$

$$=\int dx-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}\ dx$$
$$=x-\frac{4}{\pi}\left\{(x-\tfrac{1}{2})\cos^{-1}\sqrt{x}-\frac{\sqrt{x-x^2}}{2}+c\,\frac{\pi}{4}\right\}$$

[See Example 4 (iv)]

$$=x-\frac{2}{\pi}\{(2x-1)\cos^{-1}\sqrt{x}-\sqrt{x-x^2}\}+c$$

Ex. 2. Integrate : $\int\frac{x+\sin x}{1+\cos x}dx.$ [C. U. '75]

$$\int\frac{x+\sin x}{1+\cos x}dx=\int\left(\frac{x}{1+\cos x}+\frac{\sin x}{1+\cos x}\right)dx$$
$$=\int\left(x.\frac{1}{2\cos^2\frac{x}{2}}+\frac{2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx$$
$$=\tfrac{1}{2}\int x\sec^2\tfrac{x}{2}\ dx+\int\tan\tfrac{x}{2}\ dx$$
$$=\tfrac{1}{2}x\int\sec^2\tfrac{x}{2}\ dx-\int(\tfrac{1}{2}.1.\int\sec^2\tfrac{x}{2}\ dx)dx+\int\tan\tfrac{x}{2}\ dx$$

(Integrating the first integral by parts]

$$=\tfrac{1}{2}x.2\tan\tfrac{x}{2}+c-\int\tan\tfrac{x}{2}\ dx+\int\tan\tfrac{x}{2}\ dx=x\tan\tfrac{x}{2}+c.$$

Ex. 3. Evaluate : $\int e^x\log\ (e^{2x}+3e^x+2)dx.$

Let $e^x=z\quad\therefore\quad e^xdx=dz$

Hence given integral $=\int\log\ (z^2+3z+2)\ dz=\int\log\ \{(z+1)(z+2)\}\ dz$

$$=\int\log\ (z+1)\ dz+\int\log\ (z+2)\ dz.$$

Now, $\int\log\ (z+1)\ dz=z.\log\ (z+1)-\int\frac{z}{z+1}dz$

$$=z\log\ (z+1)-\int dz+\int\frac{dz}{z+1}$$
$$=z\log\ (z+1)-z+\log\ (z+1).$$

Similarly, $\int \log (z+2)\, dz = z \log (z+2) - z + 2 \log (z+2)$,

Hence the given integral

$$= z \log (z+1) + z \log (z+2) - 2z + \log (z+1) + \log (z+2)^2$$

$$= z \log (z^2+3z+2) - 2z + \log \{(z+1)(z+2)^2\}$$

$$= e^x \log (e^{2x}+3e^x+2) - 2e^x + \log \{(e^x+1)(e^{2x}+4e^x+4)\}.$$

Ex 4. Integrate : $\int \cos x \sqrt{\sin^2 x - 4 \sin x + 5}\, dx$

Let $\sin x = z$, $\therefore$ $\cos x\, dx = dz$.

$$\therefore \int \cos x \sqrt{\sin^2 x - 4 \sin x + 5}\, dx$$

$$= \int \sqrt{z^2-4z+5}\, dz = \int \sqrt{(z-2)^2+1}\, dz$$

$$= \tfrac{1}{2}(z-2) \sqrt{(z-2)^2+1} + \tfrac{1}{2} \log (z-2+\sqrt{(z-2)^2+1}) + c$$

$$= \tfrac{1}{2}(\sin x - 2) \sqrt{\sin^2 x - 4 \sin x + 5}$$

$$+ \tfrac{1}{2} \log (\sin x - 2 + \sqrt{\sin^2 x - 4 \sin x + 5}) + c.$$

Ex. 5. Integrate : $\int 2^x \sin 3x\, dx$.

$$\int 2^x \sin 3x\, dx = \int e^{x \log 2} \sin 3x\, dx.$$

$$[\because \; 2^x = e^{\log 2^x} = e^{x \log 2}]$$

$$= \frac{e^{x \log 2}}{(\log 2)^2 + 3^2} \{\log 2 \sin 3x - 3 \cos 3x\} + c$$

$$[\text{ Here } a = \log 2,\ b = 3 \,]$$

Ex. 6. Integrate : $\int x(\sin^{-1} x)^2\, dx$.

$$\int x(\sin^{-1} x)^2 dx = \int \theta^2 . \sin \theta . \cos \theta\, d\theta,$$

$$[\text{ Let } \sin^{-1} x = \theta \quad \therefore \quad x = \sin \theta;\ \cos \theta\, d\theta = dx \,]$$

$$= \tfrac{1}{2}[\int \theta^2 \sin 2\theta\, d\theta \,]$$

$$= \tfrac{1}{2}\left[\theta^2 \frac{-\cos 2\theta}{2} - \int 2\theta \frac{-\cos 2\theta}{2} d\theta\right]$$

$$= -\frac{\theta^2}{4} \cos 2\theta + \tfrac{1}{2}\int \theta \cos 2\theta\, d\theta$$

$$= -\tfrac{1}{4}\theta^2 \cos 2\theta + \tfrac{1}{2}\left\{\theta . \frac{\sin 2\theta}{2} - \int 1 . \frac{\sin 2\theta}{2} d\theta\right\}$$

$$= -\tfrac{1}{4}\, \theta^2 \cos 2\theta + \tfrac{1}{4}\, \theta \sin 2\theta + \frac{\cos 2\theta}{8} + c$$

$$=-\tfrac{1}{4}\left\{(1-2\sin^2\theta).\theta^2+2\sin\theta\,.\cos\theta.\theta+\frac{1-2\sin^2\theta}{2}\right\}+c'$$

$$=-\tfrac{1}{4}\{(1-2x^2)(\sin^{-1}x)^2+2x\sqrt{1-x^2}\,\sin^{-1}x-x^2\}+c'.$$

Ex. 7. Integrate : $\displaystyle\int\frac{e^{m\tan^{-1}x}}{(1+x^2)^2}dx$ [C. U.]

$$=\int\frac{e^{m\tan^{-1}x}}{(1+x^2)^2}dx=\int\frac{e^{m\tan^{-1}x}}{1+x^2}\cdot\frac{dx}{1+x^2}=\int\frac{e^{mz}}{1+\tan^2z}dz$$

[Putting $z=\tan^{-1}x,\ dz=\dfrac{1}{1+x^2}dx$]

$$=\int\cos^2z.e^{mz}\,dz=\int\frac{1+\cos 2z}{2}e^{mz}\,dz$$

$$=\tfrac{1}{2}\int e^{mz}dz+\tfrac{1}{2}\int e^{mz}.\cos 2z\,dz$$

$$=\tfrac{1}{2}\frac{e^{mz}}{m}+\tfrac{1}{2}\frac{e^{mz}}{m^2+2^2}(m\cos 2z+2\sin 2z)+c$$

$$=\frac{e^{m\tan^{-1}x}}{2m}\left\{1+\frac{m}{m^2+4}\left(m\frac{1-x^2}{1+x^2}+\frac{4x}{1+x^2}\right)\right\}+c.$$

$\left[\text{As }\cos 2z=\dfrac{1-\tan^2z}{1+\tan^2z}\text{ and }\sin 2z=\dfrac{2\tan z}{1+\tan^2z}\right]$

Ex. 8. Integrate : $\displaystyle\int\left\{\frac{1}{\log x}-\frac{1}{(\log x)^2}\right\}dx.$

$$\int\left\{\frac{1}{\log x}-\frac{1}{(\log x)^2}\right\}dx=\int\frac{1}{\log x}dx-\int\frac{1}{(\log x)^2}dx$$

$$=x.\frac{1}{\log x}+c-\int x\frac{d}{dx}\left(\frac{1}{\log x}\right)dx-\int\frac{1}{(\log x)^2}dx$$

[integrating the first integral by parts]

$$=\frac{x}{\log x}+c-\int x.\frac{-1}{(\log x)^2}.\frac{1}{x}dx-\int\frac{1}{(\log x)^2}dx$$

$$=\frac{x}{\log x}+c+\int\frac{1}{(\log x)^2}dx-\int\frac{1}{(\log x)^2}dx=\frac{x}{\log x}+c.$$

Ex. 9. Integrate : $\displaystyle\int\sin^{-1}\sqrt{\frac{x}{a+x}}dx$ [Joint Entrance, 1979]

$$\int\sin^{-1}\sqrt{\frac{x}{a+x}}dx$$

Let $\dfrac{x}{x+a}=\sin^2\theta\quad\therefore\quad x=a\tan^2\theta$

4. $\int \log (\sin x) \cos x \, dx.$

5. (*a*) $\int \sin^{-1}(2x\sqrt{1-x^2})dx$ (*b*) $\int \tan^{-1} \dfrac{2x}{1-x^2}dx$

(*c*) $\int \cos^{-1} \dfrac{1-x^2}{1+x^2}dx.$

6. (*a*) $\int \sin^{-1} \sqrt{x} \, dx$ (*b*) $\int (\sin^{-1} x)^3 \, dx.$

7. (*a*) $\int \cot^{-1} x \, dx$ [without determining $\int \tan^{-1} x \, dx$]

(*b*) $\int \operatorname{cosec}^{-1} x \, dx$ [without determining $\int \sec^{-1} x \, dx$]

(*c*) $\int \cos^{-1} (\frac{1}{x}) \, dx$ [without determining $\int \sec^{-1} x \, dx$]

8. (*a*) $\int e^x (\cos x - \sin x) \, dx$ (*b*) $\int e^x (x^2+2x) \, dx$

(*c*) $\int e^x \left(\tan^{-1} x + \dfrac{1}{1+x^2}\right)dx$ (*d*) $\int e^x (\log \sin x + \cot x) \, dx.$

9. $\int e^x \dfrac{x^2+1}{(x+1)^2}dx.$ **10.** $\int e^x \cos ax \, dx.$

11. (*a*) $\int e^x \sin^2 x \, dx$ (*b*) $\int e^x \sin 3x \cos x \, dx.$

12. (*a*) $\int e^{2x} \cos^3 x \, dx$ (*b*) $\int e^{2x} \sin^3 x \, dx.$

13. (*a*) $\int \sqrt{x^2+9} \, dx$ (*b*) $\int \sqrt{16-9x^2} \, dx$ (*c*) $\int \sqrt{1-a^2x^2} \, dx.$

14. (*a*) $\int \dfrac{x^2}{\sqrt{1-x^2}}dx$ (*b*) $\int \dfrac{x^2}{\sqrt{x^2+1}}dx.$

15. (*a*) $\int \sqrt{4-3x-2x^2} \, dx$ (*b*) $\int \sqrt{5-2x+x^2} \, dx.$ [C. U. '66]

Miscellaneous Exercise 3

Integrate :

1. (i) $\int \dfrac{1-\sin x}{x+\cos x}dx$ (ii) $\int \dfrac{x+\cos x}{1-\sin x}dx$

2. (i) $\int \dfrac{1-\cos x}{x-\sin x}dx$ (ii) $\int \dfrac{x-\sin x}{1-\cos x}dx$

3. (i) $\int \dfrac{x}{1+\cos x}dx$ (ii) $\int \dfrac{x}{1-\cos x}dx$

(iii) $\int \dfrac{x}{1-\sin x}dx$ (iv) $\int \dfrac{x(1+\sin x)}{\cos^2 x}dx$

4. (i) $\int e^x \sqrt{(e^{2x}-3e^x+1)} \, dx$

(ii) $\int \dfrac{\sqrt{1-2e^x+2e^{2x}}}{e^{2x}}dx$ (iii) $\int x^2 \sqrt{(x^6+x^3+1)}dx$

(iv) $\int (x+a)(x^2+b^2)\,dx$ [C. U. '66] (v) $\int \frac{\sqrt{1+x+x^2}}{x^3}dx$

5. (i) $\int 3^x \cos 4x\,dx$ (ii) $\int e^{2x}\sin x\cos x\,dx$ [C. U. '74]

(iii) $\int e^x \sin\left(\frac{\pi}{4}+x\right)dx$ [C. U. '64]

(iv) $\int e^{mx}\sin^3 x\,dx$ (v) $\int e^{-2x}\cos\frac{1}{2}x\,dx$

6. (i) $\int x(\tan^{-1}x)^2\,dx$ (ii) $\int x^2\tan^{-1}x\,dx$

7. (i) $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)^{\frac{3}{2}}}dx$ (ii) $\int \frac{e^{2\tan^{-1}x}}{(1+x^2)^{\frac{5}{2}}}dx$

(iii) $\int \frac{xe^{\sin^{-1}x}}{(1-x^2)^{\frac{1}{2}}}dx$ (iv) $\int \frac{x^3\,e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx$

8. (i) $\int \frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}dx$ (ii) $\int \frac{x^2\sin^{-1}x}{(1-x^2)^{\frac{5}{2}}}dx$ (iii) $\int \frac{x\tan^{-1}x}{(1+x^2)^{\frac{3}{2}}}dx$

9. (i) $\int\left\{\frac{1}{(\log x)^2}-\frac{2}{(\log x)^3}\right\}dx$

(ii) $\int\left\{\frac{1}{(\log x)^n}-\frac{n}{(\log x)^{n+1}}\right\}dx$

(iii) $\int\left\{\log(\log x)+\frac{1}{(\log x)^2}\right\}dx$

10. (i) $\int \sin^{-1}(3x-4x^3)\,dx$ (ii) $\int \tan^{-1}\frac{3x-x^3}{1-3x^2}dx$

(iii) $\int \tan^{-1}\sqrt{\frac{x}{x+1}}dx$ (iv) $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$

11. (i) $\int\sqrt{\frac{x}{a+x}}dx$ (ii) $\int\sqrt{\frac{x+a}{x}}dx$

(iii) $\int\sqrt{\frac{a-x}{x}}dx$ (iv) $\int\sqrt{\frac{x}{a-x}}dx$ [C. U. '62]

12. (i) $\int\frac{1}{x^2}(\tan^{-1}x)\,dx$ (ii) $\int x^6\sin^{-1}x\,dx$

13. Show that $\int\cos^n x\,dx=\frac{\sin x\cos^{n-1}x}{n}+\frac{n-1}{n}\int\cos^{n-2}x\,dx$

and hence evaluate $\int\cos^6 x\,dx$.

$\because$ $A+B=1$ and $3A+2B=0$.

Solving we get $A=-2$ and $B=3$.

So $\dfrac{x}{x^2-5x+6}=\dfrac{3}{x-2}-\dfrac{2}{x-3}$.

From the above example we get the following rule :—

If $\alpha_1, \alpha_2, \cdots, \alpha_n$ be different from one another and the degree of $f(x)$ be less than n, then $\dfrac{f(x)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)}$ can be expressed as a sum of n partial fractions in the form $\dfrac{A_1}{x-\alpha_1}+\dfrac{A_2}{x-\alpha_2}+\cdots+\dfrac{A_n}{x-\alpha_n}$.

The values of $A_1, A_2, \cdots, A_n$ can be determined by equating the coefficients of x^{n-1}, x^{n-2}, $\cdots$ and x^0 (i e., the constant term) on both sides.

Ex. 2. Express $\dfrac{x^2}{(x-1)(x-2)(x-3)}$ into sum of partial fractions and hence determine $\displaystyle\int\frac{x^2dx}{(x-1)(x-2)(x-3)}$.

Let, $\dfrac{x^2}{(x-1)(x-2)(x-3)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$

Now, $\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$

$$=\frac{A(x-2)(x-3)+B(x-3)(x-1)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$=\frac{x^2(A+B+C)-x(5A+4B+3C)+6A+3B+2C}{(x-1)(x-2)(x-3)}$$

$\therefore$ $\dfrac{x^2}{(x-1)(x-2)(x-3)}$

$$=\frac{x^2(A+B+c)-x(5A+4B+3C)+6A+3B+2C}{(x-1)(x-2)(x-3)}$$

$\therefore$ $x^2=x^2(A+B+C)-x(5A+4B+3C)+6A+3B+2C$

Now, the coefficients of x^2, x and the constant terms on both sides are equal. $\therefore$ $A+B+C=1\cdots(1)$, $5A+4B+3C=0\cdots(2)$ and $6A+3B+2C=0\cdots(3)$.

Solving the three equations we get $A=\frac{1}{2}$, $B=-4$ and $C=\frac{9}{2}$.

$$\therefore \quad \frac{x^2}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{4}{x-2}+\frac{9}{2(x-3)}$$

So, $\displaystyle\int\frac{x^2dx}{(x-1)(x-2)(x-3)}=\int\left\{\frac{1}{2(x-1)}-\frac{4}{x-2}+\frac{9}{2(x-3)}\right\}dx$

$$=\tfrac{1}{2}\int\frac{dx}{(x-1)}-4\int\frac{dx}{(x-2)}+\tfrac{9}{2}\int\frac{dx}{(x-3)}$$

$$=\tfrac{1}{2}\log(x-1)-4\log(x-2)+\tfrac{9}{2}\log(x-3)+c.$$

2. Rule of expressing a rational algebraic function when the denominator of the function can be expressed as the product of linear factors, some of which are repeated.

Rule. When the denominator is of the form $(x-a)(x-b)^m(x-c)^n$, then the function can be expressed as the sum of $(1+m+n)$ partial fractions in the form $\frac{A}{x-a}+\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\cdots+\frac{B_m}{(x-b)^m}+\frac{C_1}{(x-c)}+\frac{C_2}{(x-c)^2}+\cdots+\frac{C_n}{(x-c)^n}$. The constants $A, B_1, B_2, \cdots B_m, C_1, C_2, \cdots C_n$ can be determined from the equality of the coefficients of the different powers of x on both sides.

Ex. 3. Integrate $\frac{x^2}{(x+1)(x+2)^2}$ w. r. to x by expressing it as the sum of partial fractions.

Let $\displaystyle\frac{x^2}{(x+1)(x+2)^2}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$

Now $\displaystyle\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$

$$=\frac{A(x+2)^2+B(x+1)(x+2)+C(x+1)}{(x+1)(x+2)^2}$$

$$=\frac{x^2(A+B)+x(4A+3B+C)+4A+2B+C}{(x+1)(x+2)^2}$$

$$\therefore \quad \frac{x^2}{(x+1)(x+2)^2}=\frac{x^2(A+B)+x(4A+3B+C)+4A+2B+C}{(x+1)(x+2)^2}$$

$$\therefore \quad x^2=x^2(A+B)+x(4A+3B+C)+4A+2B+C.$$

Now the coefficients of x^2, x and the constant terms on both the sides are equal.

So, $A+B=1 \cdots (1)$, $4A+3B+C=0 \cdots (2)$ and $4A+2B+C=0 \cdots (3)$

Solving equations (1), (2) and (3) we get $A=1$, $B=0$ and $C=-4$.

$$\therefore \quad \frac{x^2}{(x+1)(x+2)^2}=\frac{1}{x+1}-\frac{4}{(x+2)^2}$$

$$\text{So,} \quad \int \frac{x^2 dx}{(x+1)(x+2)^2}=\int \left\{\frac{1}{x+1}-\frac{4}{(x+2)^2}\right\} dx$$

$$=\int \frac{1}{x+1}\,dx-4\int \frac{dx}{(x+1)^2}=\log\,(x+1)-4\left(-\frac{1}{x+2}\right)+c$$

$$=\log\,(x+1)+\frac{4}{x+2}+c.$$

Ex. 4. Integrate : $\int \frac{dx}{x(x+1)^2}$.

$$\text{Let,} \quad \frac{1}{x(x+1)^2}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$\text{Now,} \quad \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}=\frac{A(x+1)^2+Bx(x+1)+C.x}{x(x+1)^2}$$

$$=\frac{x^2(A+B)+x(2A+B+C)+A}{x(x+1)^2}$$

$$\therefore \quad \frac{1}{x(x+1)^2}=\frac{x^2(A+B)+x(2A+B+C)+A}{x(x+1)^2}$$

Now, the coefficients of x^2, x and the constant terms on both the sides are equal.

$\therefore$ $A+B=0 \cdots (1)$, $2A+B+C=0 \cdots (2)$, $A=1 \cdots (3)$

Solving equations (1), (2) and (3) we obtain,

$A=1$, $B=-1$, $C=-1$.

$$\therefore \quad \frac{1}{x(x+1)^2}=\frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}$$

$$\therefore \quad \int \frac{dx}{x(x+1)^2}=\int \frac{dx}{x}-\int \frac{dx}{x+1}-\int \frac{dx}{(x+1)^2}$$

$$=\log x-\log(x+1)+\frac{1}{x+1}+c$$

$$=\log \frac{x}{x+1}+\frac{1}{x+1}+c.$$

Ex. 5. Integrate : $\int \frac{x^2+x-1}{x^3+x^2-6x}dx$ [P. P. 1931]

$x^3+x^2-6x=x(x^2+x-6)=x(x+3)(x-2)$

Let $\frac{x^2+x-1}{x(x+3)(x-2)}=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-2}$

Now, $\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-2}=\frac{A(x+3)(x-2)+Bx(x-2)+Cx(x+3)}{x(x+3)(x-2)}$

$\therefore \quad \frac{x^2+x-1}{x(x+3)(x-2)}=\frac{A(x+3)(x-2)+Bx(x-2)+Cx(x+3)}{x(x+3)(x-2)}$

$\therefore \quad x^2+x-1=A(x+3)(x-2)+Bx(x-2)+Cx(x+3)$

Now this equality is true for all values of x. So putting $x=0$, -3 and 2 successively on the both sides we get $A=\frac{1}{6}$, $B=\frac{1}{3}$, $C=\frac{1}{2}$.

$\therefore \quad \int \frac{x^2+x-1}{x^3+x_2-6x}dx=\frac{1}{6}\int\frac{dx}{x}+\frac{1}{3}\int\frac{dx}{x+3}+\frac{1}{2}\int\frac{dx}{x-2}$

$=\frac{1}{6}\log x+\frac{1}{3}\log(x+3)+\frac{1}{2}\log(x-2)+c.$

Note. In the first four examples, the values of A, B, C etc. were determined by equating coefficients of different powers of x on both sides. In example 5 above we have used an alternative method. In example 6 below, we shall use both the methods. But the method used in the first four examples (i.e., the method of equating coefficients) is the general method. But solving the corresponding equations frequently becomes troublesome.

Ex. 6. Integrate : $\int \frac{dx}{(x-a)^2(x-b)}$

Let $\frac{1}{(x-a)^2(x-b)}=\frac{A}{(x-a)^2}+\frac{B}{x-a}+\frac{C}{x-b}$

$$=\frac{A(x-b)+B(x-a)(x-b)+C(x-a)^2}{(x-a)^2(x-b)}$$

$\therefore \quad 1=A(x-b)+B(x-a)(x-b)+C(x-a)^2$

Putting $x=a$ on both sides we get $1=A(a-b)$ or, $A=\frac{1}{a-b}$.

Again putting $x=b$ on both sides we get $1=C(b-a)^2$

$\therefore \quad C=\frac{1}{(b-a)^2}$.

Also coefficients of x^2 on both sides are equal

$\therefore \quad B+C=0. \quad \therefore \quad B=-C=-\frac{1}{(b-a)^2}$.

$$\therefore \quad \frac{1}{(x-a)^2(x-b)}$$

$$=\frac{1}{(a-b)}\cdot\frac{1}{(x-a)^2}-\frac{1}{(b-a)^2}\cdot\frac{1}{x-a}+\frac{1}{(b-a)^2}\cdot\frac{1}{x-b}$$

$$\therefore \quad \int\frac{dx}{(x-a)^2(x-b)}$$

$$=\frac{1}{a-b}\int\frac{dx}{(x-a)^2}-\frac{1}{(b-a)^2}\int\frac{dx}{x-a}+\frac{1}{(b-a)^2}\int\frac{dx}{x-b}$$

$$=\frac{1}{a-b}\left(-\frac{1}{x-a}\right)-\frac{1}{(b-a)^2}\log(x-a)+\frac{1}{(b-a)^2}\log(x-b)+k$$

$$=\frac{1}{(b-a)(x-a)}+\frac{1}{(b-a)^2}\log\frac{x-b}{x-a}+k.$$

3. If the denominator contains one or more different quadratic factors, the algebraic rational function will have corresponding to every quadratic factor x^2+bx+c $(c\neq 0)$,

a partial fraction of the form $\frac{Ax+B}{x^2+bx+c}$.

Ex. 7. Integrate : $\int\frac{x dx}{x^3-1}$.

Let $\frac{x}{x^3-1}=\frac{x}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$;

$\therefore \quad x=A(x^2+x+1)+(Bx+C)(x-1)$

Putting $x=1$ on both sides we get $1=3A$ $\therefore$ $A=\frac{1}{3}$.

Again coefficients of x^2 on both sides are equal.

$\therefore$ $A+B=0$, $\therefore$ $B=-A=-\frac{1}{3}$.

Again constant terms on both side are equal.

$A-C=0$ $\therefore$ $C=A=\frac{1}{3}$.

$$\therefore \quad \int\frac{x\,dx}{x^3-1}=\tfrac{1}{3}\int\left(\frac{1}{x-1}-\frac{x-1}{x^2+x+1}\right)dx$$

$$=\tfrac{1}{3}\int\frac{dx}{x-1}-\tfrac{1}{3}\int\frac{x-1}{x^2+x+1}dx=\tfrac{1}{3}\int\frac{dx}{x-1}-\tfrac{1}{6}\int\frac{2x+1-3}{x^2+x+1}dx$$

$$=\tfrac{1}{3}\int\frac{dx}{x-1}-\tfrac{1}{6}\int\frac{2x+1}{x^2+x+1}dx+\tfrac{1}{2}\int\frac{dx}{x^2+x+1}$$

$$=\tfrac{1}{3}\log(x-1)-\tfrac{1}{6}\log(x^2+x+1)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+c.$$

Ex. 8. Integrate : $\int \frac{x\,dx}{(1+x)(1+x^2)}$.

Let $\frac{x}{(1+x)(1+x^2)}=\frac{A}{1+x}+\frac{Bx+C}{1+x^2}$

$\therefore\ x=A(1+x^2)+(Bx+C)(1+x)$.

Putting $x=-1$ on both sides we get $-1=2A$, $\therefore\ A=-\frac{1}{2}$.

Equating coefficients of x^2 on both sides we get,

$$0=A+B,\quad \therefore\ B=-A=\tfrac{1}{2}.$$

Again the constant terms on both side are equal.

$\therefore\ 0=A+C,\ \therefore\ C=-A=\frac{1}{2}$.

$$\therefore\ \frac{x}{(1+x)(1+x^2)}=\frac{1}{2}\left(\frac{x+1}{1+x^2}-\frac{1}{1+x}\right)$$

$$\therefore\ \int\frac{x\,dx}{(1+x)(1+x^2)}=\int\frac{1}{2}\left(\frac{x+1}{1+x^2}-\frac{1}{1+x}\right)dx$$

$$=\frac{1}{4}\int\frac{2x}{1+x^2}dx+\frac{1}{2}\int\frac{dx}{1+x^2}-\frac{1}{2}\int\frac{dx}{1+x}$$

$$=\frac{1}{4}\log(1+x^2)+\frac{1}{2}\tan^{-1}x-\frac{1}{2}\log(1+x)+c.$$

§ 4·3. Integration of rational algebraic function when the degree of the numerator is greater than or equal to the degree of the denominator.

Let $\frac{f(x)}{g(x)}$ be an algebraic rational function. To integrate $\frac{f(x)}{g(x)}$ with respect to x divide $f(x)$ by $g(x)$ untill the degree of the remainder is less than the degree of the denominator. Let in such a division $q(x)$ and $r(x)$ be respectively the quotient and the remainder where $q(x)$ and $r(x)$ are polynomials.

$$\therefore\ \frac{f(x)}{g(x)}=q(x)+\frac{r(x)}{g(x)}\text{ and }\int\frac{f(x)}{g(x)}dx$$

$$=\int\left\{q(x)+\frac{r(x)}{g(x)}\right\}dx=\int q(x)dx+\int\frac{r(x)}{g(x)}dx.$$

Now, as $q(x)$ is a polynomial, so $\int q(x)\,dx$ can be easily determined. Also $\int\frac{r(x)}{g(x)}\,dx$ can be determined following the rules discussed in § 42.

Example 1. Integrate : $\int \frac{x^3\,dx}{x^2+7x+12}$

$$
\begin{array}{r@{\,}l}
x^2+7x+12\big) & x^3 \qquad\qquad \big(x-7 \\
& \underline{x^3+7x^2+12x} \\
& \quad -7x^2-12x \\
& \quad \underline{-7x^2-49x-84} \\
& \qquad\qquad 37x+84
\end{array}
$$

$$\therefore\quad \frac{x^3}{x^2+7x+12}=x-7+\frac{37x+84}{x^2+7x+12}=x-7+\frac{37x+84}{(x+3)(x+4)}$$

Now, let $\frac{37x+84}{(x+3)(x+4)}=\frac{A}{x+3}+\frac{B}{x+4}$

$\therefore\quad 37x+84=A(x+4)+B(x+3)=x(A+B)+4A+3B$

$\therefore\quad A+B=37$ and $4A+3B=84$

Solving we get $A=-27,\ B=64.$

$$\therefore\quad \frac{37x+84}{(x+3)(x+4)}=-\frac{27}{x+3}+\frac{64}{x+4}$$

$$\therefore\quad \int\frac{x^3\,dx}{x^2+7x+12}=\int\left(x-7-\frac{27}{x+3}+\frac{64}{x+4}\right)dx$$

$$=\int x\,dx-7\int dx-27\int\frac{dx}{x+3}+64\int\frac{dx}{x+4}$$

$$=\frac{x^2}{2}-7x-27\log(x+3)+64\log(x+4)+c.$$

Ex. 2. Integrate : $\int\frac{x^4+x^2+1}{(x^2+1)(x+1)}dx.$

$$(x^2+1)(x+1)=x^3+x^2+x+1$$

$$
\begin{array}{r@{\,}l}
x^3+x^2+x+1\big) & x^4+x^2+1 \qquad \big(x-1 \\
& \underline{x^4+x^3+x^2+x} \\
& \quad -x^3-x+1 \\
& \quad \underline{-x^3-x^2-x-1} \\
& \qquad\quad x^2+2
\end{array}
$$

$$\therefore\quad \frac{x^4+x^2+1}{(x^2+1)(x+1)}=x-1+\frac{x^2+2}{(x^2+1)(x+1)}$$

Now, let $\frac{x^2+2}{(x^2+1)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}$

$$\therefore \quad x^2+2=(\text{A}x+\text{B})(x+1)+\text{C}(x^2+1)$$

Putting $x=-1$ on both sides we get, $2\text{C}=3$, $\therefore \ \text{C}=\frac{3}{2}$.

Again as the constant terms on both sides are equal, $2=\text{B}+\text{C}$.

$$\therefore \quad \text{B}=2-\text{C}=2-\tfrac{3}{2}=\tfrac{1}{2}.$$

Also the coefficients of x^2 on the both sides are equal.

$$\therefore \quad 1=\text{A}+\text{C} \qquad \therefore \quad \text{A}=1-\text{C}=1-\tfrac{3}{2}=-\tfrac{1}{2}.$$

$$\therefore \quad \frac{x^2+2}{(x^2+1)(x+1)}=\frac{3}{2(x+1)}-\frac{x-1}{2(x^2+1)}$$

So, $\displaystyle\int\frac{x^4+x^2+1}{(x^2+1)(x+1)}dx=\int\left\{x-1+\frac{3}{2(x+1)}-\frac{x-1}{2(x^2+1)}\right\}dx$

$$=\int x\,dx-\int dx+\tfrac{3}{2}\int\frac{dx}{x+1}-\tfrac{1}{4}\int\frac{2x\,dx}{x^2+1}+\tfrac{1}{2}\int\frac{dx}{x^2+1}$$

$$=\frac{x^2}{2}-x+\tfrac{3}{2}\log(x+1)-\tfrac{1}{4}\log(x^2+1)+\tfrac{1}{2}\tan^{-1}x+c.$$

§ 4·4. A special technique in a special case.

If both the numerator and denominator contain only even powers of x, then it is convenient to express the function as the sum of partial fractions by putting $x^2=t$. Note that here the variable t does not replace the variable x. Before integration, replace t by x^2.

Example 1. Integrate : $\displaystyle\int\frac{x^2dx}{(x^2+a^2)(x^2+b^2)}$

$$\frac{x^2}{(x^2+a^2)(x^2+b^2)}=\frac{t}{(t+a^2)(t+b^2)} \qquad [\text{Putting } x^2=t]$$

$$=\frac{1}{a^2-b^2}\left(\frac{a^2}{t+a^2}-\frac{b^2}{t+b^2}\right)=\frac{1}{a^2-b^2}\left(\frac{a^2}{x^2+a^2}-\frac{b^2}{x^2+b^2}\right)$$

$$\therefore \quad \int\frac{x^2dx}{(x^2+a^2)(x^2+b^2)}=\int\frac{1}{a^2-b^2}\left\{\frac{a^2dx}{x^2+a^2}-\frac{b^2dx}{x^2+b^2}\right\}dx$$

$$=\frac{a^2}{a^2-b^2}\int\frac{dx}{x^2+a^2}-\frac{b^2}{a^2-b^2}\int\frac{dx}{x^2+b^2}$$

$$=\frac{a^2}{a^2-b^2}\frac{1}{a}\tan^{-1}\frac{x}{a}-\frac{b^2}{a^2-b^2}\frac{1}{b}\tan^{-1}\frac{x}{b}+c$$

$$=\frac{1}{a^2-b^2}\left(a\tan^{-1}\frac{x}{a}-b\tan^{-1}\frac{x}{b}\right)+c.$$

Ex. 2. Integrate : $\displaystyle\int\frac{(x^2+1)dx}{x^4-3x^2+2}$.

Let, $x^2=t$.

$$\therefore \quad \frac{x^2+1}{x^4-3x^2+2}=\frac{t+1}{t^2-3t+2}=\frac{t+1}{(t-1)(t-2)}=\frac{A}{t-1}+\frac{B}{t-2}$$

$$\therefore \quad t+1=A(t-2)+B(t-1)$$

Puting $t=1$ and 2 successively on both sides we get $A=-2$, $B=3$.

$$\therefore \quad \frac{x^2+1}{x^4-3x^2+2}=\frac{3}{t-2}-\frac{2}{t-1}=\frac{3}{x^2-2}-\frac{2}{x^2-1}$$

$$\therefore \quad \int\frac{x^2+1}{x^4-3x^2+2}dx=3\int\frac{dx}{x^2-2}-2\int\frac{dx}{x^2-1}$$

$$=\frac{3}{2\sqrt{2}}\log\left[\frac{x-\sqrt{2}}{x+\sqrt{2}}\right]-\log\left[\frac{x-1}{x+1}\right]+c.$$

§ 4·5. A special substitution.

If in an algebraic rational fraction the numerator and denominator contain respectively only odd powers of x and only even powers of x, then the substitution $x^2=t$ is frequently found convenient. In this case the integrand is expressed in terms of t. The new integrand is then broken up as sums of partial fractions.

Example 1. Integrate : $\int\frac{x^3dx}{x^4+x^2+1}$.

Let $x^2=t$; $\therefore$ $2x\,dx=dt$ and $x^3dx=x^2x dx=\frac{1}{2}t dt$

Also $x^4+x^2+1=t^2+t+1$.

$$\therefore \quad \int\frac{x^3dx}{x^4+x^2+1}=\tfrac{1}{2}\int\frac{t\,dt}{t^2+t+1}=\tfrac{1}{4}\int\frac{2t+1}{t^2+t+1}dt-\tfrac{1}{4}\int\frac{dt}{t^2+t+1}$$

$$=\tfrac{1}{4}\log(t^2+t+1)-\tfrac{1}{4}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}}+c$$

$$=\tfrac{1}{4}\log(x^4+x^2+1)-\frac{1}{2\sqrt{3}}\tan^{-1}\frac{2x^2+1}{\sqrt{3}}+c.$$

Ex. 2. Integrate : $\int\frac{x^5\,dx}{1+x^4}$.

$$\frac{x^5}{1+x^4}=x-\frac{x}{1+x^4}$$

$$\therefore \quad \int\frac{x^5dx}{1+x^4}=\int\left(x-\frac{x}{1+x^4}\right)dx=\int x\,dx-\int\frac{x\,dx}{1+x^4}$$

Now, $\int x\,dx=\frac{x^2}{2}+c_1$

For determination of $\int \frac{x\,dx}{1+x^4}$

Put $x^2=t \quad \therefore \quad 2xdx=dt \quad$ or, $\quad xdx=\frac{dt}{2}$.

Also $1+x^4=1+t^2$.

$$\therefore \quad \int \frac{x\,dx}{1+x^4}=\tfrac{1}{2}\int \frac{dt}{1+t^2}=\tfrac{1}{2}\tan^{-1} t+c_2=\tfrac{1}{2}\tan^{-1} x^2+c_2.$$

$$\therefore \quad \int \frac{x^5 dx}{1+x^4}=\frac{x^2}{2}+c_1-\tfrac{1}{2}\tan^{-1} x^2+c_2=\frac{x^2}{2}+\tfrac{1}{2}\tan^{-1} x^2+c.$$

Examples 4

Example 1. Integrate : $\int \frac{x^3}{(x-a)(x-b)(x-c)}dx.$ [C. U. 1954]

Let $\frac{x^3}{(x-a)(x-b)(x-c)}=1+\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$

$$\therefore \quad x^3=(x-a)(x-b)(x-c)+A(x-b)(x-c)+B(x-c)(x-a)$$
$$+C(x-a)(x-b).$$

Putting $x=a$, $x=b$, $x=c$ successively in both sides we get

$$A=\frac{a^3}{(a-b)(a-c)},\ B=\frac{b^3}{(b-c)(b-a)}\ \&\ C=\frac{c^3}{(c-a)(c-b)}$$

$$\therefore \quad \int \frac{x^3}{(x-a)(x-b)(x-c)}dx=\int\left\{1+\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}\right\}dx$$

$$=\int dx+A\int\frac{dx}{x-a}+B\int\frac{dx}{x-b}+C\int\frac{dx}{x-c}$$

$$=x+A\log(x-a)+B\log(x-b)+C\log(x-c)+k$$

$$=x+\frac{a^3}{(a-b)(a-c)}\log(x-a)+\frac{b^3}{(b-c)(b-a)}\log(x-b)$$
$$+\frac{c^3}{(c-a)(c-b)}\log(x-c)+k.$$

[Putting the values of A, B and C]

Note. Notice that here the numerator and denominator are both of degree 3. So, the numerator is divided by the denominator. The quotient $q(x)$ is 1 and the remainder is written in the form

$$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}.$$

Ex. 2. Integrate : $\int \frac{(x-1)dx}{(x+2)(x-3)}$ [C. U.]

Let $\frac{x-1}{(x+2)(x-3)}=\frac{\text{A}}{x+2}+\frac{\text{B}}{x-3}$.

$\therefore \quad x-1=\text{A}(x-3)+\text{B}(x+2)$

Putting $x=-2$ on both sides we get $-3=-5\text{A}$ or, $\text{A}=\frac{3}{5}$.

Putting $x=3$ on both sides we get $2=5\text{B}$ $\therefore$ $\text{B}=\frac{2}{5}$.

So $\frac{x-1}{(x+2)(x-3)}=\frac{3}{5(x+2)}+\frac{2}{5(x-3)}$

$$\therefore \int \frac{x-1}{(x+2)(x-3)}dx=\int \frac{3dx}{5(x+2)}+\int \frac{2dx}{5(x-3)}$$

$$=\tfrac{3}{5}\log(x+2)+\tfrac{2}{5}\log(x-3)+c.$$

Ex. 3. Integrate : $\int \frac{x\,dx}{(x+a)^2(x+b)}$

Let $\frac{x}{(x+a)^2(x+b)}=\frac{\text{A}}{(x+a)^2}+\frac{\text{B}}{(x+a)}+\frac{\text{C}}{(x+b)}$

$\therefore \quad x=\text{A}(x+b)+\text{B}(x+a)(x+b)+\text{C}(x+a)^2$.

Putting $x=-a$ on both sides we get.

$$-a=\text{A}(b-a) \quad \therefore \quad \text{A}=\frac{a}{a-b}.$$

Putting $x=-b$ on both sides we get,

$$-b=\text{C}(a-b)^2 \quad \text{or,} \quad \text{C}=-\frac{b}{(a-b)^2}.$$

Again equating the coefficients of x^2 on both sides we get $0=\text{B}+\text{C}$

$$\therefore \quad \text{B}=-\text{C}=\frac{b}{(a-b)^2}.$$

Now $\int \frac{x\,dx}{(x+a)^2(x+b)}=\int \left\{\frac{\text{A}}{(x+a)^2}+\frac{\text{B}}{x+a}+\frac{\text{C}}{x+b}\right\}dx$

$$=\text{A}\int \frac{dx}{(x+a)^2}+\text{B}\int \frac{dx}{x+a}+\text{C}\int \frac{dx}{x+b}$$

$$=-\frac{\text{A}}{x+a}+\text{B}\log(x+a)+\text{C}\log(x+b)+k$$

$$=-\frac{a}{(a-b)(x+a)}+\frac{b}{(a-b)^2}\log(x+a)-\frac{b}{(a-b)^2}\log(x+b)+k$$

$$=\frac{a}{(b-a)(x+a)}+\frac{b}{(a-b)^2}\log\frac{x+a}{x+b}+k.$$

Ex. 4. Integrate : $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$ [C. U. '28, '31, '37]

$$\frac{1}{(x^2+a^2)(x^2+b^2)}=\frac{1}{a^2-b^2}\left[\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right]$$

$$\therefore \int\frac{dx}{(x^2+a^2)(x^2+b^2)}=\int\frac{1}{a^2-b^2}\left[\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right]dx$$

$$=\frac{1}{a^2-b^2}\int\frac{dx}{x^2+b^2}-\frac{1}{a^2-b^2}\int\frac{dx}{x^2+a^2}$$

$$=\frac{1}{a^2-b^2}\,\frac{1}{b}\tan^{-1}\frac{x}{b}-\frac{1}{a^2-b^2}\,\frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

$$=\frac{1}{a^2-b^2}\left[\frac{1}{b}\tan^{-1}\frac{x}{b}-\frac{1}{a}\tan^{-1}\frac{x}{a}\right]+c.$$

Ex. 5. Integrate : $\int\frac{dx}{x^3+1}$

$$\frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}\ \text{[say]}$$

$\therefore\ 1=A(x^2-x+1)+(Bx+C)(x+1).$

Putting $x=-1$ on both sides we get $1=3A$ or, $A=\frac{1}{3}$.

Coefficients of x^2 and x on both sides are equal.

$\therefore\ 0=A+B$ and $0=-A+B+C.$

$\therefore\ B=-A=-\frac{1}{3}$ and $C=A-B=2A=\frac{2}{3}.$

$$\therefore\ \frac{1}{x^3+1}=\frac{1}{3}\left[\frac{1}{x+1}+\frac{-x+2}{x^2-x+1}\right]$$

$$\therefore\ \int\frac{dx}{x^3+1}=\frac{1}{3}\left\{\int\frac{dx}{x+1}-\int\frac{x-2}{x^2-x+1}dx\right\}$$

$$=\frac{1}{3}\left\{\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{2x-1}{x^2-x+1}dx+\frac{3}{2}\int\frac{dx}{x^2-x+1}\right\}$$

$$=\frac{1}{3}\left\{\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{2x-1}{x^2-x+1}dx+\frac{3}{2}\int\frac{dx}{(x-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$=\frac{1}{3}\log(x+1)-\frac{1}{6}\log(x^2-x+1)+\frac{1}{2}\,\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}+C$$

$$=\frac{1}{3}\log(x+1)-\frac{1}{6}\log(x^2-x+1)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+C$$

Now let $\frac{u}{1+u^3}=\frac{\text{A}}{1+u}+\frac{\text{B}u+\text{C}}{1-u+u^2}$

or, $u=\text{A}(1-u+u^2)+(1+u)(\text{B}u+\text{C})$

Putting $u=-1$ on both sides, we find $-1=3\text{A}$ or, $\text{A}=-\frac{1}{3}$.

Putting $u=0$ on both sides, we get $0=\text{A}+\text{C}$ $\therefore$ $\text{C}=-\text{A}=\frac{1}{3}$.

Comparing coefficients of u^2 on both sides we get,

$$0=\text{A}+\text{B} \quad \therefore \quad \text{B}=-\text{A}=\tfrac{1}{3}.$$

So, given integral

$$=\frac{1}{2}\left\{-\int\frac{du}{1+u}+\int\frac{u+1}{1-u+u^2}du\right\}$$

$$=\frac{1}{2}\left\{-\int\frac{du}{1+u}+\frac{1}{2}\int\frac{2u-1+3}{1-u+u^2}du\right\}$$

$$=-\frac{1}{2}\int\frac{du}{1+u}+\frac{1}{4}\int\frac{2u-1}{1-u+u^2}du+\frac{3}{4}\int\frac{du}{\left(u-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=-\tfrac{1}{2}\log(1+u)+\tfrac{1}{4}\log(1-u+u^2)+\frac{\sqrt{3}}{2}\tan^{-1}\frac{2u-1}{\sqrt{3}}+c$$

$$=-\tfrac{1}{2}\log(1+z^2)+\tfrac{1}{4}\log(z^4-z^2+1)+\frac{\sqrt{3}}{2}\tan^{-1}\frac{2z^2-1}{\sqrt{3}}+c$$

$$=-\tfrac{1}{2}\log(1+\tan^{\frac{2}{3}}x)+\tfrac{1}{4}\log(\tan^{\frac{4}{3}}x-\tan^{\frac{2}{3}}x+1)$$

$$+\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2\tan^{\frac{2}{3}}x-1}{\sqrt{3}}\right)+c.$$

Exercise 4

Integrate :

1. $\int\frac{dx}{x^2-3x+2}$
2. $\int\frac{dx}{(3x+2)(4x+3)}$
3. $\int\frac{(x-1)dx}{(x-2)(x-3)}$ [C. U.]
4. $\int\frac{3x\,dx}{x^2-x-2}$ [C. U.]
5. $\int\frac{x\,dx}{(x-a)(x-b)}$ [C. U.]
6. $\int\frac{x\,dx}{(x+1)^2(x+2)}$
7. $\int\frac{x^2dx}{(x+1)^2(x+2)}$
8. $\int\frac{dx}{(x-1)^2(x-3)}$

9. $\int \frac{(x+2)dx}{(1-x)(4+x^2)}$ 10. $\int \frac{dx}{1-x^3}$ 11. $\int \frac{x\,dx}{1+x^3}$

12. $\int \frac{x^3+2}{(x-1)(x-2)}dx$ 13. $\int \frac{x^3 dx}{(x+a)(x^2+a^2)}$

14. $\int \frac{x^4+x^2+1}{(x+1)(x-1)}dx$ 15. $\int \frac{(x^2-3)dx}{(x-1)(x-2)(x+3)}$

16. $\int \frac{x^2 dx}{x^4+x^2-12}$ 17. $\int \frac{dx}{(x^2+1)(x^2+2)}$

18. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$ 19. $\int \frac{x^2 dx}{x^4+x^2+12}$

20. $\int \frac{2x^4+3}{x^4+5x^2+6}dx$ 21. $\int \frac{x^3}{1-x^2}dx$ 22. $\int \frac{x\,dx}{x^4-1}$

23. $\int \frac{t^3 dt}{t^4+5t^2+6}$ 24. $\int \frac{x\,dx}{x^4-x^2-2}$

25. $\int \frac{x+1}{x^4(x-1)}dx$ 26. $\int \frac{dx}{\sin 2x-\sin x}$

27. $\int \frac{dx}{\sin x(3+2\cos x)}$.

Area OKAC$=F(a)-F(0)$ ...(8)

and area ONBC$=F(b)-F(0)$ ...(9)

Subtracting (8) from (9) we get, area AKNB$=F(b)-F(a)$

Hence the measure of the area between the curve $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$, is $F(b)-F(a)$, where $F'(x)=f(x)$.

So, the measure of the area enclosed between the curve $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$ [where the function $f(x)$ is continuous in $a \leqslant x \leqslant b$] can be obtained by evaluating $\int f(x)\,dx$ in the form $F(x)$ and then subtracting $F(a)$ from $F(b)$.

$F(b)-F(a)$ is said to be the value of $\int_a^b f(x)\,dx$. $\int_a^b f(x)\,dx$ is the definite integral of $f(x)$ with respect to x from the limit a to the limit b. a and b are respectively said to be the lower and upper limits of x.

Note. 1. In the above discussion, the curve $y=f(x)$ is above the x-axis i.e., y has been assumed to be positive. If the curve is situated below the x- axis, then the value of the area will become negative. If the value of an area or a definite integral be zero, then numerical values of the areas of the portions above and below the x-axis are equal.

2. Though the indefinite integral of a function is not unique, the definite integral of a function cannot have more than one value.

Note that in the value of the definite integral of a function, there is no constant of integration.

3. In the above discussion it has been assumed that the value of the integral can be determined i.e., the function $f(x)$ is an integrable function. When $\int_a^b f(x)\,dx$ can be determined, then the function is said to be integrable in the interval $a \leqslant x \leqslant b$. A function may not be integrable in an interval. But in this book we shall discuss only integrable functions.

4. In $\int_a^b f(x)dx$, the upper limit b is greater than the lower limit a.

If $b>a$, then $\int_b^a f(x)\,dx$ is defined as follows :

$$\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx.$$

5. To determine $\int_a^b f(x)\,dx$ we write

$$\int_a^b f(x)\,dx=\Big[F(x)\Big]_a^b=F(b)-F(a).$$

$\Big[\quad\Big]_a^b$ means that put b and a successively for x in the function $F(x)$ within the third bracket and then subtract $F(a)$ from $F(b)$.

6. $\int\limits_a^b \{f(x)+g(x)\}dx=\int\limits_a^b f(x)\,dx+\int\limits_a^b g(x)\,dx.$

7. $\int\limits_a^a f(x)\,dx=0.$

5·2. Substitution of variable in definite integral.

In the last section we have found that to evaluate the definite integral of a function, one has to determine its indefinite integral first. You have frequently determined indefinite integrals by substitution of variables ; the indefinite integral is finally expressed in terms of the original variable. But in case of evaluation of definite integrals, one may not express the result in terms of the original variable. After substitution of the original variable, one may determine the corresponding limits of the new variable and evaluate the definite integral of the new variable between these new limits. This process is also frequently found convenient. Hence in case of evaluation of definite integrals by substitution of variables, the integrand, the differential and the limits of integration all are to be substituted.

Examples 5A

1. Evaluate :

(i) $\int_a^b x^2dx$ (ii) $\int_1^4 \frac{4}{\sqrt{x}}dx$ [Tripura 1981]

(iii) $\int_1^2 x^n dx\ (n\neq-1)$ (iv) $\int_0^1 (x^2+3)dx$ (v) $\int_0^{\frac{\pi}{2}} \cos x\,dx$

(vi) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ [Tripura 1982] (vii) $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$

[Tripura 1979]

(viii) $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ [H. S. 1981] (ix) $\int_0^{\frac{\pi}{8}} \tan^2 2x \, dx$

[Tripura 1981]

(x) (*a*) $\int_0^{\frac{\pi}{2}} \sin 3x \cos 2x \, dx$ [Tripura 1980]

(*b*) $\int_0^{\frac{\pi}{2}} \sin \theta \sin 2\theta \, d\theta$ [Tripura 1983]

(xi) $\int_0^{\frac{\pi}{4}} \sec^4\theta \, d\theta$ [Tripura 1984]

(xii) (*a*) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{4-x^2}}$ [H. S. 1981] (*b*) $\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$

[Tripura 1983]

(i) $\int x^2 \, dx = \frac{x^3}{3}$ $\therefore$ $\int_a^b x^2 \, dx = \left[\frac{x^3}{3}\right]_a^b = \frac{b^3}{3} - \frac{a^3}{3} = \frac{1}{3}(b^3 - a^3)$.

(ii) $\int \frac{4}{\sqrt{x}} dx = 4\int x^{-\frac{1}{2}} dx = 4\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 8x^{\frac{1}{2}}$

$\therefore$ $\int_1^4 \frac{4}{\sqrt{x}} dx = \left[8x^{\frac{1}{2}}\right]_1^4 = 8.4^{\frac{1}{2}} - 8.1^{\frac{1}{2}} = 8.2 - 8.1$
$= 16 - 8 = 8.$

(iii) $\int x^n \, dx \; (n \neq -1) = \frac{x^{n+1}}{n+1}$.

$\therefore$ $\int_1^2 x^n \, dx = \left[\frac{x^{n+1}}{n+1}\right]_1^2 = \frac{2^{n+1}}{n+1} - \frac{1^{n+1}}{n+1} = \frac{2^{n+1}-1}{n+1}$.

(iv) $\int (x^2+3)dx = \int x^2 \, dx + 3\int dx = \frac{x^3}{3} + 3x$

$\therefore$ $\int_0^1 (x^2+3)dx = \left[\frac{x^3}{3} + 3x\right]_0^1 = \frac{1}{3} + 3 - 0 = 3\frac{1}{3}$.

(v) $\int \cos x \, dx = \sin x$ $\therefore$ $\int_0^{\frac{\pi}{2}} \cos x \, dx = \left[\sin x\right]_0^{\frac{\pi}{2}}$
$= \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1.$

(vi) $\int \cos^2 x \, dx = \frac{1}{2}\int 2\cos^2 x \, dx = \frac{1}{2}\int (1+\cos 2x)dx$
$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x \, dx = \frac{1}{2}x + \frac{\sin 2x}{4}$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \left[\tfrac{1}{2}x + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{\sin \pi}{4} = \frac{\pi}{4}.$$

(vii) $\int \tan^2 x\, dx = \int (\sec^2 x - 1)dx = \int \sec^2 x\, dx - \int dx$

$$= \tan x - x \quad \therefore \int_0^{\frac{\pi}{4}} \tan^2 x\, dx = \Big[\tan x - x\Big]_0^{\frac{\pi}{4}}$$

$$= \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - \left(\tan 0 - 0\right) = \left(1 - \frac{\pi}{4}\right) - \left(0 - 0\right) = 1 - \frac{\pi}{4}.$$

(viii) $\int \sin^2 x\, dx = \frac{1}{2}\int 2\sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)dx$

$$= \tfrac{1}{2}\left\{\int dx - \int \cos 2x\, dx\right\} = \tfrac{1}{2}\left(x - \frac{\sin 2x}{2}\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \left[\tfrac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - \tfrac{1}{2}\left(0 - \frac{\sin 0}{2}\right) = \frac{1}{2}\left(\frac{\pi}{2} - \frac{0}{2}\right) - \frac{1}{2}.0 = \frac{\pi}{4}.$$

(ix) $\int \tan^2 2x\, dx = \int (\sec^2 2x - 1)dx = \int \sec^2 2x\, dx - \int dx$

$$= \frac{\tan 2x}{2} - x.$$

$$\therefore \int_0^{\frac{\pi}{3}} \tan^2 2x\, dx = \left[\frac{\tan 2x}{2} - x\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{\tan\frac{2\pi}{3}}{2} - \frac{\pi}{3}\right) - \left(\frac{\tan 0}{2} - 0\right)$$

$$= \frac{-\sqrt{3}}{2} - \frac{\pi}{3} - 0 = -\frac{(3\sqrt{3} + 2\pi)}{6}.$$

(x) (*a*) $\int \sin 3x \cos 2x\, dx = \frac{1}{2}\int 2\sin 3x \cos 2x\, dx$

$$= \tfrac{1}{2}\int (\sin 5x + \sin x)dx = \tfrac{1}{2}\left\{\int \sin 5x\, dx + \int \sin x\, dx\right\}$$

$$= \frac{1}{2}\left(-\frac{\cos 5x}{5} - \cos x\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin 3x \cos 2x\, dx = \frac{1}{2}\left[-\frac{\cos 5x}{5} - \cos x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(-\frac{\cos 5\frac{\pi}{2}}{5} - \cos\frac{\pi}{2}\right) - \left(-\frac{\cos 0}{5} - \cos 0\right)\right]$$

$$= \tfrac{1}{2}[(0 - (-\tfrac{1}{5} - 1)] = \tfrac{3}{5}.$$

(*b*) $\int \sin\theta \sin 2\theta\, d\theta = \frac{1}{2}\int(\cos\theta - \cos 3\theta)d\theta$

$= \frac{1}{2}\int \cos\theta\, d\theta - \frac{1}{2}\int \cos 3\theta\, d\theta = \frac{1}{2}\sin\theta - \frac{1}{2}\frac{\sin 3\theta}{3}$

$$\therefore \int_0^{\frac{\pi}{2}} \sin\theta \sin 2\theta\, d\theta = \frac{1}{2}\left[\sin\theta - \frac{\sin 3\theta}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\sin\frac{\pi}{2} - \frac{\sin 3.\frac{\pi}{2}}{3}\right) - \frac{1}{2}\left(\sin 0 - \frac{\sin 0}{3}\right)$$

$$= \frac{1}{2}\left(1 - \frac{-1}{3}\right) - 0 = \frac{2}{3}.$$

(xi) $\int \sec^4\theta\, d\theta = \int \sec^2\theta \sec^2\theta\, d\theta = \int(\tan^2\theta + 1)\sec^2\theta\, d\theta$

Let, $\tan\theta = z$ $\therefore$ $\sec^2\theta\, d\theta = dz.$

$$\therefore \int(\tan^2\theta + 1)\sec^2\theta\, d\theta = \int(z^2 + 1)dz = \frac{z^3}{3} + z = \frac{\tan^3\theta}{3} + \tan\theta.$$

$$\therefore \int_0^{\frac{\pi}{4}} \sec^4\theta\, d\theta = \left[\frac{\tan^3\theta}{3} + \tan\theta\right]_0^{\frac{\pi}{4}} = \left(\frac{\tan^3\frac{\pi}{4}}{3} + \tan\frac{\pi}{4}\right)$$

$$- \left(\frac{\tan^3 0}{3} + \tan 0\right) = \frac{1}{3} + 1 - 0 = \frac{4}{3}.$$

(xii) (*a*) $\int \frac{dx}{\sqrt{4 - x^2}} = \int \frac{dx}{\sqrt{2^2 - x^2}} = \sin^{-1}\frac{x}{2}$

$$\therefore \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{4 - x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^{\frac{1}{2}} = \sin^{-1}\frac{1}{4} - \sin^{-1} 0$$

$$= \sin^{-1}\frac{1}{4} - 0 = \sin^{-1}\frac{1}{4}.$$

(*b*) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$

$$\therefore \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \left[\sin^{-1}\frac{x}{a}\right]_0^a = \sin^{-1} 1 - \sin^{-1} 0$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Ex. 2. Show that

$$\left.\begin{array}{l} \int_0^\pi \sin mx \sin nx = 0. \text{ if } m \neq n \\ \qquad\qquad\qquad\quad = \frac{\pi}{2} \text{ if } m = n \end{array}\right\} m \text{ and } n \text{ are positive integers}$$

[Joint Entrance 1982]

First let $m \neq n$.

$$\therefore \int \sin mx \sin nx dx = \tfrac{1}{2} \int 2 \sin mx \sin nx dx$$
$$= \tfrac{1}{2} \int \{\cos (m-n) x - \cos (m+n)x\} dx$$
$$= \frac{1}{2}\left\{\frac{\sin (m-n) x}{(m-n)} - \frac{\sin (m+n) x}{m+n}\right\}$$
$$\therefore \int_0^{\pi} \sin mx \sin nx\, dx = \frac{1}{2}\left[\frac{\sin (m-n) x}{(m-n)} - \frac{\sin (m+n) x}{(m+n)}\right]_0^{\pi}$$
$$= \frac{1}{2}\left[\left\{\frac{\sin (m-n) \pi}{m-n} - \frac{\sin (m+n) \pi}{(m+n)}\right\} - \left\{\frac{\sin 0}{m-n} - \frac{\sin 0}{m+n}\right\}\right]$$
$$= \frac{1}{2}\left[\left(\frac{0}{m-n} - \frac{0}{m+n}\right) - \left(\frac{0}{m-n} - \frac{0}{m+n}\right)\right] = 0$$

[As m and n are integers, so $(m-n)$ and $(m+n)$ are two integers and $\sin (m \pm n) \pi = 0$]

Now let $m = n$.

$$\therefore \int \sin mx \sin nx\, dx = \int \sin^2 nx\, dx = \tfrac{1}{2} \int 2 \sin^2 nx dx$$
$$= \tfrac{1}{2} \int (1 - \cos 2nx)\, dx = \tfrac{1}{2} \int dx - \tfrac{1}{2} \int \cos 2nx\, dx$$
$$= \frac{1}{2}x - \frac{1}{2} \cdot \frac{\sin 2nx}{2n}$$
$$\therefore \int_0^{\pi} \sin mx \sin nx = \left[\frac{1}{2}x - \frac{\sin 2nx}{4n}\right]_0^{\pi}$$
$$= \left(\frac{\pi}{2} - \frac{\sin 2n\pi}{4n}\right) - \left(0 - \frac{\sin 0}{4n}\right) = \left(\frac{\pi}{2} - 0\right) - 0 = \frac{\pi}{2}.$$

Ex. 3. Evaluate.

(i) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{3-2x}}$ [H. S. '79] (ii) $\int_0^1 \frac{1-x}{1+x}\, dx$ [H. S. 1980]

(iii) $\int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{1+\sin^2 x}$ [H. S. 1978, Tripura 1980, '84]

(iv) $\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\, dx$ [H. S. 1982]

(v) $\int_0^a \frac{x\, dx}{\sqrt{a^2-x^2}}$ [H. S. 1983] (vi) $\int_0^1 \frac{x+1}{x^2+1}\, dx$ [H. S. 1987]

(vii) $\int_0^2 \frac{x^2 dx}{x^3+1}$ [Tripura 1978]

(viii) $\int_0^1 x^2(1+x^3)^2 dx$ [Tripura 1982]

(ix) $\int_0^3 \frac{4x dx}{\sqrt{2x^2+9}}$ [Tripura 1979]

(x) $\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,dx$ [Tripura 1987]

(xi) $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$ [Tripura 1981]

(xii) $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x\,dx}{1+m^2\tan^2 x}$ [Tripura 1983]

(xiii) $\int_1^{e} \frac{1+\log_e x}{x}\,dx$ [H. S. 1987]

(xiv) $\int_0^2 \frac{x^4 dx}{(4+x^5)^{\frac{3}{2}}}$ [H. S. 1986]

(i) Let, $3-2x=z \quad \therefore \quad -2dx=dz$ or, $dx=-\frac{dz}{2}$.

$\therefore \int\frac{dx}{\sqrt{3-2x}}=-\int\frac{dz}{2\sqrt{z}}=-\frac{1}{2}\,\frac{z^{\frac{1}{2}}}{\frac{1}{2}}=-\sqrt{3-2x}$

$\therefore \int_0^{\frac{1}{2}}\frac{dx}{\sqrt{3-2x}}=\Big[-\sqrt{3-2x}\Big]_0^{\frac{1}{2}}=-(\sqrt{3-2.\frac{1}{2}}-\sqrt{3-2.0})$

$=-(\sqrt{2}-\sqrt{3})=\sqrt{3}-\sqrt{2}.$

(ii) $\int\frac{1-x}{1+x}dx=\int\frac{-1-x+2}{1+x}dx=-\int dx+2\int\frac{dx}{1+x}$

$=-x+2\log(1+x).$

$\therefore \int_0^1\frac{1-x}{1+x}dx=\Big[-x+2\log(1+x)\Big]_0^1$

$=(-1+2\log 2)-(0+2\log 1)=-1+\log 2^2=\log 4-1.$

(iii) Let, $\sin x=z \quad \therefore \quad \cos x\,dx=dz$

Again, when $x=0$, then $z=\sin 0=0$

when $x=\frac{\pi}{2}$, then $z=\sin\frac{\pi}{2}=1.$

$\therefore \int_0^{\frac{\pi}{2}}\frac{\cos x\,dx}{1+\sin^2 x}=\int_0^1\frac{dz}{1+z^2}=\Big[\tan^{-1}z\Big]_0^1=\tan^{-1}1-\tan^{-1}0$

$=\frac{\pi}{4}-0=\frac{\pi}{4}.$

(iv) $\int_0^2\frac{x^2dx}{\sqrt{1+x^3}}$.

Let, $\sqrt{1+x^3}=z$ or, $1+x^3=z^2$

$\therefore \quad 3x^2dx=2zdz$ or, $x^2dx=\frac{2}{3}zdz$

when $x=0$, then $z=\sqrt{1+x^3}=\sqrt{1+0}=\sqrt{1}=1$
and when $x=2$, then $z=\sqrt{1+2^3}=\sqrt{1+8}=\sqrt{9}=3$

$\therefore$ Given integral$=\int_1^3 \frac{2}{3}\frac{zdz}{z}=\frac{2}{3}\int_1^3 dz$

$=\frac{2}{3}\Big[z\Big]_1^3=\frac{2}{3}(3-1)=\frac{4}{3}.$

(v) $\int_0^a \frac{xdx}{\sqrt{a^2-x^2}}$.

Let $\sqrt{a^2-x^2}=z$ or, $a^2-x^2=z^2$

$\therefore$ $-2x\,dx=2z\,dz$ or, $x\,dx=-z\,dz$.

Again when $x=0$, then $z=\sqrt{a^2-x^2}=\sqrt{a^2-0}=\sqrt{a^2}=a$.
and when $x=a$, then $z=\sqrt{a^2-a^2}=\sqrt{0}=0$

$\therefore$ Given integral$=\int_a^0 \frac{-z\,dz}{z}=-\int_a^0 dz$

$=\int_0^a dz=\Big[z\Big]_0^a=a-0=a$

Alternative method : Let $x=a\sin\theta$ $\therefore$ $dx=a\cos\theta\,d\theta$

$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=a\cos\theta$

when $x=0$, then $0=a\sin\theta$ or, $\sin\theta=0$ or, $\theta=0$

when $x=a$, then $a=a\sin\theta$ or, $\sin\theta=1$ or, $\theta=\frac{\pi}{2}$

$\therefore$ Given integral$=\int_0^{\frac{\pi}{2}} \frac{a\sin\theta\, a\cos\theta\,d\theta}{a\cos\theta}$

$=a\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta=a\Big[-\cos\theta\Big]_0^{\frac{\pi}{2}}$

$=a\Big[-\cos\frac{\pi}{2}-(-\cos 0)\Big]=a[0-(-1)]=a.1=a.$

(vi) $\int_0^1 \frac{x+1}{x^2+1}dx=\int_0^1 \frac{x\,dx}{x^2+1}+\int_0^1 \frac{dx}{x^2+1}$

Now for $\int_0^1 \frac{x\,dx}{x^2+1}$ Let $x^2+1=z$

$\therefore$ $2x\,dx=dz$ or, $x\,dx=\frac{dz}{2}$

$\therefore$ $\int\frac{x\,dx}{x^2+1}=\frac{1}{2}\int\frac{dz}{z}=\frac{1}{2}\log z=\frac{1}{2}\log(x^2+1)$

$$\therefore\ \int_0^1 \frac{x\,dx}{x^2+1}=\left[\frac{1}{2}\log(x^2+1)\right]_0^1=\frac{1}{2}\Big[\log(1+1)-\log(0+1)\Big]$$

$$=\tfrac{1}{2}(\log 2-\log 1)=\tfrac{1}{2}\log 2=\log\sqrt{2}$$

Also, $\displaystyle\int_0^1 \frac{dx}{x^2+1}=\Big[\tan^{-1}x\Big]_0^1=\tan^{-1}1-\tan^{-1}0=\frac{\pi}{4}-0=\frac{\pi}{4}.$

Given integral $=\log\sqrt{2}+\dfrac{\pi}{4}.$

(vii) $\displaystyle\int_0^2 \frac{x^2dx}{x^3+1}$

Let $x^3+1=z$ $\quad\therefore\quad 3x^2dx=dz,\quad$ or, $\quad x^2\,dx=\dfrac{dz}{3}$

So, $\displaystyle\int\frac{x^2dx}{x^3+1}=\frac{1}{3}\int\frac{dz}{z}=\tfrac{1}{3}\log(x^3+1)$

$$\therefore\ \int_0^2 \frac{x^2dx}{x^3+1}=\frac{1}{3}\Big[\log(2^3+1)-\log(0+1)\Big]=\tfrac{1}{3}(\log 9-\log 1)$$

$$=\tfrac{1}{3}\log 3^2=\tfrac{2}{3}\log 3.$$

(viii) Let $x^3+1=z$ Now as in (vii) above

$$\int x^2(1+x^3)^2dx=\tfrac{1}{3}\int z^2dz=\frac{z^3}{9}=\frac{(1+x^3)^3}{9}$$

$$\therefore\ \int_0^1 x^2(1+x^3)^2=\left[\frac{(1+x^3)^3}{9}\right]_0^1=\frac{(1+1)^3}{9}-\frac{(1+0)^3}{9}=\frac{8}{9}-\frac{1}{9}=\frac{7}{9}.$$

(ix) $\displaystyle\int_0^3 \frac{4x\,dx}{\sqrt{2x^2+9}}$

Let $\sqrt{2x^2+9}=z\quad$ or, $\quad 2x^2+9=z^2$

$\therefore\ 4x\,dx=2z\,dz$

$$\therefore\ \int\frac{4x\,dx}{\sqrt{2x^2+9}}=\int\frac{2z\,dz}{z}=2\int dz=2z=2\sqrt{2x^2+9}$$

$$\therefore\ \int_0^3 \frac{4x\,dx}{\sqrt{2x^2+9}}=2\Big[\sqrt{2x^2+9}\Big]_0^3=2\left[\sqrt{2.3^2+9}-\sqrt{9}\right]$$

$$=2(\sqrt{27}-3)=2(3\sqrt{3}-3)=6(\sqrt{3}-1).$$

(x) $\displaystyle\int\sqrt{\frac{1-x}{1+x}}\,dx=\int\frac{1-x}{\sqrt{1-x^2}}dx=\int\frac{dx}{\sqrt{1-x^2}}-\int\frac{x\,dx}{\sqrt{1-x^2}}$

Now, $\displaystyle\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x.$

Again for $\displaystyle\int\frac{xdx}{\sqrt{1-x^2}}$ let $\sqrt{1-x^2}=z$

or, $1-x^2=z^2$ or, $-2x\,dx=2z\,dz$ or, $x\,dx=-z\,dz$.

$$\therefore \int \frac{x\,dx}{\sqrt{1-x^2}}=\int \frac{-z\,dz}{z}=-\int dz=-z=-\sqrt{1-x^2}$$

$$\therefore \int \sqrt{\frac{1-x}{1+x}}=\sin^{-1}x+\sqrt{1-x^2}$$

$$\therefore \int_{-1}^{1} \sqrt{\frac{1-x}{1-x}}=\left[\sin^{-1}x+\sqrt{1-x^2}\right]_{-1}^{1}$$

$$=(\sin^{-1}1+\sqrt{1-1^2})-\{\sin^{-1}(-1)+\sqrt{1-(-1)^2}\}$$

$$=\frac{\pi}{2}+0-\left(-\frac{\pi}{2}+0\right)=\pi.$$

(xi) $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx.$

Let, $\tan^{-1}x=z$ $\therefore$ $\frac{1}{1+x^2}dx=dz.$

when, $x=0$, then, $z=\tan^{-1}0=0$

and when $x=1$, then $z=\tan^{-1}1=\frac{\pi}{4}.$

$$\therefore \text{ Given integral}=\int_0^{\frac{\pi}{4}} z\,dz=\left[\frac{z^2}{2}\right]_0^{\frac{\pi}{4}}=\frac{\left(\frac{\pi}{4}\right)^2}{2}-0=\frac{\pi^2}{32}$$

(xii) $\int_0^{\frac{\pi}{4}} \frac{\sec^2x\,dx}{1+m^2\tan^2x}$

Let, $m\tan x=z$ $\therefore$ $m\sec^2x\,dx=dz$ $\therefore$ $\sec^2x\,dx=\frac{dz}{m}.$

$$\text{So, } \int \frac{\sec^2x\,dx}{1+m^2\tan^2x}=\int \frac{dz}{m(1+z^2)}=\frac{1}{m}\tan^{-1}z$$

$$=\frac{1}{m}\tan^{-1}(m\tan x)$$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\sec^2x\,dx}{1+m^2\tan^2x}=\left[\frac{1}{m}\tan^{-1}(m\tan x)\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{m}\tan^{-1}\left(m\tan\frac{\pi}{4}\right)-\frac{1}{m}\tan^{-1}(0)=\frac{1}{m}\tan^{-1}(m.1)=\frac{1}{m}(\tan^{-1}m).$$

(xiii) Let, $1+\log_e x=z$ $\therefore$ $\frac{dx}{x}=dz$

$$\therefore \int \frac{1+\log_e x}{x}dx=\int z\,dz=\frac{z^2}{2}=\frac{(1+\log_e x)^2}{2}$$

$$\therefore \int_1^e \frac{1+\log_e x}{x}dx=\left[\frac{(1+\log_e x)^2}{2}\right]_1^e=\frac{(1+\log_e e)^2}{2}-\frac{(1+\log_e 1)^2}{2}$$

$$=\frac{(1+1)^2}{2}-\left(\frac{1}{2}\right)=2-\frac{1}{2}=\frac{3}{2}.$$

(xiv) Let, $4+x^5=z$ $\therefore$ $5x^4dx=dz$ or, $x^4dx=\frac{dz}{5}$

$$\therefore \int\frac{x^4dx}{(4+x^5)^{\frac{3}{2}}}=\int\frac{dz}{5z^{\frac{3}{2}}}=\frac{1}{5}\left(\frac{-2}{z^{\frac{1}{2}}}\right)=-\frac{2}{5}\frac{1}{(4+x^5)^{\frac{1}{2}}}$$

$$\therefore \int_0^2\frac{x^4dx}{(4+x^5)^{\frac{3}{2}}}=\left[-\frac{2}{5}\frac{1}{(4+x^5)^{\frac{1}{2}}}\right]_0^2$$

$$=-\frac{2}{5}\left[\frac{1}{(4+2^5)^{\frac{1}{2}}}-\frac{1}{(4)^{\frac{1}{2}}}\right]=-\frac{2}{5}\left[\frac{1}{36^{\frac{1}{2}}}-\frac{1}{2}\right]=\frac{2}{5}\left[\frac{1}{2}-\frac{1}{6}\right]=\frac{2}{5}\cdot\frac{1}{3}=\frac{2}{15}.$$

Ex. 4. Evaluate :—

(i) $\int_0^{\frac{\pi}{2}}\sin^4x\cos^3x\,dx.$ [H. S. 1979]

(ii) $\int_0^{\frac{\pi}{2}}\cos^4x\,dx\sin^3x\,dx.$ [H. S. 1984]

(iii) $\int_0^{\pi}\sin^3x\cos^3x\,dx.$ [H. S. 1980]

(iv) $\int_0^{\frac{\pi}{2}}\sin^2x\cos^3x\,dx.$ [H. S. 1982]

(v) $\int_0^{\frac{\pi}{2}}\sin^4x\,dx.$ [H. S. 1985]

(vi) (*a*) $\int_0^{\frac{\pi}{2}}\sin^2x\cos^2x\,dx.$ [Tripura 1986]

(*b*) $\int_0^{\frac{\pi}{4}}\sin^2x\cos^2x\,dx.$ [Joint Entrance 1983]

(i) Let, $\sin x=z$ $\therefore$ $\cos x\,dx=dz$
when $x=0$, then $z=\sin 0=0$

and when $x=\frac{\pi}{2}$, then $z=\sin\frac{\pi}{2}=1$

$$\therefore \int_0^{\frac{\pi}{2}}\sin^4x\cos^3x\,dx=\int_0^{\frac{\pi}{2}}\sin^4x\cos^2x\cos x\,dx$$

$$=\int_0^1 z^4(1-z^2)dz \quad [\because \ \cos^2 x=1-\sin^2 x=1-z^2\ ]$$

$$=\left[\frac{z^5}{5}-\frac{z^7}{7}\right]_0^1=\tfrac{1}{5}-\tfrac{1}{7}=\tfrac{2}{35}$$

(ii) Let, $\cos x=z \quad \therefore \quad -\sin x\, dx=dz$ or, $\sin x\, dx=-dz$

when $x=0$, then $z=\cos 0=1$

and when $x=\frac{\pi}{2}$, then $z=\cos\frac{\pi}{2}=0$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^4 x \sin^3 x\, dx=\int_0^{\frac{\pi}{2}} \cos^4 x(1-\cos^2 x)\sin x\, dx$$

$$[\because \ \sin^2 x=1-\cos^2 x=1-z^2]$$

$$=\int_1^0 z^4(1-z^2)(-dz)=-\int_1^0 (z^4-z^6)dz$$

$$=\int_0^1 (z^4-z^6)\, dz=\left[\frac{z^5}{5}-\frac{z^7}{7}\right]_0^1=\tfrac{1}{5}-\tfrac{1}{7}=\tfrac{2}{35}.$$

(iii) $\int \sin^3 x \cos^3 x\, dx$

$$=\tfrac{1}{8}\int 8\sin^3 x\cos^3 x\, dx=\tfrac{1}{8}\int(2\sin x\cos x)^3\, dx$$
$$=\tfrac{1}{8}\int \sin^3 2x\, dx=\tfrac{1}{8}.\tfrac{1}{4}\int(3\sin 2x-\sin 6x)\, dx$$
$$=\tfrac{3}{32}\int \sin 2x\, dx-\tfrac{1}{32}\int\sin 6x\, dx$$
$$=-\tfrac{3}{64}\cos 2x+\tfrac{1}{192}\cos 6x$$

$$\therefore \int_0^{\pi} \sin^3 x\cos^3 .x=\left[-\tfrac{3}{64}\cos 2x+\tfrac{1}{192}\cos 6x\right]_0^{\pi}$$

$$=(-\tfrac{3}{64}+\tfrac{1}{192})-(\tfrac{3}{64}+\tfrac{1}{192})=0.$$

Alternative method See § 5·7 (4)

Let, $I=\int_0^{\pi}\sin^3 x\cos^3 x\, dx=\int_0^{\pi}\sin^3(\pi-x)\cos^3(\pi-x)\, dx$

$$=\int_0^{\pi}\sin^3 x(-\cos^3 x)dx=-\int_0^{\pi}\sin^3 x\cos^3 x\, dx=-I$$

$\therefore \quad 2I=0$ or, $I=0.$

(iv) $\int_0^{\frac{\pi}{2}}\sin^2 x\cos^3 x\, dx=\int_0^{\frac{\pi}{2}}\sin^2 x\cos^2 x\cos x\, dx$

$$=\int_0^{\frac{\pi}{2}}\sin^2 x(1-\sin^2 x)\cos x\, dx$$

Let, $\sin x=z \quad \therefore \quad \cos x\, dx=dz.$

when $x=0$, then $z=0$

and when $x=\frac{\pi}{2}$, then $z=1$

$\therefore$ Given integral$=\int_0^1 z^2(1-z^2)dz=\int_0^1 (z^2-z^4)dz$

$=\left[\frac{z^3}{3}-\frac{z^5}{5}\right]_0^1=\frac{1}{3}-\frac{1}{5}=\frac{2}{15}.$

(v) $\int \sin^4 x\, dx=\frac{1}{4}\int (2\sin^2 x)^2\, dx=\frac{1}{4}\int (1-\cos 2x)^2\, dx$

$=\frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)\, dx$

$=\frac{1}{4}\int dx-\frac{1}{2}\int \cos 2x+\frac{1}{8}\int 2\cos^2 2x\, dx$

$=\frac{1}{4}\int dx-\frac{1}{2}\int \cos 2x+\frac{1}{8}\int (1+\cos 4x)\, dx$

$=\frac{1}{4}x-\frac{1}{4}\sin 2x+\frac{1}{8}x+\frac{1}{32}\sin 4x$

$\therefore\ \int_0^{\frac{\pi}{2}} \sin^4 x\, dx=\left[\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x\right]_0^{\frac{\pi}{2}}$

$=\left(\frac{3}{8}\cdot\frac{\pi}{2}-\frac{1}{4}\sin \pi+\frac{1}{32}\sin 2\pi\right)-0=\frac{3}{16}\pi$

(vi) $\int \sin^2 x\cos^2 x\, dx=\frac{1}{4}\int 4\sin^2 x\cos^2 x\, dx=\frac{1}{4}\int (2\sin x\cos x)^2$

$=\frac{1}{8}\int 2\sin^2 2x\, dx=\frac{1}{8}\int (1-\cos 4x)\, dx=\frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)$

$\therefore$ (*a*) $\int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\, dx=\left[\frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)\right]_0^{\frac{\pi}{2}}$

$=\frac{1}{8}\left\{\left(\frac{\pi}{2}-\frac{\sin 2\pi}{4}\right)-0\right\}=\frac{\pi}{16}$

(*b*) $\int_0^{\frac{\pi}{4}} \sin^2 x\cos^2 x\, dx=\left[\frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)\right]_0^{\frac{\pi}{4}}$

$=\frac{1}{8}\left\{\left(\frac{\pi}{4}-\frac{\sin \pi}{4}\right)-0\right]=\frac{\pi}{32}.$

Ex. 5. Evaluate :—

(i) $\int_1^2 xe^x\, dx$ [H. S. 1980] (ii) $\int_0^1 x^2e^x\, dx$

[H. S. 1978,' 83 ; Tripura 1979,' 85]

(iii) $\int_0^1 x^2e^{2x}\, dx$ [H. S. 1987] (iv) $\int_0^{\frac{\pi}{4}} x\cos x\, dx$

[H. S. 1978]

(v) $\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx$ [H. S. 1981] (vi) $\int_0^{\pi} x \cos^2 x \, dx$

[H. S. 1984]

(vii) $\int_1^2 \log x \, dx$ [H. S. 1981 ; Joint Entrance 1983]

(viii) (*a*) $\int_1^2 x \log x \, dx$ [H. S. 1980 ; Tripura 1983, 1984]

(*b*) $\int_1^{\sqrt{e}} x \log x dx$ [Tripura 1986]

(ix) $\int_1^e (\log x)^3 \, dx$ [H. S. 1984] (x) $\int_1^2 (\log x)^2 \, dx$

(xi) $\int_0^1 x \log (1+2x) \, dx$ [Tripura 1987]

(xii) $\int_0^1 x \tan^{-1} x dx$ [H. S. 1982]

(xiii) $\int_0^a \sin^{-1} \frac{2x}{1+x^2} \, dx$ [H. S. 1985]

(xiv) $\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x} \, dx$ [H. S. 1986]

(i) $\int xe^x \, dx = x \int e^x dx - \int \left\{ \frac{d}{dx} (x) \int e^x \, dx \right\} dx$

$$= xe^x - \int 1.e^x \, dx = xe^x - e^x = e^x(x-1)$$

$\therefore \int_1^2 xe^x \, dx = \Big[e^x (x-1) \Big]_1^2 = e^2(2-1) - e^1(1-1) = e^2.$

(ii) $\int x^2 e^x \, dx = x^2 \int e^x \, dx - \int \left\{ \frac{d}{dx}(x^2) \int e^x dx \right\} dx$

$= x^2 e^x - 2 \int xe^x \, dx = x^2 e^x - 2e^x \, (x-1) = e^x(x^2 - 2x + 2)$

$\therefore \int_0^1 x^2 e^x \, dx = \Big[\, e^x(x^2 - 2x + 2) \Big]_0^1$

$= e(1^2 - 2.1 + 2) - e^0(2) = e - 2.$

(iii) $\int x^2 e^{2x} \, dx = x^2 \int e^{2x} dx - \int \left\{ \frac{d}{dx}(x^2) \int e^{2x} dx \right\} dx$

$= \frac{x^2}{2} e^{2x} - \int 2x. \frac{e^{2x}}{2} dx = \frac{x^2}{2} e^{2x} - \int xe^{2x} \, dx$

$$=\frac{x^2}{2}e^{2x}-\left\{x\int e^{2x}\,dx-\int\frac{d}{dx}(x)\int e^{2x}dx\ dx\right\}$$

$$=\frac{x^2}{2}e^{2x}-\left\{x\frac{e^{2x}}{2}-\int\frac{e^{2x}}{2}dx\right\}$$

$$=\frac{x^2}{2}e^{2x}-\frac{x}{2}e^{2x}+\frac{e^{2x}}{4}=e^{2x}\left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right).$$

$$\therefore\ \int_0^1 x^2e^{2x}\,dx=\left[e^{2x}\left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right)\right]_0^1$$

$$=e^2\left(\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{4}\right)-e^0\left(\tfrac{1}{4}\right)=e^2\ \tfrac{1}{4}-\tfrac{1}{4}=\tfrac{1}{4}(e^2-1)$$

(iv) $\int x\cos x\,dx=x\int\cos x\,dx-\left\{\frac{d}{dx}(x)\int\cos x\,dx\right\}dx$

$$=x\sin x-\int\sin x\,dx=x\sin x+\cos x$$

$$\therefore\ \int_0^{\frac{\pi}{4}} x\cos x\,dx=\Big[\,x\sin x+\cos x\,\Big]_0^{\frac{\pi}{4}}$$

$$=\left(\frac{\pi}{4}.\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)-\left(0.\sin 0+\cos 0\right)$$

$$=\frac{\pi}{4}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1=\frac{1}{\sqrt{2}}\left(\frac{\pi}{4}+1\right)-1.$$

(v) $\int x^2\sin x\,dx=x^2\int\sin x\,dx-\int\left[\frac{d}{dx}(x^2)\int\sin x\,dx\right]dx$

$$=-x^2\cos x+\int 2x\cos x\,dx$$

$$=-x^2\cos x+2\left[x\int\cos x\,dx-\int\left\{\frac{d}{dx}(x)\int\cos x\,dx\right\}dx\right]$$

$$=-x^2\cos x+2x\sin x-2\int\sin x\,dx$$

$$=-x^2\cos x+2x\sin x+2\cos x$$

$$\therefore\ \int_0^{\frac{\pi}{2}} x^2\sin x\,dx=\Big[-x^2\cos x+2x\sin x+2\cos x\Big]_0^{\frac{\pi}{2}}$$

$$=\left(-\frac{\pi^2}{4}.\cos\frac{\pi}{2}+2.\frac{\pi}{2}.\sin\frac{\pi}{2}+2\cos\frac{\pi}{2}\right)$$

$$-(0.\cos 0+2.0\sin 0+2\cos 0)$$

$$=2.\frac{\pi}{2}-2=\pi-2\quad\left[\because\ \cos\frac{\pi}{2}=0,\ \sin\frac{\pi}{2}=1\right]$$

(vi) $\int x\cos^2 x\,dx=\frac{1}{2}\int x\,2\cos^2 x\,dx$

$$=\tfrac{1}{2}\int x\,(1+\cos 2x)\,dx=\tfrac{1}{2}\int x\,dx+\tfrac{1}{2}\int x\cos 2x\,dx$$

$$=\frac{1}{2}\,\frac{x^2}{2}+\tfrac{1}{2}\left[x\int\cos 2x\,dx-\int\left\{\frac{d}{dx}(x)\int\cos 2x\;dx\right\}dx\right]$$

$$=\tfrac{1}{4}x^2+\tfrac{1}{2}\left[x\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}dx\right]$$

$$=\tfrac{1}{4}x^2+\tfrac{1}{2}\left[\frac{x\sin 2x}{2}+\tfrac{1}{4}\cos 2x\right]$$

$$\therefore\quad\int_0^{\pi}x\cos^2 x\,dx=\left[\tfrac{1}{4}x^2+\tfrac{1}{2}\left(\frac{x\sin 2x}{2}+\tfrac{1}{4}\cos 2x\right)\right]_0^{\pi}$$

$$=(\tfrac{1}{4}\pi^2+\tfrac{1}{4}\pi.\,\sin 2\pi+\tfrac{1}{8}\;\mathrm{ces}\;2\pi)$$

$$-(0+0+\tfrac{1}{8}.1)=\frac{\pi^2}{4}+\tfrac{1}{4}\,\pi.0+\tfrac{1}{8}.1-\tfrac{1}{8}=\frac{\pi^2}{4}.$$

(vii) $\int\log x\,dx=\log x\int 1\;dx-\int\frac{d}{dx}(\log x)\int 1\,dx\,dx$

$$=x\log x-\int\frac{1}{x}.x\,dx=x\log x-\int dx=x\log x-x$$

$$\therefore\quad\int_1^2\log x\,dx=\Big[x\log x-x\Big]_1^2=(2\log 2-2)-(1.\log 1-1)$$

$$=2\log 2-2+1=\log 2^2-1=\log 4-1.$$

(viii) $\int x\log x\,dx=\log x\int x\,dx-\int\left\{\frac{d}{dx}(\log x)\int x\,dx\right\}dx$

$$=\frac{x^2}{2}\log x-\int\frac{1}{x}.\frac{x^2}{2}dx=\frac{x^2}{2}\log x-\frac{x^2}{4}$$

$\therefore$ (*a*) $\int_1^2 x\log x\,dx=\left[\frac{x^2}{2}\log x-\frac{x^2}{4}\right]_1^2$

$$=\left(\frac{2^2}{2}\log 2-\frac{2^2}{4}\right)-(\tfrac{1}{2}\log 1-\tfrac{1}{4})$$

$$=2\log 2-1+\tfrac{1}{4}\qquad[\because\;\log 1=0]=\log 4-\tfrac{3}{4}.$$

(*b*) $\int_1^{\sqrt{e}}x\log x\,dx=\left[\frac{x^2}{2}\log x-\frac{x^2}{4}\right]_1^{\sqrt{e}}$

$$=\left(\frac{e}{2}\log\sqrt{e}-\frac{e}{4}\right)-(\tfrac{1}{2}\log 1-\tfrac{1}{4})$$

$$=\frac{e}{2}.\frac{1}{2}\log e-\frac{e}{4}+\frac{1}{4}\quad\left[\because\;\log 1=0\right]$$

$$=\frac{e}{4}-\frac{e}{4}+\frac{1}{4}=\frac{1}{4}\quad[\;\because\;\log e=1\,]$$

(ix) $\int(\log x)^3\,dx=\int 1.(\log x)^3\,dx$

$$=(\log x)^3\int 1\,dx-\int\left\{\frac{d}{dx}(\log x)^3\int 1\,dx\right\}dx$$

$$=x\,(\log x)^3-\int 3(\log x)^2\,\frac{1}{x}.x\,dx$$

$$=x\,(\log x)^3-3\int(\log x)^2\,dx$$

$$=x\,(\log x)^3-3\left[(\log x)^2\int 1\,dx-\int\frac{d}{dx}(\log x)^2\int 1\,dx\,dx\right]$$

$$=x\,(\log x)^3-3x\,(\log x)^2+3\int 2\log x.\frac{1}{x}.x\,dx$$

$$=x\,(\log x)^3-3x\,(\log x)^2+6\int\log x\,dx$$

$$=x\,(\log x)^3-3x\,(\log x)^2+6x\,(\log x-1)$$

$$\therefore\quad\int_1^e(\log x)^3\,dx=\Big[x\,(\log x)^3-3x\,(\log x)^2+6x\log x-6x\Big]_1^e$$

$$=\{e\,(\log e)^3-3e\,(\log e)^2+6e\log e-6e\}$$

$$-\{1\,(\log 1)^3-3.1(\log 1)^2+6.1\log 1-6.1\}$$

$$=e(\log e)^3-3\,e(\log e)^2+6e-6e+6$$

$$=6-2e\,[\text{as }\log e=1\text{ and }\log 1=0]$$

(x) $\int(\log x)^2\,dx=(\log x)^2\int 1\,dx-\int\left\{\frac{d}{dx}(\log x)^2\int 1\,dx\right\}dx$

$$=x\,(\log x)^2-\int 2\log x.\frac{1}{x}.x\,dx$$

$$=x\,(\log x)^2-2\int\log x\,dx=x\,(\log x)^2-2x\,(\log x-1)$$

$$\therefore\quad\int_1^2(\log x)^2\,dx=\{2(\log 2)^2-2.2(\log 2-1)\}$$

$$-\{1(\log 1)^2-2.1(\log 1-1)\}$$

$$=2(\log 2)^2-4\log 2+4-2\qquad[\because\ \log 1=0]$$

$$=2(\log 2)^2-4\log 2-2.$$

(xi) $\int x\log(1+2x)\,dx$

$$=\log(1+2x)\int x\,dx-\int\left\{\frac{d}{dx}\log(1+2x)\int x\,dx\right\}dx$$

$$=\frac{x^2}{2}\log(1+2x)-\int\frac{2}{1+2x}.\frac{x^2}{2}dx=\frac{x^2}{2}\log(1+2x)-\int\frac{x^2}{1+2x}dx$$

$$=\frac{x^2}{2}\log(1+2x)-\tfrac{1}{2}\int\left(x-\frac{x}{1+2x}\right)dx$$

$$=\frac{x^2}{2}\log(1+2x)-\tfrac12\int\left(x-\tfrac12\,\frac{1+2x-1}{1+2x}\right)dx$$

$$=\frac{x^2}{2}\log(1+2x)-\tfrac12\int\left(x-\tfrac12+\tfrac12\,\frac{1}{1+2x}\right)dx$$

$$=\frac{x^2}{2}\log(1+2x)-\tfrac12\left\{\frac{x^2}{2}-\tfrac12x+\tfrac14\log(1+2x)\right\}$$

$\therefore\ \int_0^1 x\log(1+2x)\,dx$

$$=\left[\frac{x^2}{2}\log(1+2x)-\tfrac12\left\{\frac{x^2}{2}-\tfrac12x+\tfrac14\log(1+2x)\right]_0^1\right.$$

$$=\{\tfrac12\log 3-\tfrac12(\tfrac12-\tfrac12+\tfrac14\log 3)\}-\{\tfrac14\log 1\}$$

$$=\tfrac12\log 3-\tfrac18\log 3=\tfrac38\log 3.$$

(xii) $\int x\tan^{-1}x\,dx=\tan^{-1}x\int x\,dx-\int\left\{\frac{d}{dx}(\tan^{-1}x)\int x\,dx\right\}dx$

$$=\frac{x^2}{2}\tan^{-1}x-\tfrac12\int\frac{x^2}{1+x^2}dx=\frac{x^2}{2}\tan^{-1}x-\tfrac12\int\left\{1-\frac{1}{1+x^2}\right\}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\tfrac12(x-\tan^{-1}x)$$

$\therefore\ \int_0^1 x\tan^{-1}x\,dx=\left[\frac{x^2}{2}\tan^{-1}x-\tfrac12(x-\tan^{-1}x)\right]_0^1$

$$=\{\tfrac12\tan^{-1}1-\tfrac12(1-\tan^{-1}1)\}-\tfrac12\tan^{-1}0$$

$$=\frac12\cdot\frac{\pi}{4}-\frac12+\frac12\cdot\frac{\pi}{4}-0=\frac{\pi}{4}-\frac12.$$

(xiii) Let, $x=\tan\theta$ $\quad\therefore\ dx=\sec^2\theta\,d\theta$

$\therefore$ when $x=0$, then $\theta=0$; when $x=a$, then $\theta=\tan^{-1}a$

$\therefore\ \sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}\sin 2\theta=2\theta$

$\therefore\ \int\sin^{-1}\frac{2x}{1+x^2}=\int 2\theta\sec^2\theta\,d\theta$

$$=2\left[\theta\int\sec^2\theta\,d\theta-\int\left\{\frac{d}{d\theta}(\theta)\int\sec^2\theta\,d\theta\right\}d\theta\right]$$

$$=2[\theta\tan\theta-\int\tan\theta\,d\theta]=2\theta\tan\theta-\log\sec^2\theta$$

$$=2\theta\tan\theta-\log(1+\tan^2\theta)$$

$\therefore\ \int_0^a\sin^{-1}\frac{2x}{1+x^2}dx=\left[2\theta\tan\theta-\log(1+\tan^2\theta)\right]_0^{\tan^{-1}a}$

$=2a\tan^{-1}a-\log(1+a^2)$ [$\because$ $\log 1=0$]

(xiv) $\int \frac{x\,dx}{1+\cos x}=\int \frac{x}{2\cos^2 \frac{x}{2}}dx=\int \frac{x}{2}\sec^2 \frac{x}{2}\,dx$

$=x.\int \frac{1}{2}\sec^2 \frac{x}{2}\,dx-\int\left\{\frac{d}{dx}(x)\int \frac{1}{2}\sec^2 \frac{x}{2}\,dx\right\}dx$

$=x.\tan \frac{x}{2}-\int 1.\tan \frac{x}{2}\,dx=x\tan \frac{x}{2}+2\log\cos \frac{x}{2}$

$\therefore \int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x}=\left[x\tan \frac{x}{2}+2\log\cos \frac{x}{2}\right]_0^{\frac{\pi}{2}}$

$=\left(\frac{\pi}{2}\tan \frac{\pi}{4}+2\log\cos \frac{\pi}{4}\right)-(0.\tan 0+2\log\cos 0)$

$=\frac{\pi}{2}+2\log \frac{1}{\sqrt{2}}$ $\quad\left[\because\ 2\log\cos 0=2\log 1=0\right]$

$=\frac{\pi}{2}-2.\frac{1}{2}\log 2=\frac{\pi}{2}-\log 2.$

Ex. 6. Evaluate :—

(i) $\int_0^1 \frac{dx}{x^2+4x+5}$ [H. S. 1988] (ii) $\int_0^1 \frac{dx}{1+x+x^2}$ [Tripura 1978]

(iii) $\int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}}$ [H. S. 1985 ; c.f. Joint Entrance 1980]

(iv) $\int_0^a \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}$

(i) $\int \frac{dx}{x^2+4x+5}=\int \frac{dx}{(x+2)^2+1^2}=\tan^{-1}(x+2)$

$\therefore \int_0^1 \frac{dx}{x^2+4x+5}=\left[\tan^{-1}(x+2)\right]_0^1$

$=\tan^{-1} 3-\tan^{-1} 2=\tan^{-1}\frac{1}{7}.$

(ii) $\int \frac{dx}{1+x+x^2}=\int \frac{dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$

$\therefore \int_0^1 \frac{dx}{1+x+x^2}=\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2x+1}{\sqrt{3}}\right]_0^1$

$=\frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3}-\tan^{-1}\frac{1}{\sqrt{3}}\right)$

$=\frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}.\frac{\pi}{6}=\frac{\pi}{3\sqrt{3}}.$

(iii) Let $x+1=z^2$ or, $dx=2z\,dz$

Again $x=z^2-1$ $\therefore$ $x-3=z^2-1-3=z^2-4.$

when $x=8$, then $z=\sqrt{8+1}=3.$

and when $x=15$, then $z=\sqrt{15+1}=4$

$$\therefore \text{ Given integral}=\int_3^4 \frac{2\,z dz}{(z^2-4)z}=2\int_3^4 \frac{dz}{z^2-4}$$

$$=\tfrac{2}{4}\left[\log \frac{z-2}{z+2}\right]_3^4$$

$$=\tfrac{1}{2}(\log \tfrac{2}{6}-\log \tfrac{1}{5})=\tfrac{1}{2}\log\left(\frac{\frac{2}{6}}{\frac{1}{5}}\right)=\tfrac{1}{2}\log \tfrac{5}{3}.$$

(iv) Let $x=a\tan\theta$ $\therefore$ $dx=a\sec^2\theta\, d\theta$

and $(a^2+x^2)^{\frac{3}{2}}=(a^2+a^2\tan^2\theta)^{\frac{3}{2}}=(a^2\sec^2\theta)^{\frac{3}{2}}=a^3\sec^3\theta.$

Again when $x=0$, then $a\tan\theta=0$ or $\tan\theta=0$ or, $\theta=0$

and when $x=a$, then $a\tan\theta=a$ or, $\tan\theta=1$ or, $\theta=\frac{\pi}{4}$

$$\therefore \text{ Given integral}=\int_0^{\frac{\pi}{4}} \frac{a\sec^2\theta d\theta}{a^3\sec^3\theta}=\frac{1}{a^2}\int_0^{\frac{\pi}{4}} \cos\theta d\theta$$

$$=\frac{1}{a^2}\Big[\sin\theta\Big]_0^{\frac{\pi}{4}}=\frac{1}{a^2}\cdot\Big[\sin\tfrac{\pi}{4}-\sin 0\Big]=\frac{1}{\sqrt{2}a^2}.$$

Ex. 7. Evaluate :—

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x+1}-\sqrt{x}}$ [H. S. 1982] (ii) $\displaystyle\int_0^2 \frac{dx}{\sqrt{x+3}-\sqrt{x+1}}$ [H. S. 1984]

(i) $$\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}=\int \frac{\sqrt{x+1}+\sqrt{x}}{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}dx$$

$$=\int \frac{\sqrt{x+1}+\sqrt{x}}{x+1-x}dx=\int \sqrt{x+1}dx+\int \sqrt{x}\,dx=\tfrac{2}{3}[(x+1)^{\frac{3}{2}}+x^{\frac{3}{2}}]$$

$$\therefore \int_0^1 \frac{dx}{\sqrt{x+1}-\sqrt{x}}=\tfrac{2}{3}\Big[(x+1)^{\frac{3}{2}}+x^{\frac{3}{2}}\Big]_0^1$$

$$=\tfrac{2}{3}\{(2^{\frac{3}{2}}+1)-1\}=\tfrac{2}{3}.2^{\frac{3}{2}}=\frac{4\sqrt{2}}{3}$$

(ii) $$\int \frac{dx}{\sqrt{x+3}-\sqrt{x+1}}=\int \frac{\sqrt{x+3}+\sqrt{x+1}}{(\sqrt{x+3}-\sqrt{x+1})(\sqrt{x+3}+\sqrt{x+1})}dx$$

$$=\int \frac{\sqrt{x+3}+\sqrt{x+1}}{x+3-x-1}dx=\tfrac{1}{2}\int \left\{(x+3)^{\frac{1}{2}}+(x+1)^{\frac{1}{2}}\right\}dx$$

$$=\tfrac{1}{2}.\tfrac{2}{3}[(x+3)^{\frac{3}{2}}+(x+1)^{\frac{3}{2}}]$$

$$\therefore \int_0^2 \frac{dx}{\sqrt{x+3}-\sqrt{x+1}}=\tfrac{1}{3}\left[(x+3)^{\frac{3}{2}}+(x+1)^{\frac{3}{2}}\right]_0^2$$

$$=\tfrac{1}{3}[(5^{\frac{3}{2}}+3^{\frac{3}{2}})-(3^{\frac{3}{2}}+1^{\frac{3}{2}})]=\tfrac{1}{3}(5\sqrt{5}-1).$$

Ex. 8. (*a*) Evaluate :—

(i) $\int_2^e \left\{\frac{1}{\log x}-\frac{1}{(\log x)^2}\right\}dx$ (ii) $\int_1^e \frac{(x+1)^3}{x^2}\log x\,dx$ [Joint Entrance 1978]

(iii) $\int_0^{\frac{\pi}{4}} \sec x\sqrt{\frac{1-\sin x}{1+\sin x}}dx$ [Joint Entrance 1984]

(iv) $\int_0^1 \frac{dx}{e^x+e^{-x}}$ [Joint Entrance 1985]

(v) $\int_0^{\frac{\pi}{4}} \frac{\sin^2 x\cos^2 x\,dx}{(\sin^3 x+\cos^3 x)^2}$ [Joint Entrance 1987]

(vi) $\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}$ $(a>0,\ b>0)$ [Joint Entrance 1988]

(vii) $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ $(\beta>\alpha)$ [Joint Entrance, 1979]

(viii) $\int_1^2 \sqrt{(x-1)(2-x)}\,dx$ [Tripura 1985]

(ix) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x\,dx}{(\cos\frac{x}{2}+\sin\frac{x}{2})^2}$

(*b*) Prove that $\int_{-3}^{-2} \frac{dx}{x^2-1}=\frac{1}{2}\log\frac{3}{2}$ [Joint Entrance 1981]

(*a*) (i) $\int_2^e \left\{\frac{1}{\log x}-\frac{1}{(\log x)^2}\right\}dx=\int_2^e \frac{dx}{\log x}-\int_2^e \frac{dx}{(\log x)^2}$

$$=\left[\frac{x}{\log x}\right]_2^e-\int_2^e\left\{-\frac{1}{(\log x)^2}\cdot\frac{1}{x}.x\right\}dx-\int_2^e\frac{dx}{(\log x)^2}$$

$$=\frac{e}{\log e}-\frac{2}{\log 2}+\int_2^e\frac{1}{(\log x)^2}dx-\int_2^e\frac{dx}{(\log x)^2}$$

$$=e-\frac{2}{\log 2}\qquad\left[\because\ \log e=1\right].$$

(ii) $\int_1^e \frac{(x+1)^3}{x^2}\log x\,dx=\int_1^e \frac{x^3+3x^2+3x+1}{x^2}\log x\,dx$

$$=\int_1^e x\log x\ dx+3\int_1^e \log x\ dx+3\int_1^e \frac{\log x}{x}dx+\int_1^e \frac{\log x}{x^2}dx$$

Now, $\int x\log x\ dx=\log x\int x\ dx-\int\left\{\frac{d}{dx}(\log x)\int x\ dx\right\}dx$

$$=\frac{x^2}{2}\log x-\int\frac{1}{x}.\frac{x^2}{2}dx=\frac{x^2}{2}\log x-\frac{x^2}{4}$$

$$\therefore\quad \int_1^e x\log x\ dx=\left[\frac{x^2}{2}\log x-\frac{x^2}{4}\right]_1^e$$

$$=\left(\frac{e^2}{2}\log e-\frac{e^2}{4}\right)-(\tfrac{1}{2}\log 1-\tfrac{1}{4})=\left(\frac{e^2}{2}.1-\frac{e^2}{4}\right)-(\tfrac{1}{2}.0-\tfrac{1}{4})=\frac{e^2}{4}+\tfrac{1}{4}.$$

Again, $\int\log x\ dx=\log x\int 1.\ dx-\int\left\{\frac{d}{dx}(\log x)\int 1\ dx\right\}dx$

$$=x\log x-\int\frac{1}{x}.\ x\ dx=x\log x-x.$$

$$\therefore\quad \int_1^e \log x\ dx=\Big[x\log x-x\Big]_1^e$$

$$=(e\log e-e)-(1\log 1-1)=(e.1-e)-(1.0-1)=1.$$

$$\int\frac{1}{x}\log x\ dx=\log x\int\frac{dx}{x}-\int\left\{\frac{d}{dx}(\log x)\int\frac{dx}{x}\right\}dx$$

$$=\log x.\ \log x-\int\frac{1}{x}.\log x\ dx$$

$\therefore\quad 2\int\frac{\log x}{x}dx=(\log x)^2$ বা, $\int\frac{\log x}{x}dx=\tfrac{1}{2}(\log x)^2$

$$\therefore\quad \int_1^e\frac{\log x}{x}dx=\tfrac{1}{2}\{(\log e)^2-(\log 1)^2\}=\frac{1^2-0}{2}=\tfrac{1}{2}$$

$$\int\frac{\log x}{x^2}dx=\log x\int\frac{dx}{x^2}-\int\left\{\frac{d}{dx}(\log x)\int\frac{dx}{x^2}\right\}dx$$

$$=-\frac{1}{x}\log x-\int\frac{1}{x}\left(-\frac{1}{x}\right)dx=-\frac{1}{x}\log x+\int\frac{dx}{x^2}$$

$$=-\frac{1}{x}\log x-\frac{1}{x}$$

$$\therefore\quad \int_1^e\frac{\log x}{x^2}dx=\left[-\frac{1}{x}\log x-\frac{1}{x}\right]_1^e$$

$$=\left(-\frac{1}{e}\log e-\frac{1}{e}\right)-(-\tfrac{1}{1}\log 1-\tfrac{1}{1})=-\frac{2}{e}+1=1-\frac{2}{e}.$$

∴ Given integral

$$=\int_1^e x\log x\,dx+3\int_1^e \log x\,dx+3\int_1^e \frac{1}{x}\log x\,dx+\int_1^e \frac{\log x}{x^2}dx$$

$$=\frac{e^2}{4}+\tfrac{1}{4}+3.1+3.\tfrac{1}{2}+1-\frac{2}{e}=\frac{e^2}{4}-\frac{2}{e}+5\tfrac{3}{4}.$$

(iii) $\displaystyle\int \sec x\sqrt{\frac{1-\sin x}{1+\sin x}}dx=\int\sqrt{\frac{(1-\sin x)^2}{(1-\sin x)(1+\sin x)\cos^2 x}}dx$

$$=\int\frac{1-\sin x}{\sqrt{\cos^4 x}}dx=\int\frac{1}{\cos^2 x}dx-\int \sec x\tan x\,dx=\tan x-\sec x$$

$$\therefore\ \int_0^{\frac{\pi}{4}}\sec x\sqrt{\frac{1-\sin x}{1+\sin x}}dx=\Big[\tan x-\sec x\Big]_0^{\frac{\pi}{4}}$$

$$=\left(\tan\frac{\pi}{4}-\sec\frac{\pi}{4}\right)-(\tan 0-\sec 0)=(1-\sqrt{2})-(0-1)$$

$$=1-\sqrt{2}+1=2-\sqrt{2}.$$

(iv) $\displaystyle\int_0^1\frac{dx}{e^x+e^{-x}}=\int_0^1\frac{dx}{e^x+\frac{1}{e^x}}=\int_0^1\frac{e^x\,dx}{1+e^{2x}}$

Let, $e^x=z$ ∴ $e^x\,dx=dz$; $e^{2x}=(e^x)^2=z^2$.

when $x=0$, then $z=e^0=1$ when $x=1$, then $z=e^1=e$

$$\therefore\ \int_0^1\frac{dx}{e^x+e^{-x}}=\int_1^e\frac{dz}{1+z^2}=\Big[\tan^{-1}z\Big]_1^e=\tan^{-1}e-\tan^{-1}1$$

$$=\tan^{-1}e-\frac{\pi}{4}.$$

(v) $\displaystyle\frac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}=\frac{\sin^2 x\cos^2 x}{\cos^6 x\,(1+\tan^3 x)^2}=\frac{\sec^2 x\tan^2 x}{(1+\tan^3 x)^2}$

Now let, $\tan x=z$ ∴ $\sec^2 x\,dx=dz$.

when $x=0$, then $z=\tan 0=0$

and when $x=\frac{\pi}{4}$, then $z=\tan\frac{\pi}{4}=1$.

∴ Given integral

$$=\int_0^1\frac{z^2\,dz}{(1+z^3)^2}=-\frac{1}{3}\left[\frac{1}{1+z^2}\right]_0^1\quad(1+z^3=t\text{ say})$$

$$=-\tfrac{1}{3}(\tfrac{1}{2}-1)=-\tfrac{1}{3}.(-\tfrac{1}{2})=\tfrac{1}{6}.$$

(vi) $\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2x+b^2\sin^2x)^2}$

$$=\int_0^{\frac{\pi}{2}} \frac{\sec^4 dx}{\sec^4x(a^2\cos^2x+b^2\sin^2x)^2}=\int_0^{\frac{\pi}{2}}\frac{\sec^2x.\sec^2x\,dx}{(a^2+b^2\tan^2x)^2}$$

$$=\int_0^{\frac{\pi}{2}}\frac{(1+\tan^2x)\sec^2x\,dx}{(a^2+b^2\tan^2x)}$$

Let $b\tan x=a\tan\theta$ ∴ $b\sec^2xdx=a\sec^2\theta d\theta$

when $x=0$, then $\tan\theta=\frac{b}{a}\tan 0=0$ or, $\theta=0$

and when $x=\frac{\pi}{2}$ then $\tan\theta=\frac{b}{a}\tan\frac{\pi}{2}=$which is undefined

∴ $\theta=\frac{\pi}{2}$

$$\therefore \int_0^{\frac{\pi}{2}}\frac{dx}{(a^2\cos^2x+b^2\sin^2x)^2}=\int_0^{\frac{\pi}{2}}\frac{\left(1+\frac{a^2}{b^2}\tan^2\theta\right)\frac{a}{b}\sec^2\theta d\theta}{(a^2+a^2\tan^2\theta)^2}$$

$$=\int_0^{\frac{\pi}{2}}\frac{(b^2\cos^2\theta+a^2\sin^2\theta)\frac{a}{b}\sec^2\theta d\theta}{a^4b^2\cos^2\theta.\sec^4\theta}$$

$$=\frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}}(b^2\cos^2\theta+a^2\sin^2\theta)d\theta$$

$$=\frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}}\left\{\frac{b^2}{2}(1+\cos 2\theta)+\frac{a^2}{2}(1-\cos 2\theta)\right\}d\theta$$

$$=\frac{1}{2a^3b^3}\left[b^2\left(\theta+\frac{\sin 2\theta}{2}\right)+a^2\left(\theta-\frac{\sin 2\theta}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2a^3b^3}\Big[\Big\{b^2\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)+a^2\left(\frac{\pi}{2}-\frac{\sin\pi}{2}\right)-\Big\{b^2\left(0+\frac{\sin 0}{2}\right)+a^2\left(0-\frac{\sin 0}{2}\right)\Big\}\Big\}\Big]$$

$$=\frac{1}{2a^3b^3}\cdot\frac{\pi}{2}\left(a^2+b^2\right)=\frac{\pi}{4}\cdot\frac{a^2+b^2}{a^3b^3}$$

(vii) Let, $\alpha\cos^2\theta+\beta\sin^2\theta=x$

∴ $-2\alpha\cos\theta\sin\theta d\theta+2\beta\sin\theta\cos\theta d\theta=dx$

or, $2(\beta-\alpha)\sin\theta\cos\theta\, d\theta=dx$

$x-\alpha=\alpha\cos^2\theta+\beta\sin^2\theta-\alpha=\beta\sin^2\theta-\alpha(1-\cos^2\theta)=(\beta-\alpha)\sin^2\theta.$

$\beta-x=\beta-\alpha\cos^2\theta-\beta\sin^2\theta=\beta(1-\sin^2\theta)-\alpha\cos^2\theta=(\beta-\alpha)\cos^2\theta$

$$\therefore \quad \sqrt{(x-\alpha)(\beta-x)}=\sqrt{(\beta-\alpha)^2 \sin^2\theta \cos^2\theta}$$
$$=(\beta-\alpha) \sin\theta \cos\theta$$

when $x=\alpha$, then $\alpha \cos^2\theta+\beta \sin^2\theta=\alpha$

or, $\beta \sin^2\theta-\alpha(1-\cos^2\theta)=0$

or, $(\beta-\alpha) \sin^2\theta=0$; or, $\sin^2\theta=0$ $(\because \beta>\alpha)$

$\therefore$ $\sin\theta=0$ or, $\theta=0$

when $x=\beta$, then $\alpha \cos^2\theta+\beta \sin^2\theta=\beta$.

or, $\alpha \cos^2\theta-\beta(1-\sin^2\theta)=0$

or, $(\alpha-\beta) \cos^2\theta=0$; or, $\cos^2\theta=0$ $[\because \beta>\alpha]$

$\therefore$ $\cos\theta=0$ or, $\theta=\frac{\pi}{2}$

$$\therefore \quad \int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}=\int_0^{\frac{\pi}{2}} \frac{2(\beta-\alpha) \sin\theta \cos\theta d\theta}{(\beta-\alpha) \sin\theta \cos\theta}$$

$$=2\int_0^{\frac{\pi}{2}} d\theta=2\Big[\theta\Big]_0^{\frac{\pi}{2}}=2.\frac{\pi}{2}=\pi.$$

(viii) $\int_1^2 \sqrt{(x-1)(2-x)}dx$

Let, $x=\cos^2\theta+2 \sin^2\theta$

$$\therefore \quad dx=-2 \cos\theta \sin\theta d\theta+4 \sin\theta \cos\theta d\theta$$
$$=2 \sin\theta \cos\theta d\theta$$

$\sqrt{(x-1)(x-2)}=(2-1) \sin\theta \cos\theta$

when $x=1$, then $\theta=0$

when $x=2$, then $\theta=\frac{\pi}{2}$ } See (vii) above

$$\therefore \quad \int_1^2 \sqrt{(x-1)(2-x)}\, dx=\int_0^{\frac{\pi}{2}} \sin\theta \cos\theta . 2 \sin\theta \cos\theta d\theta.$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{2}} 2 \sin^2 2\theta d\theta=\tfrac{1}{4}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta)d\theta$$

$$=\tfrac{1}{4}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\tfrac{1}{4}\left\{\left(\frac{\pi}{2}-\frac{\sin\pi}{2}\right)-\left(0-\frac{\sin 0}{2}\right)\right\}=\tfrac{1}{4}.\frac{\pi}{2}=\frac{\pi}{8}.$$

(ix) $$\int\frac{\cos x\, dx}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}=\int\frac{\cos x\, dx}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\int\frac{\cos x dx}{1+\sin x}$$

$$=\int\frac{d(1+\sin x)}{1+\sin x}=\int\frac{dz}{z}\quad[\text{ Let } 1+\sin x=z\ ]$$

$$=\log z=\log(1+\sin x)$$

$$\therefore\quad\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos x dx}{1+\sin x}=\Big[\log(1+\sin x)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$=\log\left(1+\sin\frac{\pi}{3}\right)-\log\left(1+\sin\frac{\pi}{4}\right)=\log\frac{1+\sin\frac{\pi}{3}}{1+\sin\frac{\pi}{4}}$$

$$=\log\frac{1+\frac{\surd 3}{2}}{1+\frac{1}{\surd 2}}=\log\frac{2+\surd 3}{\surd 2+2}$$

(*b*) $\int\frac{dx}{x^2-1}=\int\frac{1}{2}\left\{\frac{1}{x-1}-\frac{1}{x+1}\right\}dx=\frac{1}{2}\log\frac{x-1}{x+1}$

$$\therefore\quad\int_{-3}^{-2}\frac{dx}{x^2-1}=\frac{1}{2}\Big[\log\frac{x-1}{x+1}\Big]_{-3}^{-2}=\frac{1}{2}\left(\log\frac{-2-1}{-2+1}-\log\frac{-3-1}{-3+1}\right)$$

$$=\tfrac{1}{2}(\log 3-\log 2)=\tfrac{1}{2}\log\tfrac{3}{2}.$$

Ex. 9. Evaluate :—

(i) $\int_0^{\frac{1}{2}}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ [I. I. T. 1984]

(ii) $\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}dx$ [I. I. T. 1983]

(i) Let $\sin^{-1}x=z\ \therefore\ \sin z=x$ and $\frac{dx}{\sqrt{1-x^2}}=dz$

Again when $x=0$, then $z=\sin^{-1}0=0$.

and when $x=\frac{1}{2}$, then $z=\sin^{-1}\frac{1}{2}=\frac{\pi}{6}$.

So given integral $=\int_0^{\frac{\pi}{6}}z\sin z\,dz$

Now, $\int z\sin z dz=z\int\sin z-\int\left\{\frac{d}{dz}(z)\int\sin z dz\right\}dz$

$=-z\cos z+\int\cos z dz=-z\cos z+\sin z$

$\therefore$ Given integral $=\int_0^{\frac{\pi}{6}}z\sin z dz=\Big[-z\cos z+\sin z\Big]_0^{\frac{\pi}{6}}$

$$=\left(-\frac{\pi}{6}\cos\frac{\pi}{6}+\sin\frac{\pi}{6}\right)-(-0.\cos 0+\sin 0)$$

$$=\left(-\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2}+\tfrac{1}{2}\right)-0=-\frac{\sqrt{3}\pi}{12}+\tfrac{1}{2}$$

(ii) Let $\sin x-\cos x=z$ $\therefore$ $(\cos x+\sin x)dx=dz$

Again when $x=0$, then $z=\sin 0-\cos 0=-1$

and when $x=\frac{\pi}{4}$, then $z=\sin\frac{\pi}{4}-\cos\frac{\pi}{4}=0$.

Again $(\sin x-\cos x)^2=1-\sin 2x$ or, $z^2=1-\sin 2x$

$\therefore$ $\sin 2x=1-z^2$

$$\therefore \int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}$$

$$=\int_{-1}^{0}\frac{dz}{9+16(1-z^2)}=\int_{-1}^{0}\frac{dz}{25-16z^2}=\left[\tfrac{1}{4}\cdot\frac{1}{2.5}\log\frac{5+4z}{5-4z}\right]_{-1}^{0}$$

$$=\tfrac{1}{40}\{\log 1-\log\tfrac{1}{9}\}=\tfrac{1}{40}(\log 1-\log 1+\log 9)=\tfrac{1}{40}\log 9.$$

Exercise 5A

Evaluate :—

1. (i) $\int_1^{10}x^8\,dx$ (ii) $\int_1^3 x^2\,dx$ (iii) $\int_a^b dx$ (iv) $\int_2^5(x+5)dx$

(v) $\int_0^1(px+q)dx$ (vi) $\int_{-1}^{1}\frac{3t+2}{4}dt$ (vii) $\int_{-2}^{2}(x+2)^3dx$ [C. U.]

2. $\int_0^9(x^{\frac{1}{2}}+x^{-\frac{1}{2}})dx.$

3. (i) $\int_0^{\frac{\pi}{6}}\cos 3x\,dx$ (ii) $\int_0^{\frac{\pi}{2}}\cos^2 nz\,dz$ ($n\neq 0$, an integer)

(iii) $\int_{\frac{\pi}{2}}^{\pi}(x+\sin 2x)dx$ (iv) $\int_0^{\pi}\sin mx\,dx$ (v) $\int_0^{\frac{\pi}{4}}\tan x\,dx$

4. $\int_0^{\frac{\pi}{2}}(\cos\theta-\sin\theta)d\theta$

5. $\int_0^{\frac{\pi}{4}}\sin^2 nx\,dx$

6. $\int_0^1\tan^{-1}x\,dx$

7. $\int_0^4\sqrt{1+2x}\,dx$

8. $\int_{-2}^{-1}\frac{dx}{(x-2)^3}$

9. $\int_0^{\frac{\pi}{4}}\sec\theta(\sec\theta-\tan\theta)d\theta$

10. $\int_0^1 x(\tan^{-1} x)^2 dx.$

If m and n be integers (Ex. 11-13)

11. $\int_0^{\pi} \sin mx \sin nx\, dx$ 12. $\int_0^{\pi} \sin mx \cos nx\, dx$

13. $\int_0^{\pi} \cos mx \cos nx\, dx$ 14. $\int_a^b e^{mx}\, dx$ 15. $\int_1^4 \log x\, dx$

16. $\int_0^{\frac{1}{2}} \sin^{-1} x\, dx$ [C. U. 1970] 17. $\int_0^{\pi} x \sin x\, dx$

18. $\int_0^{\frac{\pi}{2}} x \sin^2 x\, dx$ 19. $\int_0^1 x \log (x+3) dx$ 20. $\int_0^1 x^2 \tan^{-1} x\, dx$

21. $\int_0^{\frac{\pi}{2}} e^x (\sin x + \cos x) dx$ 22. $\int_{\frac{\pi}{2}}^{\pi} (x + \sin 2x) dx$

23. $\int_0^{\frac{\pi}{4}} x \cos x \cos 3x\, dx$ 24. (i) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{5+4 \sin \theta}$

(ii) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{4+5 \sin \theta}$ (iii) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{4+5 \cos \theta}$ (iv) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{5+4 \cos \theta}$

25. $\int_0^a \sqrt{a^2-x^2} dx$ 26. $\int_0^1 \frac{3x\, dx}{4-x^2}$ 27. $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

28. (*a*) $\int_0^1 \sqrt{4-3x} dx$ (*b*) $\int_0^1 x^3 \sqrt{1+3x^4 dx}$

29. $\int_1^2 \left(\frac{x^2-1}{x^2}\right) e^{x+\frac{1}{x}} dx$ 30. $\int_0^{2a} \sqrt{2x-x^2} dx$

31. $\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$ 32. $\int_0^1 \frac{dx}{(x+2)(x^2+1)}$

33. $\int_0^2 \frac{dx}{(x+2)\sqrt{x+1}}$ 34. $\int_0^1 \frac{dx}{(x+2)(x^2+1)}$

35. $\int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$ [C.U.] 36. $\int_1^{e^2} \frac{dx}{x(1+\log x)^2}$

37. $\int_0^3 \frac{x\, dx}{\sqrt{x+1}+\sqrt{5x+1}}$ 38. $\int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}} = \frac{\pi}{3}$

39. $\int_0^1 x^{\frac{3}{2}} (1-x)^{\frac{1}{2}} dx$ 40. $\int_{-1}^1 \frac{x^2-1}{(x^2+1)^2} dx$

41. $\int_0^1 x \log (1+\frac{1}{2}x)dx$ **42.** $\int_0^1 \frac{x\, dx}{(1+x^2)\sqrt{1-x^2}}$

43. $\int_1^{\sqrt{2}} \frac{x^2+1}{x^4+1}dx$ **44.** $\int_0^1 \frac{dx}{1-x+x^2}$

45. $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}dx$ **46.** $\int_2^3 \frac{dx}{\sqrt{(x-1)(5-x)}}$

Show that (Ex. 47—55),

47. $\int_a^b \frac{\log x}{x}dx=\frac{1}{2}\log\left(\frac{b}{a}\right)\log (ab)$ **48.** $\int_1^5 \sqrt{(x-1)(5-x)}=2\pi$

49. $\int_0^{\log 2} \frac{e^x\, dx}{1+e^x}=\log \frac{3}{2}$ **50.** $\int_0^{\frac{\pi}{2}} \cos^3 x \sqrt[4]{\sin x}\; dx=\frac{32}{65}$

51. $\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}=\frac{1}{2}\log (2+\sqrt{3})$ [C. U]

52. $\int_0^a \frac{a^2-x^2}{(a^2+x^2)^2}dx=\frac{1}{2a}$

53. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+4\cot^2\theta}=\frac{\pi}{6}$ **54.** $\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\cos\alpha\cos\theta}=\frac{\alpha}{\sin\alpha}$

55. $\int_{\frac{1}{2}}^2 (4x^2+1)dx=\int_{\frac{1}{2}}^2 \frac{8}{x^2}dx$ [C. U.]

56. If $\int_0^a \frac{dx}{\sqrt{x+a}+\sqrt{x}}=\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^2\theta}d\theta$

find the value of a. [State Council (W. B. 1987]

Prove that (57—60)

57. $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}=\frac{\pi}{2ab}$ ($a, b>0$)

58. $\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x\, dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}=\frac{\pi}{4ab^2(a+b)}$ ($a, b>0$)

59. $\int_0^{\frac{\pi}{4}} \frac{\sin 2x dx}{\sin^4 x+\cos^4 x}=\frac{\pi}{4}$

60. $\int_0^{\frac{\pi}{4}} \frac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx=\frac{1}{6}$

§5·3. Definition of definite integral as the limit of the sum of a special class of series :

In § 5·1 definite integral of a function has been defined as an area. In this article we shall give a more generalised definition of definite integral. In the next article it will be shown that these definitions are consistent with each other.

Bounded Function : If in an interval $a \leqslant x \leqslant b$, a function $f(x)$ is defined and if there exist two finite numbers M and m such that for all values of $f(x)$ in the interval, $m \leqslant f(x) \leqslant M$ then the function $f(x)$ is said to be bounded in the interval $a \leqslant x \leqslant b$.

If corresponding to every value of x in an interval $a \leqslant x \leqslant b$, one can get one and only one value of $f(x)$, then the function $f(x)$ is said to be single valued in the interval.

Let a and b be two finite quantities and $b > a$. Let $f(x)$ be a bounded, single valued and continuous function of x defined in the interval $a \leqslant x \leqslant b$. Divide the interval $a \leqslant x \leqslant b$ into n equal subintervals each of length h,

$$a \leqslant x \leqslant a+h,\ a+h \leqslant x \leqslant a+2h,\ \cdots,\ a+(n-1)h \leqslant x \leqslant a+nh=b$$

$$\therefore\quad nh=b-a.$$

$$\underset{h\to 0}{Lt}\ h\{f(a)+f(a+h)+f(a+2h)+\cdots+f(a+\overline{n-1}h)\} \qquad (1)$$

or, $\underset{h\to 0}{Lt}\ h\sum\limits_{r=0}^{n-1} f(a+rh)$ is defined as the definite integral of the function $f(x)$ with respect to x and is written as $\int_a^b f(x)\,dx$.

$$\therefore\quad \int_a^b f(x)dx = \underset{h\to 0}{Lt}\sum_{r=0}^{n-1} h f(a+rh) \text{ where } nh=b-a$$

Note 1. It can be easily proved that

$$\underset{h\to 0}{Lt}\ h\{f(a)+f(a+h)+f(a+2h)+\cdots+f(a+\overline{n-1}h)\} \qquad \cdots(1)$$

$$= \underset{h\to 0}{Lt}\ h\{f(a+h)+f(a+2h)+\cdots+f(a+\overline{n-1}h+f(a+nh)\}$$

$$\therefore\quad \underset{h\to 0}{Lt}\ h\{f(a+h)+f(a+2h)+\cdots+f(a+\overline{n-1}h) + f(a+nh)\} \qquad \cdots(2)$$

$$=\int_a^b f(x)dx.$$

Any of the limits (1) and (2) can be taken as the definition of $\int_a^b f(x)\,dx$.

Note 2. As $nh=b-a$. $h=\dfrac{b-a}{n}$;

$\therefore$ when $h\to 0$, then $n\to\infty$.

§ 5·4. Geometrical Interpretation of the definition of definite Integral as the limit of a sum.

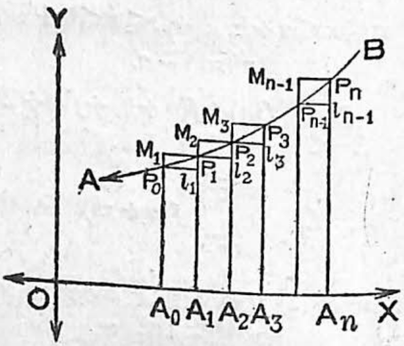
Let a function $f(x)$ be finite and continuous everywhere within the interval $a\leqslant x\leqslant b$ and the curve AB be the graph of the function $y=f(x)$. The ordinates A_0P_0 and A_nP_n at the points $A_0(a, 0)$ and $A_n(b, 0)$ intersect the curve at P_0 and P_n respectively.

Now, $A_0A_n=OA_n-OA_0=b-a$.

Divide the line segment A_0A_n into n equal parts each of length h.

$\therefore$ $nh=b-a$, or, $a+nh=b$.

Draw perpendicular to the x-axis at each of the points $(a+h, 0)$, $(a+2h, 0)\cdots\{a+(n-1)h, 0\}$ and complete the rectangles below and above the curve as shown in the figure.



Let A be the measure of the area enclosed by the x-axis, the curve $y=f(x)$ and the ordinates $x=a$ and $x=b$.

Let A_1 and A_2 be respectively the sum of the areas of the lower rectangles and the upper rectangles.

From figure it is evident that $A_1<A<A_2$ ...(1)

Now, $A_1=hf(a)+hf(a+h)+\cdots+hf\{a+(n-1)h\}$

$$=h\sum_{r=0}^{n-1} f(a+rh)$$

and $A_2=hf(a+h)+hf(a+2h)+\cdots+hf(a+nh)$

$$=h\sum_{r=0}^{n-1} f(a+rh)-hf(a)+hf(b)$$

Now, if the value of h be very small i.e., $h \to 0$, then n will be very large i.e., n will tend to infinity.

Hence when $n \to \infty$, then as $f(a)$ and $f(b)$ are finite $hf(a)$ and $hf(b)$ will both tend to zero.

$\therefore$ A_1 and A_2 will respectively approach

$$\underset{h \to 0}{Lt}\ h\sum_{r=0}^{n-1} f(a+rh) = \int_a^b f(x)dx$$

and $$\underset{h \to 0}{Lt}\ h\sum_{r=0}^{n-1} f(a+rh) = \int_a^b f(x)dx$$

Hence from (1) we get,

$$A = \int_a^b f(x)dx$$

Hence $\int_a^b f(x) \cdot dx$ is the measure of the area enclosed by the curve $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$.

Note. 1. From § 5·1 and this article, you can now understand that the two definitions of definite integral as the limit of a sum and as an area have same geometrical meaning. Hence the two definitions are consistent with each other. The definition of definite integral as the limit of a sum is a more generalised definition than that given as an area.

§ 5.5. Fnndamental Theorem of Integral Calculus.

At the very beginning we have indicated that the mutual relationship of indefinite and definite integrals are found in the Fundamental Theorem of Integral Calculus. We now state the theorem without proof. The proof is outside the scope of the syllabus.

Fundamental Theorem of Integral Calculus. If two functions $f(x)$ and $\phi(x)$ be such that the function $f(x)$ is integrable in the interval $a \leqslant x \leqslant b$ and $\phi'(x)=f(x)$ everywhere in the interval, then

$$\int_a^b \mathbf{f(x)dx} = \boldsymbol{\phi}\mathbf{(b)} - \boldsymbol{\phi}\mathbf{(a)}.$$

Properties of Definite Integrals :

(1) By definition, $\int_b^a f(x)dx = -\int_a^b f(x)dx.$

(2) $\int_a^b f(x)dx=\int_a^b f(z)\, dz.$

Proof : If $\int f(x)dx=\phi(x)$, then $\int f(z)dz=\phi(z)$.

Now, $\int_a^b f(x)dx=\phi(b)-\phi(a)$ and $\int_a^b f(z)dz=\phi(b)-\phi(a)$

$\therefore \int_a^b f(x)dx=\int_a^b f(z)dz.$

(3) If $a<c<b$

$\int_a^b f(x)\, dx=\int_a^c f(x)dx+\int_c^b f(x)dx.$

Proof : Let $\int f(x)dx=\phi(x)$

$\therefore \int_a^b f(x)dx=\phi(b)-\phi(a)$;

$\int_a^c f(x)dx=\phi(c)-\phi(a)$ and $\int_c^b f(x)dx=\phi(b)-\phi(c).$

$\therefore \int_a^b f(x)dx=\phi(b)-\phi(a)=\{\phi(c)-\phi(a)\}+\{\phi(b)-\phi(c)\}$

$=\int_a^c f(x)dx+\int_c^b f(x)dx.$

Cor. : If $a<c_1<c_2<\cdots\cdots<c_n<b$, then

$$\int_a^b f(x)dx=\int_a^{c_1} f(x)dx+\int_{c_1}^{c_2} f(x)dx+\cdots\cdots+\int_{c_{n-1}}^{c_n} f(x)dx+\int_{c_n}^b f(x)dx$$

(4) $\int_0^a f(x)dx=\int_0^a f(a-x)dx.$

Proof : Let $a-x=z \quad \therefore \quad -dx=dz.$

Again, when $x=0$ then $z=a$; when $x=a$ then, $z=0$.

$\therefore \int_0^a f(a-x)dx=-\int_a^0 f(z)dz$

$=\int_0^a f(z)\, dz$ [By (1)] $=\int_0^a f(x)dx.$ [By (2)].

(5) If $f(x)=f(a+x)$ then

$\int_0^{na} f(x)dx=n\int_0^a f(x)dx.$

Proof : Let $x=a+z \quad \therefore \quad dx=dz.$
when $x=a$, then $z=0$ when $x=2a$, then $z=a$.

$\therefore \quad \int_a^{2a} f(x)dx=\int_0^a f(a+z)dz=\int_0^a f(a+x)dx$ [By (2)]

$=\int_0^a f(x)dx \quad [\because \; f(a+x)=f(x)]$

Similarly, $\int_{2a}^{3a} f(x)dx=\int_a^{2a} f(x)dx=\int_0^a f(x)dx.$

$\int_{3a}^{4a} f(x)dx=\int_{2a}^{3a} f(x)dx=\int_0^a f(x)dx$

$\int_{(n-1)a}^{na} f(x)dx=\int_{(n-2)a}^{(n-1)a} f(x)dx=\int_0^a f(x)dx.$

Now, $\int_0^{na} f(x)dx=\int_0^a f(x)dx+\int_a^{2a} f(x)dx+$

$\int_{2a}^{3a} f(x)dx+\cdots+\int_{(n-1)a}^{na} f(x)dx$

$=\int_0^a f(x)\,dx+\int_0^a f(x)dx+\cdots+\int_0^a f(x)dx=n\int_0^a f(x)dx.$

(6) $\int_{-a}^a f(x)dx=\int_0^a\{f(x)+f(-x)\}dx$

Proof : $\int_{-a}^a f(x)dx=\int_{-a}^0 f(x)dx+\int_0^a f(x)dx.$

Now let $x=-z \quad \therefore \quad dx=-dz$
when $x=-a$, then $z=a$ and when $x=0$, then $z=0$

$\therefore \quad \int_{-a}^0 f(x)dx=\int_a^0 f(-z)(-dz)=-\int_a^0 f(-z)dz$

$=\int_0^a f(-z)dz=\int_0^a f(-x)\,dx$

$\therefore \quad \int_{-a}^a f(x)dx=\int_0^a f(-x)dx+\int_0^a f(x)dx=\int_0^a\left\{f(x)+f(-x)\right\}dx$

Corollary 1. when $f(x)$ is an even function, $f(-x)=f(x)$

$\therefore \quad \int_{-a}^a f(x)dx=\int_0^a\{f(x)+f(x)\}dx=2\int_0^a f(x)dx.$

2. when $f(x)$ is an odd function, then $f(-x)=-f(x)$

$\therefore \quad \int_{-a}^a f(x)dx=\int_0^a\left\{f(x)-f(x)\right\}dx=0.$

Example 5B

Example 1. From the definition of a definite integral, evaluate $\int_0^1 3dx$.

Let $f(x)=3$.

$$\therefore \int_0^1 3dx = \operatorname*{Lt}_{n\to\infty} h \sum_{r=1}^{n} f(rh);$$

where $nh=1=1-0$ and as $n\to\infty$, then $h\to 0$

$$= \operatorname*{Lt}_{h\to 0} h(3+3+3+\cdots+\text{to } n \text{ terms})$$

$$= \operatorname*{Lt}_{h\to 0} 3nh = \operatorname*{Lt}_{h\to 0} 3.1 = \operatorname*{Lt}_{h\to 0} 3 = 3.$$

Ex. 2. Evaluate $\int_0^1 (ax+b)\ dx$ by the method of summation.

[Joint Entrance, 1980]

Let $f(x)=ax+b$.

$$\text{Now, } \int_0^1 (ax+b)dx = \operatorname*{Lt}_{n\to\infty} h \sum_{r=1}^{n} f(rh);$$

[where, $nh=1=1-0$ and when $n\to\infty$, then $h\to 0$]

$$= \operatorname*{Lt}_{h\to 0} h\{(ah+b)+(2ah+b)+(3ah+b)+\cdots\cdots+(nah+b)\}$$

$$= \operatorname*{Lt}_{h\to 0}\left\{ah^2(1+2+3+\cdots+n)+bh.(1+1+\cdots+1)\right\}$$

$$= \operatorname*{Lt}_{h\to 0}\left\{ah^2\frac{n(n+1)}{2}+bnh\right\} = \operatorname*{Lt}_{h\to 0}\left\{\frac{an^2h^2}{2}\left(1+\frac{1}{n}\right)+b.1\right\}$$

$$= \operatorname*{Lt}_{n\to\infty}\left\{\frac{a}{2}\cdot 1^2\left(1+\frac{1}{n}\right)+b\right\} = \frac{a}{2}\cdot 1+b = \frac{a}{2}+b.$$

Ex. 3. Evaluate from the definition, $\int_0^1 x^2 dx$

[Joint Entrance, 1979, 1981 ; Tripura, 1981, 1986]

Let $f(x)=x^2$

$$\text{Now, } \int_0^1 x^2 dx = \operatorname*{Lt}_{n\to\infty} h\sum_{r=1}^{n} f(rh)$$ [where $nh=1-0=1$ and as $n\to\infty$, then $h\to 0$]

$$= \operatorname*{Lt}_{n\to\infty} h(h^2+2^2h^2+3^2h^2+\cdots+n^2h^2)$$

$$= \underset{n\to\infty}{\text{Lt}}\; h^3(1^2+2^2+3^2+\cdots+n^2) = \underset{n\to\infty}{\text{Lt}}\; \frac{h^3 n(n+1)(2n+1)}{6}$$

$$= \frac{1}{6} \underset{n\to\infty}{\text{Lt}}\; 2n^3h^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)$$

$$= \frac{1}{3} \cdot \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right) \quad [\because \; nh=1\,]$$

$$= \tfrac{1}{3}.1.1 = \tfrac{1}{3}.$$

Ex. 4. Use the method of summation to evaluate $\int_0^1 e^{2x}dx$

[Joint Entrance, 1982]

Let, $f(x)=e^{2x}$.

$$\therefore \int_0^1 e^{2x}\,dx = \int_0^1 f(dx)dx = \underset{n\to\infty}{\text{Lt}}\; h\sum_{r=1}^{n} f(rh)$$

[where $nh=1=1-0$ and $h\to 0$, as $n\to\infty$]

$$= \underset{n\to\infty}{\text{Lt}}\; h\left[e^{2h}+e^{2.2h}+e^{3.2h}+\cdots+e^{n.2h}\right]$$

$$= \underset{n\to\infty}{\text{Lt}}\; h\left\{\frac{e^{2h}(e^{2nh}-1)}{e^{2h}-1}\right\}$$ [using the formula for the sum of a G. P.]

$$= \underset{h\to 0}{\text{Lt}}\; h\left[\frac{e^{2h}(e^2-1)}{e^{2h}-1}\right]$$

$$= \left(e^2-1\right) \underset{h\to 0}{\text{Lt}}\; \frac{e^{2h}}{\frac{e^{2h}-1}{2h}.2} = \frac{e^2-1}{2}\; \frac{\underset{h\to 0}{\text{Lt}}\; e^{2h}}{\underset{h\to 0}{\text{Lt}}\; \frac{e^{2h}-1}{2h}}$$

$$= \frac{e^2-1}{2} \cdot \frac{1}{1} = \frac{e^2-1}{2}.$$

Ex. 5. Express $\int_a^b e^x\,dx$ as the limit of a sum and hence evaluate the integral.

[Joint Entrance, 1985]

Let $f(x)=e^x$.

Now, $$\int_a^b e^x dx = \int_a^b f(x)dx = \underset{n\to\infty}{\text{Lt}}\; h\sum_{r=1}^{n} f(a+rh).$$

[where $nh=b-a$ and $h\to 0$, as $n\to\infty$]

$$= \underset{n\to\infty}{\text{Lt}}\; h\left\{e^{a+h}+e^{a+2h}+e^{a+3h}+\cdots+e^{a+nh}\right\}$$

$$=\underset{n\to\infty}{\text{Lt}}\ he^a\left\{e^h+e^{2h}+e^{3h}+\cdots+e^{nh}\right\}$$

$$=\underset{n\to\infty}{\text{Lt}}\ he^a\left\{\frac{e^h(e^{nh}-1)}{e^h-1}\right\}$$ [using the formula for the sum of a G. P.]

$$=\underset{h\to 0}{\text{Lt}}\ \frac{e^h.e^a(e^{b-a}-1)}{\frac{e^h-1}{h}}$$

$$=(e^b-e^a)\frac{\underset{h\to 0}{\text{Lt}}\ e^h}{\underset{h\to 0}{\text{Lt}}\ \frac{e^h-1}{h}}=\frac{(e^b-e^a).1}{1}=e^b-e^a$$

Ex. 6. From the definition of definite integral as the limit of a sum evaluate $\int_0^1 2e^x dx.$ [H. S. 1984 ; 1986]

Let $f(x)=2e^x$.

Now, $\int_0^1 2e^x dx=\int_0^1 f(x)dx.$

$$=\underset{n\to\infty}{\text{Lt}}\ h\sum_{r=1}^{n} f(rh)$$ [where $nh=1=1-0$ and $h\to 0$ as $n\to\infty$,]

$$=\underset{n\to\infty}{\text{Lt}}\ h\left[2e^h+2e^{2h}+2e^{3h}+\cdots+2e^{nh}\right]$$

$$=\underset{n\to\infty}{\text{Lt}}\ 2h\left\{\frac{e^h(e^{nh}-1)}{e^h-1}\right\}$$ [using the formula for the sum of a G. P.]

$$=2\underset{h\to 0}{\text{Lt}}\ \frac{e^h(e-1)}{\frac{e^h-1}{h}}$$ [as $nh=1$, $e^{nh}=e^1=e$]

$$=2(e-1)\frac{\underset{h\to 0}{\text{Lt}}\ e^h}{\underset{h\to 0}{\text{Lt}}\ \frac{e^h-1}{h}}=2(e-1).\frac{1}{1}=2(e-1).$$

Ex. 7. Evaluate $\int_0^1 e^{-x}\, dx$ from the definition of definite integral. [H. S. 1982]

Let $f(x)=e^{-x}$

$\therefore\ \int_0^1 e^{-x}\, dx=\int_0^1 f(x)dx.$

$$=\underset{n\to\infty}{\text{Lt}}\ h\sum_{r=1}^{n} f(rh) \qquad [\text{ where } nh=1=1-0 \text{ and } h\to 0 \text{ as } n\to\infty\ ]$$

$$=\underset{n\to\infty}{\text{Lt}}\ h\left\{e^{-h}+e^{-2h}+e^{-3h}+\cdots+e^{-nh}\right\}$$

$$=\underset{h\to 0}{\text{Lt}}\ h\frac{e^{-h}(e^{-nh}-1)}{e^{-h}-1} \qquad [\text{ using the formula for the sum of a G. P. }]$$

$$=-\underset{h\to 0}{\text{Lt}}\ \frac{e^{-h}(e^{-1}-1)}{\dfrac{e^{-h}-1}{-h}}$$

$$=-(e^{-1}-1)\frac{\underset{h\to 0}{\text{Lt}}\ e^{-h}}{\underset{h\to 0}{\text{Lt}}\ \dfrac{e^{-h}-1}{-h}}=\left(1-\frac{1}{e}\right).\frac{1}{1}=1-\frac{1}{e}.$$

Ex. 8. Express $\int_1^2 x dx$ as the limit of a sum and find its value.

[Joint Entrance, 1983]

Let $f(x)=x \quad \therefore \int_1^2 x dx=\int_1^2 f(x)dx.$

$$=\underset{n\to\infty}{\text{Lt}}\ h\sum_{r=1}^{n} f(1+rh) \quad [\text{ where } nh=2-1=1, \text{ and } h\to 0 \text{ as } n\to\infty.]$$

$$=\underset{n\to\infty}{\text{Lt}}\ h\left\{(1+h)+(1+2h)+(1+3h)+\cdots+(1+nh)\right\}$$

$$=\underset{n\to\infty}{\text{Lt}}\ h\left\{n+h(1+2+3+\cdots+n)\right\}$$

$$=\underset{n\to\infty}{\text{Lt}}\ \left\{nh+h^2\ \frac{n(n+1)}{2}\right\}=\underset{n\to\infty}{\text{Lt}}\ \left\{nh+\frac{n^2h^2(1+\frac{1}{n})}{2}\right\}$$

$$=\underset{n\to\infty}{\text{Lt}}\ \left\{1+\frac{(1+\frac{1}{n})}{2}\right\}=1+\tfrac{1}{2}=\tfrac{3}{2}.$$

Ex, 9. Evalute $\int_1^2 5x^2 dx$ from the definition of a definite integral as the limit of a sum. [H. S. 1983]

Let, $f(x)=5x^2 \quad \therefore \int_1^2 5x^2 dx=\int_1^2 f(x)dx$

$$=\underset{n\to\infty}{\text{Lt}}\ h\sum_{r=1}^{n} f(1+rh) \qquad [\text{ where } nh=2-1=1 \text{ and } h\to 0 \text{ as } n\to\infty\ ]$$

$$=\mathop{\text{Lt}}_{n\to\infty} h\sum_{r=1}^{n} 5(1+rh)^2=\mathop{\text{Lt}}_{n\to\infty} h\sum_{r=1}^{n} 5(1+2rh+r^2h^2)$$

$$=\mathop{\text{Lt}}_{n\to\infty} 5h\{n+2h(1+2+\cdots+n)+h^2(1^2+2^2+\cdots+n^2)\}$$

$$=5\mathop{\text{Lt}}_{n\to\infty} h\left\{n+2h\frac{n(n+1)}{2}+h^2\frac{n(n+1)(2n+1)}{6}\right\}$$

$$=5\mathop{\text{Lt}}_{n\to\infty} \left\{nh+n^2h^2(1+\tfrac{1}{n})+\frac{h^3n^3(1+\frac{1}{n})2(1+\frac{1}{2n})}{6}\right\}$$

$$=5\mathop{\text{Lt}}_{n\to\infty} \left\{1+1(1+\tfrac{1}{n})+\frac{1(1+{}^1)(1+\frac{1}{2n})}{3}\right\}$$

$$=5\{1+1+\tfrac{1}{3}\}=5.\tfrac{7}{3}=\tfrac{35}{3}.$$

Ex. 10. Show that $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx.$ [Tripura, 1979]

Let $a+b-x=z$ $\therefore$ $-dx=dz$ or, $dx=-dz.$

when $x=a$, then $z=b$ and when $x=b$, then $z=a.$

$$\therefore \int_a^b f(a+b-x)dx=\int_b^a f(z)(-dz)=-\int_b^a f(z)dz$$

$$=\int_a^b f(z)dz=\int_a^b f(x)dx.$$

Ex. 11. Evaluate : $\int_0^\pi \sin^3x \cos^3x\, dx.$ [H. S. 1980]

$$\text{Let, } I=\int_0^\pi \sin^3x\cos^3x\,dx=\int_0^\pi \sin^3(\pi-x)\cos^3(\pi-x)dx$$

$$=-\int_0^\pi \sin^3x\cos^3x\,dx \quad [\because \cos(\pi-x)=-\cos x]$$

$$=-I$$

$$\therefore 2I=0 \text{ or, } I=0 \quad \therefore \int_0^\pi \sin^3x\cos^3x\,dx=0.$$

Ex. 12. Show that $\int_0^{\frac{\pi}{2}} \log\sin x\, dx=\int_0^{\frac{\pi}{2}} \log\cos x\, dx=\frac{\pi}{2}\log\frac{1}{2}.$

$$\text{Let, } I=\int_0^{\frac{\pi}{2}} \log\sin x\,dx=\int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2}-x\right)dx=\int_0^{\frac{\pi}{2}} \log\cos x\,dx$$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\therefore \quad 2I = I + I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx + \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) dx = \int_0^{\frac{\pi}{2}} \log (\sin x \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \frac{\sin 2x}{2} dx = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} 2 dx$$

Now let $2x = z$ $\quad \therefore \quad 2dx = dz$ or, $dx = \frac{dz}{2}$

when $x = 0$, then $z = 0$; when $x = \frac{\pi}{2}$, then $z = \pi$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \log \sin 2x dx = \int_0^{\pi} \sin z \frac{dz}{2}$$

$$= \tfrac{1}{2} \int_0^{\pi} \log \sin z \, dz = \tfrac{1}{2} . 2 \int_0^{\frac{\pi}{2}} \log \sin z dz$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x dx = I \qquad \therefore \quad 2I = I - \int_0^{\frac{\pi}{2}} \log 2 dx$$

$$= I - \log 2 \Big[x \Big]_0^{\frac{\pi}{2}} = I - \frac{\pi}{2} \log 2. = I + \frac{\pi}{2} \log \tfrac{1}{2} \qquad \therefore \quad I = \frac{\pi}{2} \log \tfrac{1}{2}$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \log \sin x dx = \int_0^{\frac{\pi}{2}} \log \cos x dx = \frac{\pi}{2} \log \tfrac{1}{2}.$$

Ex. 13. Using the formula $\int_0^a f(x) dx = \int_0^a f(a - x) dx$

Prove that $\int_0^{\frac{\pi}{2}} \log \tan x dx = 0$ [H. S. 1985]

Let $I = \int_0^{\frac{\pi}{2}} \log \tan x dx$

$$= \int_0^{\frac{\pi}{2}} \log \tan \left(\frac{\pi}{2} - x \right) dx = \int_0^{\frac{\pi}{2}} \log \cot x dx$$

$$\therefore \quad 2I = I + I = \int_0^{\frac{\pi}{2}} \log \tan x dx + \int_0^{\frac{\pi}{2}} \log \cot x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \tan x + \log \cot x) \, dx = \int_0^{\frac{\pi}{2}} \log (\tan x \cot x) dx$$

$$=\int_0^{\frac{\pi}{2}} \log 1\, dx=\log 1\int_0^{\frac{\pi}{2}} dx=0.\int_0^{\frac{\pi}{2}} dx=0.$$

$$\therefore\quad I=\int_0^{\frac{\pi}{2}} \log \tan x dx=0.$$

Ex. 14. Evaluate : $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\sin x+\cos x}$ [Tripura, '87]

$$\text{Let } I=\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\sin x+\cos x}\, dx=\int_0^{\frac{\pi}{2}} \frac{\cos^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\, dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\, dx}{\cos x+\sin x} dx$$

$$\therefore\quad 2I=I+I=\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\sin x+\cos x}+\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\, dx}{\sin x+\cos x}.$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{\cos^2 x}{\sin x+\cos x}+\frac{\sin^2 x}{\sin x+\cos x}\right)dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{\cos^2 x+\sin^2 x}{\sin x+\cos x} dx=\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x+\cos x}$$

$$=\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)}$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\cos\frac{\pi}{4}\sin x+\sin\frac{\pi}{4}\cos x}=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \operatorname{cosec}\left(\frac{\pi}{4}+x\right)dx$$

$$=\frac{1}{\sqrt{2}}\left[\log\left\{\operatorname{cosec}\left(\frac{\pi}{4}+x\right)-\cot\left(\frac{\pi}{4}+x\right)\right\}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{\sqrt{2}}\left[\log\left(\operatorname{cosec}\frac{3\pi}{4}-\cot\frac{3\pi}{4}\right)-\log\left(\operatorname{cosec}\frac{\pi}{4}-\cot\frac{\pi}{4}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

So, $2I=\frac{1}{\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{1}=\frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$

$\therefore\quad I=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$.

Ex. 15. Evaluate : $\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x}\ (0<\alpha<\pi)$ [I. I. T. 1986]

Let $I=\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x}$

$$=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos\alpha\sin(\pi-x)}=\int_0^{\pi}\frac{(\pi-x)dx}{1+\cos\alpha\sin x}$$

$$=\int_0^{\pi}\frac{\pi dx}{1+\cos\alpha\sin x}-\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x}$$

$$=\pi\int_0^{\pi}\frac{dx}{1+\left(\frac{2\cos\alpha\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}-I$$

or, $2I=\pi.2\int_0^{\frac{\pi}{2}}\frac{\sec^2\frac{x}{2}dx}{1+\tan^2\frac{x}{2}+2\cos\alpha\tan\frac{x}{2}}$

$$=4\pi\int_0^1\frac{dt}{1+t^2+2t\cos\alpha}$$

[$\tan\frac{x}{2}=t$ (say) $\therefore\ \frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$ and when $x=0$, then $t=\tan 0=0$; when $x=\frac{\pi}{2}$, then $t=\tan\frac{\pi}{4}=1$]

$$=4\pi\int_0^1\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}=\frac{4\pi}{\sin\alpha}\left[\tan^{-1}\frac{t+\cos\alpha}{\sin\alpha}\right]_0^1$$

$$=\frac{4\pi}{\sin\alpha}\left[\tan^{-1}\frac{1+\cos\alpha}{\sin\alpha}-\tan^{-1}\cot\alpha\right]$$

$$=\frac{4\pi}{\sin\alpha}\left[\tan^{-1}\frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}-\tan^{-1}\tan\left(\frac{\pi}{2}-\alpha\right)\right]$$

$$=\frac{4\pi}{\sin\alpha}\left[\tan^{-1}\cot\frac{\alpha}{2}-\frac{\pi}{2}+\alpha\right]=\frac{4\pi}{\sin\alpha}\left[\tan^{-1}\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)-\frac{\pi}{2}+\alpha\right]$$

$$=\frac{4\pi}{\sin\alpha}\left[\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\pi}{2}+\alpha\right]=\frac{2\pi\alpha}{\sin\alpha}=2\pi\alpha\operatorname{cosec}\alpha.$$

$$\therefore \quad I=\pi\alpha \operatorname{cosec}\alpha \text{ or, } \int_0^{\pi} \frac{x\,dx}{1+\cos\alpha \sin x}=\pi\alpha \operatorname{cosec}\alpha.$$

Ex. 16. Show that $\int_0^{\pi} xf(\sin x)dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$ [I.I.T. 1982]

$$\int_0^{\pi} xf(\sin x)dx=\int_0^{\pi} (\pi-x)f\{\sin(\pi-x)\}dx=\int_0^{\pi}(\pi-x)f(\sin x)dx$$

$$=\pi\int_0^{\pi} f(\sin x)dx-\int_0^{\pi} xf(\sin x)dx$$

$$\text{or, } 2\int_0^{\pi} xf(\sin x)\,dx=\pi\int_0^{\pi} f(\sin x)dx$$

$$\text{or, } \int_0^{\pi} xf(\sin x)dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$$

Ex. 17. Evaluate : $\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cot x}\,dx}{\sqrt{\cot x}+\sqrt{\tan x}}$ [I. I. T. 1983]

$$\text{Let } I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cot x}\,dx}{\sqrt{\cot x}+\sqrt{\tan x}}$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}dx}{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\tan x}\,dx}{\sqrt{\tan x}+\sqrt{\cot x}}$$

$$\therefore \quad 2I=I+I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cot x}\,dx}{\sqrt{\cot x}+\sqrt{\tan x}}+\int_0^{\frac{\pi}{2}}\frac{\sqrt{\tan x}\,dx}{\sqrt{\tan x}+\sqrt{\cot x}}$$

$$=\int_0^{\frac{\pi}{2}}\left\{\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}+\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}\right\}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cot x}+\sqrt{\tan x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx=\int_0^{\frac{\pi}{2}}dx=\Big[x\Big]_0^{\frac{\pi}{2}}=\frac{\pi}{2} \quad \therefore \quad I=\frac{\pi}{4}.$$

Ex. 18. $f(x)=f(a+x)$ prove that

$\int_a^{a+t} f(x)\,dx$, is independent of a. [Joint Entrance, 1988]

$f(a+x)=f(x)$; $\therefore$ $f(a+h)=f(h), f(a+2h)=f(2h),\cdots f(a+t)=f(t).$

Now, $\int_a^{a+t} f(x)dx = \underset{h\to 0}{\text{Lt}}\; h\{f(a+h)+f(a+2h)+\cdots f(a+nh)\}$

where $a+nh=a+t$ or, $nh=t=t-0$

$= \underset{h\to 0}{\text{Lt}}\; h\{f(h)+f(2h)+\cdots+f(nh)\}$

$= \int_0^t f(x)dx$ which is independent of a.

Ex. 19. Evaluate :

(i) $\int_0^{\frac{\pi}{2}} \sin^9 x dx$ (ii) $\int_0^{\pi} (1-\cos x)^2 dx$ (iii) $\int_0^1 x(1-x)^{\frac{3}{2}} dx$

(i) $\int \sin^9 x dx = \int \sin^8 x \sin x dx$

$= \int (1-\cos^2 x)^4 \sin x dx$

Now, let $\cos x = z$ $\therefore$ $-\sin x dx = dz$

or, $\sin x dx = -dz$.

Again when $x=0$, then $z=\cos 0=1$

and when $x=\frac{\pi}{2}$, then $z=\cos\frac{\pi}{2}=0$

$\therefore$ $\int_0^{\frac{\pi}{2}} \sin^9 x dx = -\int_1^0 (1-z^2)^4\ (dz)$

$= \int_0^1 (1-4z^2+6z^4-4z^6+z^8)dz$

$= \left[z-\frac{4}{3}z^3+\frac{6}{5}z^5-\frac{4}{7}z^7+\frac{z^9}{9} \right]_0^1$

$= 1-\frac{4}{3}+\frac{6}{5}-\frac{4}{7}+\frac{1}{9} = \frac{315-420+378-180+35}{315} = \frac{128}{315}$.

(ii) $\int_0^{\pi} (1-\cos x)^2 dx = \int_0^{\pi} (1-2\cos x+\cos^2 x)\, dx$

$= \int_0^{\pi} \{1-2\cos x+\frac{1}{2}(1+\cos 2x)\}dx$

$= \left[x-2\sin x+\frac{1}{2}x+\frac{\sin 2x}{4} \right]_0^{\pi}$

$= \left[\frac{3}{2}\pi-2\sin\pi+\frac{\sin 2\pi}{4} \right] = \frac{3}{2}\pi$

(iii) Let $x=\sin^2\theta$ $\therefore$ $dx=2\sin\theta\cos\theta\, d\theta$

when $x=0$, then $\sin^2\theta=0$ $\therefore$ $\theta=0$

when $x=1$, then $\sin^2\theta=1 \quad \therefore \quad \theta=\frac{\pi}{2}$

$$\therefore \quad \text{Given integral}=\int_0^{\frac{\pi}{2}} \sin^2\theta\,(1-\sin^2\theta)^{\frac{3}{2}}\, 2\sin\theta\cos\theta\, d\theta$$

$$=2\int_0^{\frac{\pi}{2}} \sin^3\theta\, \cos^4\theta\, d\theta=2\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^4\theta\sin\theta\, d\theta$$

$$=2\int_0^1 (1-z^2)z^4dz$$

[where $z=\cos\theta$ and when $\theta=0$, then $z=1$; when $\theta=\frac{\pi}{2}$, then $z=0$]

$$=2\left[\frac{z^5}{5}-\frac{z^7}{7}\right]_0^1=2(\tfrac{1}{5}-\tfrac{1}{7})=\tfrac{4}{35}$$

Ex. 20. Evaluate the following :

(i) $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{8n}\right]$ [Tripura, 1980, '86]

(ii) $\underset{n\to\infty}{\text{Lt}} \left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots+\frac{1}{2n}\right]$

[H. S. 1986 ; Tripura, 1982, '87]

(iii) $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right]$ [H. S. 1983]

(iv) $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots+\frac{1}{n}\right]$ [Tripura, 1984]

(v) $\underset{n\to\infty}{\text{Lt}} \frac{1}{\sqrt{n}}\left[1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\right]$ [Joint Entrance, 1986]

(vi) $\underset{n\to\infty}{\text{Lt}} \frac{1^k+2^k+\cdots+n^k}{n^{k+1}}\, k>0$ [Joint Entrance, 1987]

(vii) $\underset{n\to\infty}{\text{Lt}} \frac{1}{n}\left[\tan\left(\frac{\pi}{4n}\right)+\tan\left(\frac{2\pi}{4n}\right)+\tan\left(\frac{3\pi}{4n}\right)+\cdots+\tan\left(\frac{n\pi}{4n}\right)\right]$

[Joint Entrance, 1987]

(viii) $\underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\cdots+\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$

[Joint Entrance, 1988]

(i) $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{8n}\right]$

$$=\underset{n\to\infty}{\mathrm{Lt}}\left[\frac{1}{n}+\frac{\frac{1}{n}}{\frac{1}{n^3}(n+1)^3}+\frac{\frac{1}{n}}{\frac{1}{n^3}(n+2)^3}+\cdots+\frac{\frac{1}{n}}{\frac{1}{n^3}(n+n)^3}\right]$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\ \frac{1}{n}\left[\frac{1}{1^3}+\frac{1}{\left(1+\frac{1}{n}\right)^3}+\frac{1}{\left(1+\frac{2}{n}\right)^3}+\cdots+\frac{1}{\left(1+\frac{n}{n}\right)^3}\right]$$

$$=\underset{h\to0}{\mathrm{Lt}}\left[h\left\{\frac{1}{(1+0.h)^3}+\frac{1}{(1+1.h)^3}+\frac{1}{(1+2.h)^3}+\cdots+\frac{1}{(1+nh)^3}\right\}\right]$$

[where $h=\frac{1}{n}$ $\therefore$ $nh=1=1-0$ and when $n\to\infty$, then $h\to0$]

$$=\underset{h\to0}{\mathrm{Lt}}\ h\sum_{r=0}^{n}\frac{1}{(1+rh)^3}=\int_0^1\frac{1}{(1+x)^3}dx$$

$$=\left[-\frac{1}{2(1+x)^2}\right]_0^1=\left[-\frac{1}{8}-\left(\frac{-1}{2}\right)\right]=-\tfrac{1}{8}+\tfrac{1}{2}=\tfrac{3}{8}.$$

(ii) $\underset{n\to\infty}{\mathrm{Lt}}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots+\frac{1}{2n}\right]$

$$=\underset{n\to\infty}{\mathrm{Lt}}\left[\frac{\frac{n}{n^2}}{1+\frac{1^2}{n^2}}+\frac{\frac{n}{n^2}}{1+\frac{2^2}{n^2}}+\frac{\frac{n}{n^2}}{1+\frac{3^2}{n^2}}+\cdots+\frac{\frac{1}{n}}{1+\frac{n^2}{n^2}}\right]$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\ \frac{1}{n}\left[\frac{1}{1+\frac{1^2}{n^2}}+\frac{1}{1+\frac{2^2}{n^2}}+\frac{1}{1+\frac{3^2}{n^2}}+\cdots+\frac{1}{1+\frac{n^2}{n^2}}\right]$$

$$=\underset{h\to0}{\mathrm{Lt}}\ h\left[\frac{1}{1+1^2h^2}+\frac{1}{1+2^2h^2}+\frac{1}{1+3^2h^2}+\cdots+\frac{1}{1+n^2h^2}\right]$$

[where $h=\frac{1}{n}$ or, $nh=1=1-0$ and as $n\to\infty$, then $h\to0$]

$$=\underset{h\to0}{\mathrm{Lt}}\ h\sum_{r=1}^{n}\frac{1}{1+r^2h^2}=\int_0^1\frac{1}{1+x^2}dx=\left[\tan^{-1}x\right]_0^1$$

$$=\tan^{-1}1-\tan^{-1}0=\frac{\pi}{4}.$$

(iii) $\underset{n\to\infty}{\mathrm{Lt}}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n}\right]$

$$=\underset{n\to\infty}{\text{Lt}}\left[\frac{\frac{1}{n}}{\frac{n}{n}+\frac{1}{n}}+\frac{\frac{1}{n}}{\frac{n}{n}+\frac{2}{n}}+\frac{\frac{1}{n}}{\frac{n}{n}+\frac{3}{n}}+\cdots+\frac{\frac{1}{n}}{\frac{n}{n}+\frac{n}{n}}\right]$$

$$=\underset{h\to0}{\text{Lt}}\ h\left[\frac{1}{1+h}+\frac{1}{1+2h}+\frac{1}{1+3h}+\cdots+\frac{1}{1+nh}\right]$$

[where $h=\frac{1}{n}$ or, $nh=1=1-0$ and $h\to0$ as $n\to\infty$]

$$\Rightarrow\underset{h\to0}{\text{Lt}}\ h\sum_{r=1}^{n}\frac{1}{1+rh}=\int_0^1\frac{dx}{1+x}=\Big[\log(1+x)\Big]_0^1$$

$$=\log 2-\log 1=\log 2.$$

(iv) $$\underset{n\to\infty}{\text{Lt}}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots+\frac{1}{n}\right]$$

$$=\underset{n\to\infty}{\text{Lt}}\left[\frac{\frac{1}{n}}{\sqrt{\frac{2}{n}-\frac{1^2}{n^2}}}+\frac{\frac{1}{n}}{\sqrt{\frac{4}{n}-\frac{2^2}{n^2}}}+\cdots+\frac{\frac{1}{n}}{\sqrt{\frac{2n}{n}-\frac{n^2}{n^2}}}\right]$$

$$=\underset{h\to0}{\text{Lt}}\ h\left[\frac{1}{\sqrt{2h-1^2h^2}}+\frac{1}{\sqrt{2.2h-2^2h^2}}+\cdots+\frac{1}{\sqrt{2.nh-n^2h^2}}\right]$$

$\Big[$ where $h=\frac{1}{n}$ or, $nh=1=1-0$ and $h\to0$, as $n\to\infty$ $\Big]$

$$=\underset{h\to0}{\text{Lt}}\ h\sum_{r=1}^{n}\frac{1}{\sqrt{2rh-r^2h^2}}=\int_0^1\frac{dx}{\sqrt{2x-x^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{1-(x-1)^2}}=\Big[\sin^{-1}(x-1)\Big]_0^1$$

$$=\sin^{-1}0-\{\sin^{-1}(-1)\}=0-\left(-\frac{\pi}{2}\right)=\frac{\pi}{2}.$$

(v) $$\underset{n\to\infty}{\text{Lt}}\ \frac{1}{\sqrt{n}}\left[1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\right]$$

$$=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\left[\sqrt{\frac{n}{1}}+\sqrt{\frac{n}{2}}+\sqrt{\frac{n}{2}}+\cdots+\sqrt{\frac{n}{n}}\right]$$

$$=\underset{h\to0}{\text{Lt}}\ h\left[\sqrt{\frac{1}{1.h}}+\sqrt{\frac{1}{2.h}}+\sqrt{\frac{1}{3.h}}+\cdots+\sqrt{\frac{1}{nh}}\right]$$

$\Big[$ where $h=\frac{1}{n}$ or, $nh=1=1-0$ and $h\to0$, as $n\to\infty$ $\Big]$

$$=\underset{h\to 0}{\text{Lt}}\ h\sum_{r=1}^{n}\frac{1}{\sqrt{rh}}=\int_0^1\frac{dx}{\sqrt{x}}=\Big[2\sqrt{x}\Big]_0^1=2.$$

(vi) $\underset{n\to\infty}{\text{Lt}}\ \dfrac{1^k+2^k+\cdots\cdots+n^k}{n^{k+1}}$

$$=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\left\{\left(\frac{1}{n}\right)^k+\left(\frac{2}{n}\right)^k+\cdots\cdots\left(\frac{n}{n}\right)^k\right\}$$

$$=\underset{h\to 0}{\text{Lt}}\ h[(1h)\ +(2h)^k+\cdots\cdots+(nh)^k]$$

$\left(\text{Where } h=\dfrac{1}{n}\ \text{ or } nh=1=1-0 \text{ and } \infty\to 0 \text{ as } n\to\infty\right)$

$$=\underset{h\to 0}{\text{Lt}}\ h\sum_{r=1}^{n}(rh)^k=\int_0^1 x^k=\left[\frac{x^{k+1}}{k+1}\right]_0^1=\frac{1}{k+1}.$$

(vii) $\underset{n\to\infty}{\text{Lt}}\ \dfrac{1}{n}\left[\tan\left(\dfrac{\pi}{4n}\right)+\tan\left(\dfrac{2\pi}{4n}\right)+\tan\left(\dfrac{3\pi}{4n}\right)+\cdots\cdots\tan\left(\dfrac{n\pi}{4n}\right)\right]$

$$=\underset{h\to 0}{\text{Lt}}\ h\left[\tan\left(\frac{\pi}{4}h\right)+\tan\left(\frac{\pi}{4}.\,2h\right)+\tan\left(\frac{\pi}{4}.\,3h\right)+\cdots+t\left(\frac{\pi}{4}.\,nh\right)\right]$$

$\left(\text{where } h=\dfrac{1}{n}\ \text{ or, } nh=1=1-0 \text{ and } h\to 0, \text{ as } n\to\infty\right)$

$$=\underset{h\to 0}{\text{Lt}}\ h\sum_{r=1}^{n}\tan\left(\frac{\pi}{4}rh\right)=\int_0^1\tan\left(\frac{\pi}{4}x\right)dx$$

$$=\frac{4}{\pi}\left[\log\left(\sec\frac{\pi}{4}x\right)\right]_0^1=\frac{4}{\pi}\left[\log\sec\frac{\pi}{4}-\log\sec 0\right]$$

$$=\frac{4}{\pi}[\log\ \sqrt{2}-\log 1]=\frac{4}{\pi}.\ \tfrac{1}{2}\log 2=\frac{2}{\pi}\log 2.$$

(vii) Let $\text{A}=\underset{n\to\infty}{\text{Lt}}\ \left[\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)\cdots\cdots\left(1+\dfrac{n}{n}\right)\right]^{\frac{1}{n}}$

$\therefore\ \log\text{A}=\underset{n\leftarrow\infty}{\text{Lt}}\ \dfrac{1}{n}\left[\log\left(1+\dfrac{1}{n}\right)+\log\left(1+\dfrac{2}{n}\right)+\cdots+\log\left(1+\dfrac{n}{n}\right)\right]$

$$=\underset{h\to 0}{\text{Lt}}\ h\,[\log(1+h)+\log(1+2h)+\cdots\cdots+\log(1+nh)]$$

$\left(\text{where } h=\dfrac{1}{n}\ \text{ or, } nh=1=1-0 \text{ and } h\to 0, \text{ as } n\to\infty\right)$

$$=\underset{h\to 0}{\text{Lt}}\ h\sum_{r=1}^{n}\log(1+rh)=\int_0^1\log(1+x)\ dx$$

$$=\int_0^1 1.\log(1+x)\,dx=\Big[\log(1+x)\int 1\,dx\Big]_0^1-\int_0^1\frac{1}{1+x}\int 1\,dx\,dx$$

$$=\Big[x\log(1+x)\Big]_0^1-\int_0^1\frac{x}{1+x}dx.$$

$$=\log 2-0-\int_0^1\left(1-\frac{1}{1+x}\right)dx=\log 2-\Big[\,x-\log(1+x)\Big]_0^1$$

$$=\log 2-(1-\log 2)=\log 4-\log e=\log\frac{4}{e}.$$

$\therefore\quad A=\frac{4}{e}.$ So the required limit$=\frac{4}{e}.$

Ex. 21. Show that $\underset{n\to\infty}{\mathrm{Lt}}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{6n}\right)=\log 6$

[I. I. T. 1981]

Given limit$=\underset{n\to\infty}{\mathrm{Lt}}\left(\frac{\frac{1}{n}}{1+\frac{1}{n}}+\frac{\frac{1}{n}}{1+\frac{2}{n}}+\cdots\cdots+\frac{\frac{1}{n}}{1+\frac{5n}{n}}\right)$

$$=\underset{h\to0}{\mathrm{Lt}}\left(\frac{h}{1+h}+\frac{h}{1+2h}+\cdots\cdots+\frac{h}{1+5nh}\right)$$

[Where $h=\frac{1}{n}$ and $h\to0$ when $n\to\infty$: $\therefore\quad nh=1$]

$$=\underset{h\to0}{\mathrm{Lt}}\ h\left(\frac{1}{1+h}+\frac{1}{1+2n}+\cdots\cdots+\frac{1}{1+mh}\right)$$

$=\underset{h\to0}{\mathrm{Lt}}\ h\sum_{r=1}^{m}\frac{1}{1+rh}$ [where $m=5n$ and $mh=5nh=5.1=5-0$]

$$=\int_0^5\frac{dx}{1+x}=\Big[\,\log(1+x)\Big]_0^5=\log 6-\log 1=\log 6.$$

Exercise 5B

1. Evaluate the values of the following definite integrals from the first principle (definition)

(i) $\int_a^b x^2dx$ (ii) $\int_0^1 3x^2dx$ (iii) $\int_0^1 xdx$ (iv) $\int_0^1 x^3dx$

(v) $\int_0^1\frac{2x+1}{4}dx$ (vi) $\int_{-1}^1\frac{2x+3}{4}dx$ (vii) $\int_a^b e^{mx}dx$

(viii) $\int_0^1 e^{mx}dx$ (ix) $\int_0^1(ax^2+bx+c)dx$

2. Evaluate :

(i) $\int_0^1 x^6 \sqrt{(1-x^2)}dx$ (ii) $\int_0^\pi \cos^7 xdx$ (iii) $\int_0^{\frac{\pi}{2}} \cos^6 xdx$

(iv) $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^8 xdx$ (v) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 xdx$

3. Prove that :

(i) $\int_0^1 x^3(1-x)^3 dx = \frac{1}{140}$ (ii) $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}dx = \frac{3}{16}\pi a^4$

(iii) $\int_0^{\frac{\pi}{2}} \sin^6\theta \cos^3\theta d\theta = \frac{2}{63}$.

4. Prove that : $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}dx = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$

5. Prove that : $\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8} \log 2.$

6. Prove that (i) $\int_{a-\alpha}^{b-\alpha} f(x+\alpha)dx = \int_a^b f(x)dx$

(ii) $\int_{na}^{nb} f(x)dx = n\int_a^b f(x)dx.$

7. Prove that (i) $\int_0^{\frac{\pi}{2}} \frac{x \sin x\, dx}{1+\cos^2 x} = \frac{\pi^2}{4}$

(ii) $\int_0^\pi \frac{x \tan x\, dx}{\sec x + \tan x} = \frac{1}{2}\pi(\pi-2)$

(iii) $\int_0^\pi \frac{xdx}{a^2\sin^2 x + b^2\cos^2 x}dx = \frac{\pi^2}{2ab}$ $(a, b) > 0$

8. Show that $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}dx = \frac{\pi}{2}$.

9. Prove that, $\int_0^{\frac{\pi}{2}} (a \cos^2 x + b \sin^2 x)dx = (a+b)\frac{\pi}{4}$.

10. Evaluate : (*a*) $\int_0^{\frac{\pi}{2}} \frac{\sin\theta d\theta}{\sin\theta + \cos\theta}$ (*b*) $\int_0^{\frac{\pi}{2}} \frac{\sin^3 xdx}{\sin^3 x + \cos^3 x}$.

11. Prove that : (i) $\int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x + \sin 2x} = \frac{\pi}{16} \log 2$

(ii) $\int_0^\pi \frac{xdx}{(a^2\cos^2 x + b^2\sin^2 x)} = \frac{\pi^2(a^2+b^2)}{4a^3b^3}$

(iii) $\int_0^{\pi} x \cos^4 x dx = \frac{3}{16}\pi^2.$

12. Show that : (i) $\int_{-a}^{+a} \frac{xe^{x^2}}{1+x^2} dx = 0 = \int_{-a}^{+a} \frac{xe^{x^4}}{1+x^2} dx$

(ii) $\int_{-a}^{a} x\sqrt{a^2-x^2}\, dx = 0.$

13. Prove that : (*a*) $\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x \cos x} dx = 0$

(*b*) $\int_0^{\frac{\pi}{2}} \frac{\sec x - \operatorname{cosec} x}{1+\sec x \operatorname{cosec} x} dx = 0$

14. Show that, (i) $\int_0^{\frac{\pi}{2}} \sin 2x \log \tan x = 0$

(ii) $\int_0^1 x \log \sin x dx = \frac{1}{2}\pi^2 \log \frac{1}{2}$

(iii) $\int_0^1 \log \sin \frac{1}{2}(\pi\, \theta) d\theta = \log \frac{1}{2}$

(iv) $\int_0^{\frac{\pi}{4}} \log(1+\tan\theta) d\theta = \frac{\pi}{8} \log 2$

15. Show that (i) $\int_0^{\pi} x \sin x \cos^2 x dx = \frac{\pi}{3}$

(ii) $\int_0^{\pi} \frac{\sin 4x}{\sin x} dx = 0$

16. Show that (i) $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \log \frac{1}{2}$

(ii) $\int_0^1 \frac{dx}{(x^2-2x+2)^2} = \frac{3\pi+8}{32}$

17. Show that $\int_0^a \frac{a(x-\sqrt{a^2-x^2})^2}{(2x^2-a^2)^2} = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$

18. Evaluate : (i) $\underset{n\to\infty}{\text{Lt}} \frac{1+2^{10}+3^{10}+\cdots\cdots+n^{10}}{n^{11}}$ [C. U.]

(ii) $\underset{n\to\infty}{\text{Lt}} \left[\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\cdots\cdots+\frac{n^2}{n^3+n^3}\right]$ [C. U.]

(iii) $\lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

(iv) $\lim_{n\to\infty} \frac{1}{n\sqrt{n}}\left[\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}+\cdots\cdots+\sqrt{2n}\right]$

(v) $\lim\limits_{n\to\infty} \left[\frac{n}{n^2}+\frac{n}{1^2+n^2}+\frac{n}{2^2+n^2}+\cdots\cdots+\frac{n}{(n-1)^2+n^2}\right]$

(vi) $\lim\limits_{n\to\infty} \left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{4n}\right]$

(vii) $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

(viii) $\underset{n\to\infty}{\text{Lt}} \left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\cdots\cdots+\frac{1}{n}\right]$

(ix) $\underset{n\to\infty}{\text{Lt}} \frac{1}{n}\left[\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\sin\frac{3\pi}{2n}+\cdots\cdots+\frac{n\pi}{2n}\right]$

ANSWERS

[In the first four chapters add an arbitrary constant of integration with every integral.]

Exercise 1A

1. (i) $\frac{x^{101}}{101}$ (ii) $\frac{x^8}{8}$ (iii) $-\frac{1}{x}$ (iv) $-\frac{1}{2x^2}$ (v) $\frac{-4}{\sqrt[4]{x}}$

(vi) $\frac{2}{7}x^{\frac{7}{2}}$ (vii) $2\sqrt{x}$ (viii) $n\sqrt[n]{x}$.

2. (i) $\frac{e^{2x}}{2}$ (ii) $\frac{e^{17x}}{17}$ (iii) $\frac{e^{cx}}{c}$ (iv) $\frac{5}{4}e^{\frac{4x}{5}}$ (v) $2e^{\frac{x}{2}}$

(vi) $-\frac{e^{-70x}}{70}$ (vii) $\frac{-5}{\sqrt[5]{e^x}}$ (viii) $\frac{x^2}{2}$.

3. (i) $\frac{3^x}{\log_e 3}$ (ii) $\frac{-2^{-x}}{\log_e 2}$ (iii) $\frac{a^x}{\log_e a}$ (iv) $\frac{6^{2x}}{2\log_e 6}$

(v) $\frac{10^x}{\log_e 10}$ (vi) $\frac{6^{10x}}{10\log_e 6}$.

4. (i) $-\frac{\cos 7x}{7}$ (ii) $\frac{\cos 2x}{2}$ (iii) $\frac{\sin 6x}{6}$ (iv) $\frac{\sin 4x}{4}$

(v) $\frac{-\cot 3x}{3}$ (vi) $\frac{\operatorname{cosec} 2x}{2}$.

Exercise 1B

1. (*a*) (i) $\frac{x^3}{3}+\frac{3^x}{\log_e 3}$ (ii) $8x+18x^2+18x^3+\frac{27}{4}x^4$.

(iii) $\frac{x^3}{3}-\frac{x^2}{2}+x$. (iv) $x+\frac{e^{2x}}{2}$ (v) $\frac{x^8}{8}+e^x+\frac{a^x}{\log_e a}$

(vi) $\frac{2}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}-\frac{e^{2x}}{2}$ (vii) $-\cot x-x$

(viii) $2\sin x+\tan x-x$.

(*b*) (i) $\frac{1}{2}\left(\sin x-\frac{\sin 3x}{3}\right)$ (ii) $\frac{1}{2}\left(-\frac{\cos 16x}{16}-\frac{\cos 4x}{4}\right)$

(iii) $\frac{\sin 10x}{10}+\frac{\sin 2x}{2}$ (iv) $\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)$ (v) $\frac{1}{2}\left(x+\frac{\sin 4x}{4}\right)$

(vi) $\frac{1}{2}(x+\sin x)$ (vii) $\frac{1}{12}\sin 3x+\frac{3}{4}\sin x$

(viii) $-\frac{\cos 3x}{4}+\frac{\cos 9x}{36}$ (ix) $\frac{1}{4}\left(x+\frac{\sin 2x}{2}+\frac{\sin 4x}{4}+\frac{\sin 6x}{6}\right)$

(x) $\frac{1}{4}\left(\cos x-\frac{\cos 3x}{3}+\frac{\cos 7x}{7}-\frac{\cos 5x}{5}\right)$.

2. (i) $\frac{2}{3}x^3-\frac{5}{2}x^2+2x$ (ii) $\frac{3}{22}x^{\frac{22}{3}}+\frac{3}{\sqrt[3]{x^2}}$

(iii) $\frac{2}{3}x^3+5x-\frac{2}{x}$ (iv) $-\left(\frac{1}{4}x^4+\frac{1}{x^3}+\frac{3}{2x^2}+\frac{1}{x}\right)$.

3. (i) $\frac{x^2}{2}-3x$ (ii) $\frac{x^3}{3}+\frac{x^2}{2}-6x$.

4. (i) $x-2e^{-x}-\frac{e^{-2x}}{2}$ (ii) $e^x+4e^{-x}+\frac{e^{-2x}}{2}$.

5. $\frac{e^{4x}}{4}+\frac{e^{2x}}{2}+x$. **6.** $\frac{180}{\pi}\sin x°$. **7.** $\frac{1}{2}x-\frac{\sin 2ax}{4a}$.

8. $\frac{1}{2}x+\frac{\sin 12x}{24}$. **9.** $\frac{1}{2}x-\frac{3}{4}\sin\frac{2x}{3}$. **10.** $-\frac{\cot 2x}{2}-x$.

11. $\frac{\sin 2x}{2}$. **12.** $-\operatorname{cosec}\theta$. **13.** $\sin x-\frac{\sin 7x}{7}$.

14. $\frac{\sin 9x}{18}+\frac{\sin x}{2}$. **15.** $-\left\{\frac{\cos(m+n)x}{2(m+n)}+\frac{\cos(m-n)x}{2(m-n)}\right\}$.

if $m\neq n$; $\frac{-\cos 2mx}{4m}$ if $m=n$

16. $-\frac{3}{2}\cos\frac{x}{2}+\frac{1}{6}\cos\frac{3x}{2}$.

17. $\tan x-x$. **18.** $\sec x+\operatorname{cosec} x$.

19. (i) $\tan x+x$ (ii) $2x-2\cot x+3\tan x$.

20. $\sin x-\cos x$.

21. (i) $\frac{a^x}{\log_e a}+2x-\frac{a^{-x}}{\log_e a}$ (ii) $\frac{a^x}{\log_e a}-\frac{a^{-x}}{\log_e a}$.

22. (i) $\frac{1}{2}\tan x$ (ii) $-\cot\frac{x}{2}$ (iii) $\tan x-\sec x$.

23. $2(\tan x+\sec x+\cos x-\frac{3}{2}x)$. **24.** $\sqrt{2}\sin x$.

25. $2\sin x+2x\cos\theta$. **26.** $\frac{20x+3\sin 4x}{32}$.

27. (i) $\frac{3}{8}x+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}$ (ii) $\frac{1}{32}\sin 4x-\frac{1}{4}\sin 2x+\frac{3}{8}x$.

28. $\frac{1}{4}\left\{\frac{\sin(a-b-c)x}{a-b-c}-\frac{\sin(a+b+c)x}{a+b+c}+\frac{\sin(a-b+c)x}{a-b+c}-\frac{\sin(a+b-c)x}{a+b-c}\right\}$.

29. $-\frac{\cos 2x}{2}-\frac{\cos 4x}{8}$. 30. $\frac{\sin 3x}{3}+2\cos x+\frac{x^3}{3}$.

31. $a \tan x-b \cot x$. 32. $2 \operatorname{cosec} x+3 \sin x$.

33. $\frac{1}{3}\sin 3x-\frac{1}{2}\sin 2x$.

Exercise 2A

1. (i) $\frac{1}{12a}(ax+b)^{12}$ (ii) $\frac{1}{28}(4x-5)^7$ (iii) $\frac{1}{a-x}$
(iv) $-\frac{1}{b}\log(a-bx)$ (v) $\frac{(1+x)^6}{6}$.

2. (i) $\frac{1}{a}\sin(ax+b)$ (ii) $\frac{1}{2}\tan(2x+3)$
(iii) $\frac{1}{2}t-\frac{1}{8}\sin(4t+6)$ (iv) $\frac{1}{3}\cot(2-3t)-t$ (v) $\frac{\tan^3 x}{3}$.

3. $\frac{a^{p+qt}}{q\log_e a}$. 4. (i) $\frac{1}{6}\log\left|\frac{x-3}{x+3}\right|$ (ii) $\frac{1}{4}\log\left|\frac{2+x}{2-x}\right|$
(iii) $\frac{1}{20}\log\left|\frac{2x-5}{2x+5}\right|$ 5. $\frac{2}{3(a-b)}\left\{(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right\}$.

6. $\frac{1}{b}x-\frac{a}{b^2}\log(a+bx)$. 7. (i) $\frac{1}{2}\log\left|\frac{x-6}{x-4}\right|$ (ii) $\log\left|\frac{x-4}{x-3}\right|$.

8. (i) $\log(ax^2+bx+c)+k$ (ii) $\frac{1}{2}(x^3+6x^2+5x+12)^2$.

9. $2\log(e^{\frac{x}{2}}+e^{-\frac{x}{2}})$. 10. $\log(1+x^2)$.

11. (i) $\frac{(\sin^{-1}x)^2}{2}$ (ii) $\log(\tan^{-1}x)$.

12. (i) $\frac{1}{6}(x^4+a^4)^{\frac{3}{2}}$ (ii) $\frac{1}{2}\sqrt{2x^2+3}$.

13. $\frac{1}{2}\log(1+\sin^2 x)$. 14. $\frac{(\tan x+\sin x)^3}{3}$.

15. (i) $-\frac{1}{1+\tan x}$ (ii) $\log\frac{1}{1+\cot x}$ (iii) $2\sqrt{\tan x-1}$

16. (i) $\sqrt{x^2-a^2}$ (ii) $\log(1+x^4)$ (iii) $\frac{a}{n}\log(x^n+b^n)$.

17. (i) $\log(1+\log x)$ (ii) $\frac{1}{2}\{\log(\log x)\}^2$ (iii) $\log\{\log(\log x)\}$.

18. (i) $\log(\log\sin x)$ (ii) $\log(\log\sec x)$.

19. $\log(x\sin x)$. **20.** $\log(\log\tan x)$.

21. $\tan(e^x)$. **22.** (*a*) $\dfrac{a^{\sin^{-1}x}}{\log_e a}$ (*b*) $\dfrac{a^{x^2}}{2\log_e a}$

23. $\log\sin(e^x)$. **24.** $2\sin\sqrt{x}$.

25. $-e^{\cos^{-1}x}$. **26.** $e^{\tan^{-1}x}$.

27. (*a*) $e^{-\frac{1}{x}}$ (*b*) $\frac{1}{3}e^{x^3}$. **28.** $e^{\sin x}$.

29. $e^{x-\frac{1}{x}}$. **30.** $\frac{1}{4}\log(3+4e^x)$.

31. $\frac{1}{2}e^{x^2+6x+9}$. **32.** $\frac{1}{12}(\log x)^2$.

33. (*a*) $\frac{1}{2}\sec^{-1}x^2$ (*b*) $-\dfrac{\sqrt{1-x^2}}{x}$ (*c*) $-\dfrac{\sqrt{1+x^2}}{x}$ (*d*) $\dfrac{x}{\sqrt{1-x^2}}$.

34. $\frac{1}{4}\log(3+4\sin x)$. **35.** (*a*) $\frac{1}{3}\sin x^3$ (*b*) $-\cos(\log x)$.

36. (*a*) $\frac{1}{n}\sin x^n$. (*b*) $-\sin(\frac{1}{x})$.

37. (i) $\dfrac{1}{4b}(a+bx^2)^2$ (ii) $-\dfrac{1}{bn}\cos(a+bx^n)$

(iii) $\frac{1}{24}\tan(3+4x^6)$ (iv) $\dfrac{1}{202m}(2x^m+11)^{101}$.

38. $\dfrac{(\tan x-x)^2}{2}$. **39.** $\frac{1}{3}\log\sec 3x-\frac{1}{2}\log\sec 2x-\log\sec x$.

40. $\dfrac{b^2}{q^3}\log(p+qx)-\dfrac{2b(aq-pb)}{q^3(p+qx)}-\dfrac{(aq-pb)^2}{2q^3(p+qx)^2}$.

41. $\frac{1}{3}\log\cos 3x$. **42.** $\dfrac{1}{a-b}\log(a\sin^2x-b\cos^2x)$.

43. $\dfrac{1}{b^2-a^2}\left\{\dfrac{1}{a^2\sin^2x+b^2\cos^2x}\right\}$. **44.** $\sec^{-1}(2x-5)$.

45. $\dfrac{x^4}{4}+\dfrac{x^3}{3}+\dfrac{x^2}{2}+x+\log(x-1)$.

46. $\frac{3}{4}x+\frac{1}{16}\log(x+\frac{5}{4})$. **47.** $\frac{2}{3}\sin^{-1}(\frac{x}{a})^{\frac{3}{2}}$.

48. $\frac{1}{2}\sec^{-1}x+\dfrac{\sqrt{x^2-1}}{2x^2}$ **49.** $\frac{2}{105}(x+a)^{\frac{3}{2}}(15x^2-12ax+43a^2)$.

50. $-\dfrac{2+3x^2}{2x(1+x^2)}-\frac{3}{2}\tan^{-1}x.$ 51. $\log\dfrac{x}{\sqrt{x^2+1}}.$

52. $\log\dfrac{x}{1+x}-\dfrac{x}{1+x}.$ 53. $-\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{\frac{3}{2}}.$

54. $\frac{4}{3}\{x^{\frac{3}{4}}-\log(1+x^{\frac{3}{4}})\}.$ 55. $-\frac{1}{2}x^2.$

Exercise 2B

1. (i) $\frac{1}{3}\tan^{-1}\frac{x}{3}$ (ii) $\dfrac{1}{ab}\tan^{-1}\dfrac{bx}{a}$ (iii) $\frac{1}{4}\tan^{-1}\dfrac{x}{4}$

(iv) $\frac{1}{2}\tan^{-1}\left(\dfrac{\tan x}{2}\right)$ (v) $\frac{1}{4}\tan^{-1}\dfrac{e^{2x}}{2}$ (vi) $\frac{1}{2}\tan^{-1}(\sin x)$

(vii) $\tan^{-1}(\tan^{-1}x)$ (viii) $\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{\log x}{\sqrt{3}}\right)$ (ix) $\tan^{-1}(e^x)$.

2. (i) $\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{x-\sqrt{2}}{x+\sqrt{2}}\right|$ (ii) $\frac{1}{2}\log\left|\dfrac{1+x}{1-x}\right|$ (iii) $\dfrac{1}{2a}\log\left|\dfrac{x-2a}{x}\right|$

(iv) $\dfrac{1}{2a}\log\left|\dfrac{x}{2a-x}\right|$ (v) $\frac{1}{2}\log\left|\dfrac{1+\log x}{1-\log x}\right|$ (vi) $\frac{1}{4}\log\left|\dfrac{1+e^{2x}}{1-e^{2x}}\right|$

(vii) $\frac{1}{2}\log\left|\dfrac{\tan\theta-1}{\tan\theta+1}\right|$ (viii) $\log\left|\tan\dfrac{x}{2}\right|$.

3. (i) $\log(x+\sqrt{x^2+9})$ (ii) $\dfrac{1}{b}\log(bx+\sqrt{a^2+b^2x^2})$

(iii) $\log(x^2+\sqrt{1+x^4})$ (iv) $\dfrac{1}{b}\sin^{-1}\dfrac{bx}{a}$

(v) $\sin^{-1}(\tan^{-1}x)$ (vi) $\sin^{-1}(\tan x)$.

4. (i) $\frac{1}{7}\log\left|\dfrac{x-3}{x+4}\right|$ (ii) $\log\dfrac{2+x}{3x+7}$ (iii) $\tan^{-1}(x+2)$

(iv) $\dfrac{1}{\sqrt{5}}\log\left|\dfrac{2x+\sqrt{5}+1}{\sqrt{5}-1-2x}\right|$ (v) $\frac{1}{2}\tan^{-1}\dfrac{\sin x+1}{2}$

(vi) $\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2\log x+1}{\sqrt{3}}$ (vii) $\frac{1}{4}\log\dfrac{x^2+1}{x^2+3}$

(viii) $\frac{1}{5}\log\left|\dfrac{1+2e^x}{2-e^x}\right|$ (ix) $\frac{1}{8}\log\left|\dfrac{2-\sin x}{2-5\sin x}\right|$.

5. (i) $\frac{1}{2}\log(x^2+4x+5)-\tan^{-1}(x+2)$

(ii) $x+\log\left|\dfrac{x-2}{x+2}\right|$ (iii) $\log(x^2+2x+3)-\dfrac{3}{\sqrt{2}}\tan^{-1}\dfrac{x+1}{\sqrt{2}}$

(iv) $-\frac{4}{\sqrt{5}} \log \left|\frac{\sqrt{5}+2x+1}{\sqrt{5}-2x-1}\right| - \log (1-x-x^2)$

(v) $\frac{1}{16} \log \left|\frac{2x-3}{2x+1}\right| - \frac{1}{8} \log (4x^2-4x-3)$.

6. (i) $2 \log (\sqrt{x-2}+\sqrt{x-3})$

(ii) $\frac{1}{\sqrt{3}} \sin^{-1} \frac{3x-4}{\sqrt{22}}$.

7. (i) $\log \{(x+1)+\sqrt{x^2+2x+6}\}$ (ii) $\sin^{-1} (2x-5)$

(iii) $\frac{1}{\sqrt{2}} \log \left\{4x+3+4\sqrt{x^2+\frac{3x}{2}+2}\right\}$ (iv) $\sin^{-1} \frac{2x+1}{\sqrt{5}}$

(v) $\frac{1}{\sqrt{3}} \sin^{-1} \frac{6x-1}{5}$ (vi) $\frac{1}{\sqrt{3}} \log (x-\frac{1}{6}+\sqrt{x^2-\frac{x}{3}-1})$

(vii) $\log \{(x+1)+\sqrt{x^2+2x+5}\}$

(viii) $\frac{2}{\sqrt{5}} \log (\sqrt{\tan x-\frac{2}{5}}+\sqrt{\tan x-2})$.

(ix) $\log \{(\log x+1)+\sqrt{(\log x)^2+2\log x+5}\}$.

8. (i) $2\{\sqrt{x^2+x+1}+\log (2x+1+2\sqrt{x^2+x+1})\}$

(ii) $\frac{1}{2}\sqrt{2x^2-8x+5}$

(iii) $2\sqrt{x^2+3x+1}+2\log (2x+3+2\sqrt{x^2+3x+1})$

(iv) $\sqrt{x^2+x+1}-\frac{1}{2}\log \{(2x+1)+2\sqrt{x^2+x+1}\}$

(v) $\sqrt{x^2+2x+2}+2\log (x+1+\sqrt{x^2+2x+2})$

(vi) $2\sqrt{3}\sin^{-1}\sqrt{\frac{3x+1}{4}}+2\sqrt{1+2x-3x^2}$

(vii) $-2\sqrt{3x-x^2-2}+16\sin^{-1}\sqrt{x-1}$.

9. (i) $2\tan^{-1}\sqrt{1+x}$ (ii) $\frac{1}{10}\log \frac{\sqrt{2(3x+4)}-\sqrt{5}}{\sqrt{2(3x+4)}+\sqrt{5}}$

(iii) $\log \frac{\sqrt{x+3}-1}{\sqrt{x+3}+1}$

10. (i) $\sin^{-1}\frac{1+3x}{\sqrt{5}(1+x)}$

(ii) $-\frac{1}{\sqrt{5}}\log \left\{\frac{2-x+2\sqrt{5(1+x^2)}}{(1+2x)}\right\}$

(iii) $\log \frac{x}{2+x+2\sqrt{x^2+x+1}}$ (iv) $-\frac{1}{a}\sqrt{\frac{x+a}{x-a}}$

(v) $-\sqrt{\frac{1-x}{1+x}}$ (vi) $\log \frac{\sqrt{2x+1}-\sqrt{3}}{\sqrt{2x+1}+\sqrt{3}}$.

11. $\sqrt{(x-3)(x-4)}+\log(\sqrt{x-3}+\sqrt{x-4})$

12. $x-4\tan^{-1}\frac{x}{2}$.

13. $x-\tan^{-1}x$. **14.** $-\frac{\sqrt{a^2-x^2}}{x}-\sin^{-1}\frac{x}{a}$

15. $a\sin^{-1}\sqrt{\frac{x}{a}}-\sqrt{x(a-x)}$.

16. $\frac{1}{\sqrt{2}}\log\left(\frac{1+x}{1-x}-+\frac{\sqrt{2}\sqrt{1+x^2}}{1-x}\right)$.

17. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x\sqrt{3}}{\sqrt{x^2+4}}\right)$.

18. $2\sqrt{2}\tan^{-1}\sqrt{\frac{x}{2}}-2\tan^{-1}\sqrt{x}$.

20. $2\tan^{-1}\sqrt{1+x}$.

21. $\frac{1}{4\sqrt{3}}\log\frac{2\sqrt{x^2-1}-x\sqrt{3}}{2\sqrt{x^2-1}+x\sqrt{3}}$. **22.** $\frac{1}{2}\sec^{-1}x^2$.

Exercise 2C

1. $\frac{1}{3}\log\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}$. **2.** $\frac{2}{3}\tan^{-1}\frac{1}{3}(5\tan\frac{x}{2}+4)$.

3. $\frac{1}{3}\log\frac{\tan\frac{x}{2}-2}{2\tan\frac{x}{2}-1}$. **4.** $\frac{1}{5}\log\frac{2\tan\frac{x}{2}+1}{\tan\frac{x}{2}-1}$.

5. $\sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right)$. **6.** $\frac{1}{7}\sin^7 x-\frac{1}{9}\sin^9 x$.

7. $\frac{1}{16}\left(x-\frac{\sin 2x}{4}-\frac{\sin 4x}{4}+\frac{\sin 12x}{12}\right)$

8. $\frac{5}{2}x+\frac{\tan^3 x}{3}-2\tan x-\frac{1}{4}\sin 2x$.

9. $\frac{1}{\sqrt{a^2+b^2}}\log\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{a}{b}\right)$.

10. $\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{3\tan\frac{x}{2}+2}{\sqrt{5}}\right)$. **11.** $\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{5}}\right)$.

Miscellaneous Exercise 3B

1. (i) $\log(x+\cos x)$ (ii) $x(\sec x+\tan x)$.

2. (i) $\log(x-\sin x)$ (ii) $-x\cot\frac{x}{2}$.

3. (i) $x\tan\frac{x}{2}+2\log\cos\frac{x}{2}$.

(ii) $x(\tan x-\sec x)+\log(\sec x+\tan x)$

(iii) $-x\cot\frac{1}{2}x+2\log\sin\frac{1}{2}x$

(iv) $x(\sec x+\tan x)-\log(\sec x+\tan x)-\log\sec x$.

4. (i) $\frac{(2e^x-3)\sqrt{e^{2x}-3e^x+1}}{4}-\frac{5}{8}\log(e^x-\frac{3}{2}+\sqrt{e^{2x}-3e^x+1})$

(ii) $\frac{1}{2}[e^{-2x}(e^x-1)\sqrt{2e^{2x}-2e^x+1}$

$-\log(1-e^x+\sqrt{2e^{2x}-2e^x+1})+x]$

(iii) $\frac{1}{8}\left[\frac{(2x^3+1)\sqrt{x^6+x^3+1}}{4}\right]$

(iv) $\frac{x^4}{4}+a\frac{x^3}{3}+\frac{b^2x^2}{2}+ab^2x$

(v) $-\frac{1}{2}\left[\frac{x+2}{2x^2}\sqrt{1+x+x^2}+\frac{3}{4}\log\left(\frac{x+2+2\sqrt{1+x+x^2}}{2x}\right)\right]$.

5. (i) $\frac{3^x}{(\log 3)^2+16}\{\log 3\cos 4x+4\sin 4x\}$

(ii) $\frac{e^{2x}(\sin 2x-\cos 2x)}{8}$ (iii) $\frac{1}{\sqrt{2}}e^x\sin x$

(iv) $\frac{3e^{mx}}{4(1+m^2)}(m\sin x-\cos x)-\frac{e^{mx}}{4(9+m^2)}(m\sin 3x-3\cos 3x)$

(v) $-\frac{4e^{-2x}}{17}(2\cos\frac{1}{2}x-\frac{1}{2}\sin\frac{1}{2}x)$.

6. (i) $\frac{(\tan^{-1}x)^2}{2}(x^2+1)-x\tan^{-1}x+\frac{1}{2}\log(1+x^2)$

(ii) $\frac{1}{3}x^3\tan^{-1}x+\frac{1}{6}[\log(x^2+1)-x^2]$.

7. (i) $\frac{e^{m\tan^{-1}x}}{\sqrt{m^2+1}}\cos(\tan^{-1}x-\cot^{-1}m)$

(ii) $\frac{1}{4}e^{2\theta}\left[\frac{2\cos 3\theta+3\sin 3\theta}{13}+\frac{3}{5}(2\cos\theta+\sin\theta\right]$

where $\tan^{-1}x=\theta$

(iii) $\frac{e^{\theta}}{\sqrt{2}} \sin \left(\theta-\frac{\pi}{4}\right)$ where $\sin^{-1} x=\theta$

(iv) $\frac{e^{\theta}}{8}\ [(3 \sin \theta-\cos \theta)-\frac{1}{5}(\sin 3\theta-3 \cos 3\theta)]$

where $\sin^{-1} x=\theta$.

8. (i) $\frac{x}{\sqrt{1-x^2}} \sin^{-1} x+\frac{1}{2} \log \sqrt{1-x^2}$

(ii) $\frac{1}{3}\left[\left(\frac{x}{\sqrt{1-x^2}}\right)^3 \sin^{-1} x-\frac{1}{2} \frac{x^2}{1-x^2}-\frac{1}{2} \log (1-x^2)\right]$

(iii) $-\frac{\tan^{-1} x}{\sqrt{1+x^2}}+\frac{x}{\sqrt{1+x^2}}$.

9. (i) $\frac{x}{(\log x)^2}$ (ii) $\frac{x}{(\log x)^n}$ (iii) $x\left\{\log (\log x)-\frac{1}{\log x}\right\}$.

10. (i) $3(x \sin^{-1} x+\sqrt{1-x^2})$

(ii) $3\{x \tan^{-1} x-\frac{1}{2} \log (1+x^2)\}$

(iii) $x \tan^{-1} \sqrt{\frac{x}{x+1}}+\frac{1}{2} \tan^{-1} \sqrt{\frac{x}{1+x}}-\frac{1}{4} \log \frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}-\sqrt{x}}$

(iv) $-\frac{1}{2}[-x \cos^{-1} x+\sqrt{(1-x^2)}$.

11. (i) $\sqrt{x(x+a)}-a \log (\sqrt{x}+\sqrt{x+a})$

(ii) $\sqrt{x^2+ax}+a \log (\sqrt{x}+\sqrt{x+a})$

(iii) $(a \sin^{-1}\sqrt{\frac{x}{a}}+\sqrt{ax-x^2})$

(iv) $\left(-\sqrt{ax-x^2}+a \sin^{-1}\sqrt{\frac{x}{a}}\right)$

12. (i) $-\frac{1}{x} \tan^{-1} x+\log \frac{x}{\sqrt{x^2+1}}$

(ii) $\frac{x^7}{7} \sin^{-1} x+\frac{1}{7}\,[\sqrt{1-x^2}-(1-x^2)$

$+\frac{3}{5}(1-x^2)^{\frac{5}{2}}-\frac{1}{7}(1-x^2)^{\frac{7}{2}}]$.

13. $\frac{\sin x \cos^5 x}{6}+\frac{5}{24} \sin x \cos^3 x+\frac{5}{16}(x+\sin x \cos x)$

15. $e^x \log \sin x$.

16. $\frac{1}{3}\{(\log \sqrt{x})^2\}=\frac{1}{12}(\log x)^2$.

17. $xe^x\,[\log(xe^x)-1]$.

18. $e^x \cot x$.

[Read—$\operatorname{cosec}^2 x$ in the equation]

19. $x\log(x+\sqrt{x^2+a^2})-\sqrt{x^2+a^2}$

20. $\frac{1}{2}x^2\log[x+\sqrt{a^2+x^2}]$

$$-\tfrac{1}{4}x\sqrt{x^2+a^2}-a^2\log(x+\sqrt{x^2+a^2})]$$

21. $\dfrac{2}{3(a-b)}\Big[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\Big]$.

22. (i) $(x+1)\log(x+1)-x$; (ii) $\dfrac{x^3}{3}\log x-\dfrac{x^3}{9}$.

23. $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}$.

Exercise 4

1. $\log\dfrac{x-2}{x-1}$. **2.** $\log\dfrac{3x+2}{4x+3}$. **3.** $2\log(x-3)-\log(x-2)$.

4. $\log\{(x-2)^2(x+1)\}$. **5.** $\dfrac{1}{a-b}\{a\log(x-a)-b\log(x-b)\}$.

6. $\dfrac{1}{x+1}+\log\left(\dfrac{x+1}{x+2}\right)^2$ **7.** $4\log(x+2)-3\log(x+1)-\dfrac{1}{x+1}$.

8. $\dfrac{1}{2(x-1)}+\frac{1}{4}\log\dfrac{x-3}{x-1}$.

9. $\frac{3}{10}\{\log(x^2+4)-2\log(1-x)\}-\frac{1}{5}\tan^{-1}\dfrac{x}{2}$.

10. $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}-\frac{1}{3}\log\dfrac{1}{\sqrt{1+x+x^2}}$.

11. $\dfrac{1}{6}\log\dfrac{x^2-x+1}{x^2+2x+1}+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}$.

12. $\dfrac{x^2}{2}+3x+10\log(x-2)-3\log(x-1)$.

13. $x-\frac{a}{2}\log(x+a)-\frac{a}{4}\log(x^2+a^2)-\frac{a}{2}\tan^{-1}\dfrac{x}{a}$.

14. $\frac{x^2}{3}+x+\frac{3}{2}\log(x-1)-\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x$.

15. $\frac{1}{2}\log(x-1)+\frac{1}{5}\log(x-2)+\frac{3}{10}\log(x+3)$.

16. $\frac{\sqrt{3}}{14}\log\frac{x-\sqrt{3}}{x+\sqrt{3}}+\frac{2}{7}\tan^{-1}\frac{x}{2}$.

17. $\tan^{-1}x-\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}$.

18. $\frac{1}{a^2-b^2}\left[\frac{1}{b}\tan^{-1}\frac{x}{b}-\frac{1}{a}\tan^{-1}\frac{x}{a}\right]$.

19. $\frac{1}{7}\log\frac{x-2}{x+2}+\frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}}$.

20. $2x+\frac{11}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}-7\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}$.

21. $-\frac{1}{2}x^2-\frac{1}{2}\log(1-x^2)$.

22. $\frac{1}{4}\log\frac{x^2-1}{x^2+1}$. **23.** $\frac{3}{2}\log(t^2+3)-\log(t^2+2)$.

24. $\frac{1}{6}\{\log(x^2-2)-\log(x^2+1)\}$. **25.** $\frac{1+3x+6x^2}{3x^2}+2\log\frac{x-1}{x}$.

26. $\frac{1}{6}\log(1+\cos x)+\frac{1}{2}\log(1-\cos x)-\frac{2}{3}\log(1-2\cos x)$.

27. $-\frac{1}{2}\log(1+\cos x)+\frac{1}{10}\log(1-\cos x)+\frac{2}{5}\log(3+2\cos x)$.

Exercise 5A

1. (i) $\frac{1}{9}(10^9-1)$ (ii) $\frac{26}{3}$ (iii) $b-a$ (iv) 25·5
(v) $b-a$ (vi) 1 (vii) 64.

2. 24. **3.** (i) $\frac{1}{3}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{3}{8}\pi^2-1$ (iv) $\frac{1}{m}(1-\cos m\pi)$
(v) $\log\sqrt{2}$. **4.** 0. **5.** $\frac{\pi}{8}-\frac{\sin n\frac{\pi}{2}}{4n}$.

6. $\frac{\pi}{4}-\frac{1}{2}\log 2$. **7.** $\frac{26}{3}$. **8.** $-\frac{7}{288}$. **9.** $2-\sqrt{2}$.

10. $\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log 2$. **11.** 0. **12.** 0 or, $\frac{2m}{m^2-n^2}$. **13.** 0.

14. $\frac{1}{m}\left(e^{mb}-e^{ma}\right)$. **15.** $8\log 2-3$. **16.** $\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$.

17. π. **18.** $\frac{1}{16}(\pi^2+4)$. **19.** $\frac{9}{2}\log 3-4\log 4+5$.

20. $\frac{\pi}{12}-\frac{1}{6}+\frac{1}{6}\log 2$. **21.** $e^{\frac{\pi}{2}}$. **22.** $\frac{3}{8}\pi^2-1$.

23. $\frac{1}{16}(\pi-3)$. **24.** (i) $\frac{2}{3}\tan^{-1}\frac{1}{3}$ (ii) $\frac{1}{3}\log 2$

(iii) $\frac{1}{3}\log 2$ (iv) $\frac{2}{3}\tan^{-1}\frac{1}{3}$. **25.** $\frac{\pi a^2}{4}$. **26.** $\frac{3}{2}\log\frac{4}{3}$.

27. $\frac{\pi}{6}$. **28.** (*a*) $\frac{14}{9}$ (*b*) $\frac{7}{18}$. **29.** $e^2(\sqrt{e}-1)$.

30. $(2a-1)\sqrt{a-a^2}+\frac{1}{2}\sin(2a-1)+\frac{\pi}{4}$.

31. $\frac{1}{2}\log(2+\sqrt{3})$. **32.** $1-\frac{1}{e}$. **33.** $\frac{\pi}{6}$.

34. $\frac{1}{5}\left[\log\frac{3}{2}-\frac{1}{2}\log 2+\frac{\pi}{2}\right]$. **35.** $\log\frac{4}{3}$. **36.** $\frac{2}{3}$. **37.** $\frac{14}{15}$

38. $\frac{\pi}{3}$. **39.** $\frac{\pi}{16}$. **40.** -1. **41.** $\frac{3}{4}(1-2\log\frac{3}{2})$. **42.** $\frac{\pi}{2\sqrt{2}}$.

43. $\frac{1}{\sqrt{2}}\cot^{-1}2$. **44.** $\frac{2\pi}{3\sqrt{3}}$. **45.** $\frac{1}{8}\pi(\beta-\alpha)^2$.

46. $\frac{\pi}{6}$. **56.** $\frac{9}{16}$.

Exercise 5B

1. (i) $\frac{b^3-a^3}{3}$ (ii) 1 (iii) $\frac{1}{2}$ (iv) $\frac{1}{4}$ (v) $\frac{1}{2}$ (vi) [illegible]

(vii) $\frac{1}{m}(e^{mb}-e^{ma})$ (viii) $\frac{1}{m}(e^{m}-1)$ (ix) $\frac{a}{3}+\frac{b}{2}+c$ (x) $\frac{1}{2}$.

2. (i) $\frac{5\pi}{256}$ (ii) 0 (iii) $\frac{5}{32}\pi$ (iv) $\frac{7\pi}{2048}$ (v) 0.

10. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$.

18. (i) $\frac{1}{11}$ (ii) $\frac{1}{3}\log 2$ (iii) $\frac{\pi}{2}$ (iv) $\frac{2}{3}(2\sqrt{2}-1)$ (v) $\frac{\pi}{4}$
(vi) $\log 4$ (vii) $\frac{\pi}{2}$ (viii) $\frac{\pi}{4}+\frac{1}{2}\log 2$ (ix) $\frac{2}{\pi}$.

DIFFERENTIAL EQUATIONS

CHAPTER ONE

DIFFERENTIAL EQUATIONS THEIR ORIGIN AND SOLUTIONS

§ 1.1. Equations involving derivatives or differentials of functions are called Differential equations.

Hence $x^2\frac{dy}{dx}+y-1=0$ is a differential equation due to the presence of the derivative $\frac{dy}{dx}$.

$y\,dx-xdy=xy\,dx$ is a differential equation, since the differentials dx, dy are present in the equation.

Let us now consider the three equations,

$$\frac{dy}{dx}=0,\ dy=\sin x\,dx \text{ and } y\,dy=dx$$

In the first equation, x and y are both absent, though $\frac{dy}{dx}$ is present ; y and x are respectively absent in the second and third equations. But all the three equations, due to the presence of derivatives or differentials are differential equations. So, in a differential equation both or any of the dependent and independent variables may not be directly present.

The differential equation, in which the derivatives present are obtained by differentiating the dependent variable with respect to a single independent variable only, is called an **ordinary differential equation**. In this book we shall discuss ordinary differential equation and by differential equations shall mean ordinary differential equations. So, the epithet ordinary will not be used any farther.

§ 1·2. Order and degree of differential equations.

The order of a differential equation is the order of the highest order derivative present in the equation.

Hence, $x\frac{dy}{dx}=2a$ ...(1)

$x\,dy+y\,dx=0$ ...(2)

$\left(\frac{dy}{dx}\right)^2=x^2+y^2$ ....(3)

are equations of the first order. The two equations,

$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$ ...(4)

$x\left(\frac{dy}{dx}\right)^2-\left(\frac{d^2y}{dx^2}\right)^2=x^3$ ...(5)

are examples of second order differential equations.

Similarly examples of third, fourth or *n*th-order differential equations can be obtained.

By the degree of a differential equation is meant the highest power of the highest order derivative present in the equation.

In the equation $\frac{dy}{dx}=\frac{y}{x}$, the highest order derivative present is $\frac{dy}{dx}$ and its degree is 1 and hence the degree of the equation is 1 and the equation is of the first order and degree.

In the equation $x\left(\frac{dy}{dx}\right)^2-\frac{d^2y}{dx^2}=x^3$, the highest order derivative present is $\frac{d^2y}{dx^2}$ and its degree is 1. Hence the equation is of degree 1.

The equation $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$ is an equation of the second order but of the first degree. In this equation the highest order derivative present is $\frac{d^2y}{dx^2}$ whose dergee is 1. Similarly one can find examples of differential equations of the *n*-th order and *m*-th degree. Now, let us consider the equation.

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\rho, \cdots\text{(i)}$$

This equation can be written as

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}=\rho\frac{d^2y}{dx^2}$$

or, $$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3=\rho^2\left(\frac{d^2y}{dx^2}\right)^2$$

or, $$\rho^2\left(\frac{d^2y}{dx^2}\right)^2-1-3\left(\frac{dy}{dx}\right)^2-3\left(\frac{dy}{dx}\right)^4-\left(\frac{dy}{dx}\right)^6=0\cdots(\text{ii})$$

The equation (ii) is obtained from the equation (i) by making the latter free of fractions and radicals or fractional indices. The equation (ii) is evidently an equation of the second order and second degree and so the equation (i) is also called an equation of the second order and second degree.

Similarly the equation $x+\dfrac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}=a$ can be written

as $$\frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}=a-x$$

or, $$\left(\frac{dy}{dx}\right)^2=(a-x)^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}$$

So, the equation is of the first order and second degree. Hence the degree of a differential equation free from fractions and radicals or fractional indices is the highest degree of the highest order derivative present in the equation. For determination of the degree of an equation, the equation should be first made free from fractions and radicals.

§ 1·3. Formation of Differential Equations.

You know that the equation $x^2+y^2-2ax=0\cdots(\text{i})$ is the equation of a circle passing through the origin and the centre of the circle lies on the x-axis. Now for different values of a, the equation will represent different circles passing through the origin and having centres on the x-axis. Hence the equation $x^2+y^2-2ax=0$ represents a family of circles. The constant a is called the parameter of the equation. Now, differentiating both sides of the equation (1) we get,

$$2x+2y\frac{dy}{dx}-2a=0 \quad \text{or, } a=x+y\frac{dy}{dx} \quad \cdots\quad (2)$$

Putting this value of a in equation-(1) we get,

$$x^2+y^2-2\left(x+y\frac{dy}{dx}\right)x=0$$

$$\text{or, } \quad y^2-2xy\frac{dy}{dx}-x^2=0 \quad \cdots(3)$$

Now the equation-(3) is free from a. The equation-(3) is said to be the differential equation of the family of circles represented by the equation (1). Note that in the equation-(1) there is only one arbitrary constant or parameter and equation (2) has been obtained by single differentiation of both sides of equation (1). The differential equation (3) has been obtained after eliminating a from the equations (1) and (2)

Let us now discuss equations with two parameters.

In the equation $y=mx+c\cdots(4)$ m and c are parameters. For different values of m and c equation-(5) represents different straight lines. Now, differentiating both sides of equation-(4) we obtain

$$\frac{dy}{dx}=m\cdots(5)$$

and the parameter c in equation-(4) is eliminated. To eliminate the parameter m, another differentiation is required ; differentiating both sides of equation-(5) we obtain

$$\frac{d^2y}{dx^2}=0 \qquad \cdots(6)$$

In equation-(6) both the parameters m and c are absent. Equation-(6) is the differential equation of the family of straight lines in a plane.

In general if n independent parameters be present in an equation, then the n parameters are eliminated from the total number of $(n+1)$ equations obtained after successive n times differentiation of both sides of the given equation. The differential equation obtained after elimination of the n parameters is the differential equation originated from the given equation.

§ 1·4. Solution of a differential equation.

If a relation between the dependent and independent variables of a differential equation not containing any derivative or differential satisfy the differential equation, then the relation is said to be a solution of the differential equation. So the solution of a differential equation is the equation, free from differentials or derivatives, from which the differential equation can be obtained after one or more successive differentiations.

Let us now consider the equation $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$.

Let $y=e^{2x}$.

$\therefore \quad \frac{dy}{dx}=2e^{2x}$ and $\frac{d^2y}{dx^2}=4e^{2x}$

Hence the left hand side of the given differential equation becomes $4e^{2x}-5.2e^{2x}+6.e^{2x}=0$.

Hence the relation $y=e^{2x}$ is a solution of the equation.

Now, if $y=Ae^{2x}$,

then $\frac{dy}{dx}=2Ae^{2x}$ and $\frac{d^2y}{dx^2}=4Ae^{2x}$.

Now putting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the left hand side of the given differential equation, we find that the equation is satisfied. Similarly $y=e^{3x}$ and $y=Be^{3x}$ both satisfy the differential equation and both of them are solutions of the equation.

Now, as $y=Ae^{2x}$ and $y=Be^{3x}$ both satisfy the given differential equation, so $y=Ae^{2x}+Be^{3x}$ [where A and B are arbitrary constants] will also satisfy the equation.

$y=Ae^{2x}+Be^{3x}$ is the general solution of the given differential equation.

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

We state below what is meant by the general solution of a differential equation.

If the number of independent arbitrary constants in a solution of a differential equation be equal to the order of the equation, then that solution of the equation is called the general solution of the equation.

Note 1. In the relation $y=\log\frac{x}{a}+c$, the two quantities a and c are constants. But $\log\frac{x}{a}+c$ can be written as $\log x-\log a+c=\log x+k$ (taking $c-\log a=k$). So the two constants can be absorbed in a single constant. Here the two constants a and c are not independent.

Similarly, the relation $y=Ae^{a+x}$ can be expressed as $y=A.e^a.e.^x=Be^x$ which contains only one arbitrary constant.

If a relation containing n arbitrary constants cannot be expressed as a relation containing less than n arbitrary constants, then the n arbitrary constants are mutually independent in the relation.

2. In § 1·3 it has been found that to eliminate a relation containing only one parameter one has to differentiate both sides of the equation only once and the eliminant becomes a differential equation of the first order. To eliminate two independent arbitrary constants one has to differentiate twice and the eliminant is a differential equation of the second order. Hence it is quite natural to presume that the general solution of a differential equation of the n-th order, will contain n mutually independent arbitrary constants.

3. Determination of the solution of a differential equation means determination of the general solution.

4. A particular solution of a differential equation is obtained by assigning particular values to the arbitrary constants in the general solution of the equation.

Examples 1.

Example 1. Find the order and degree of the following differential equations.

(i) $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+8y=0$ (ii) $\frac{dy}{dx}+\frac{y}{x}=0.$

(iii) $\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}+3y=0$ (iv) $\frac{d^2y}{dx}+\left(\frac{dy}{dx}\right)^2+y=0$

(v) $\frac{d^2y}{dx^2}+y=\sqrt{1+\frac{dy}{dx}}.$

(i) The equation is of the second order and first degree as the highest order derivative present is $\left(\frac{d^2y}{dx^2}\right)$ of order 2 and its degree is 1.

(ii) The equation is of the first order and of first degree. As the highest order derivative present is $\frac{dy}{dx}$ which is of order one and its degree is also one.

(iii) The equation is of first order and second degree. As the highest order derivative present in the equation is $\frac{dy}{dx}$ which is of order 1 and the highest power of $\frac{dy}{dx}$ present in the equation is $\left(\frac{dy}{dx}\right)^2$ whose degree is 2.

(iv) The equation is of the second order and first degree. As the highest order derivative present is $\frac{d^2y}{dx^2}$ and the highest degree of $\frac{d^2y}{dx^2}$ is 1. Here $\left(\frac{dy}{dx}\right)^2$ is present ; but $\frac{dy}{dx}$ is not the highest order

$y=Ae^{2x}+Be^{3x}$ is the general solution of the given differential equation.

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

We state below what is meant by the general solution of a differential equation.

If the number of independent arbitrary constants in a solution of a differential equation be equal to the order of the equation, then that solution of the equation is called the general solution of the equation.

Note 1. In the relation $y=\log\frac{x}{a}+c$, the two quantities a and c are constants. But $\log\frac{x}{a}+c$ can be written as $\log x-\log a+c=\log x+k$ (taking $c-\log a=k$). So the two constants can be absorbed in a single constant. Here the two constants a and c are not independent.

Similarly, the relation $y=Ae^{a+x}$ can be expressed as $y=A.e^{a}.e.^{x}=Be^{x}$ which contains only one arbitrary constant.

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(iii) $\left(\dfrac{dy}{dx}\right)^2+2\dfrac{dy}{dx}+3y=0$ (iv) $\dfrac{d^2y}{dx}+\left(\dfrac{dy}{dx}\right)^2+y=0$

(v) $\dfrac{d^2y}{dx^2}+y=\sqrt{1+\dfrac{dy}{dx}}.$

(i) The equation is of the second order and first degree as the highest order derivative present is $\left(\dfrac{d^2y}{dx^2}\right)$ of order 2 and its degree is 1.

(ii) The equation is of the first order and of first degree. As the highest order derivative present is $\dfrac{dy}{dx}$ which is of order one and its degree is also one.

(iii) The equation is of first order and second degree. As the highest order derivative present in the equation is $\dfrac{dy}{dx}$ which is of order 1 and the highest power of $\dfrac{dy}{dx}$ present in the equation is $\left(\dfrac{dy}{dx}\right)^2$ whose degree is 2.

(iv) The equation is of the second order and first degree. As the highest order derivative present is $\dfrac{d^2y}{dx^2}$ and the highest degree of $\dfrac{d^2y}{dx^2}$ is 1. Here $\left(\dfrac{dy}{dx}\right)^2$ is present ; but $\dfrac{dy}{dx}$ is not the highest order derivative present in the equation. So its degree is not the degree of the equation.

(v) $\frac{d^2y}{dx^2}+y=\sqrt{1+\frac{dy}{dx}}$.

Squaring both sides of the equation we make at first the equation free from radicals as $\left(\frac{d^2y}{dx^2}+y\right)^2=1+\frac{dy}{dx}$.

or, $\left(\frac{d^2y}{dx^2}\right)^2+2y\frac{d^2y}{dx^2}+y^2=1+\frac{dy}{dx}$.

The highest order derivative present in the equation is of order 2 and its highest degree is also 2. So the equation is of the second order and second degree.

Ex-2. Eliminate 'c' from the following equations.

(i) $y=cx+\frac{a}{c}$ (ii) $y^2=2cx+c^2$.

(i) $y=cx+\frac{a}{c}$

$\therefore \frac{dy}{dx}=c+0=c.$

Hence $y=x\frac{dy}{dx}+\frac{a}{\frac{dy}{dx}}$ $\left(\text{Putting } c=\frac{dy}{dx}\right)$

is the required eliminant.

(ii) $y^2=2cx+c^2$

Differentiating both sides with respect to x we get,

$$2y\frac{dy}{dx}=2c. \quad \therefore c=y\frac{dy}{dx}.$$

Putting $c=y\frac{dy}{dx}$ in the given equation we get

$$y^2=2xy\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2$$

or, $y=2x\frac{dy}{dx}+y\left(\frac{dy}{dx}\right)^2$

Which is the required eliminant.

Ex-3. Eliminate A and B from the following equations.

(i) $Ax^2+By^2=1$

(ii) $y=Ae^x+Be^{-x}+x^2$ [Joint Entrance 1984]

(i) $Ax^2+By^2=1$

Differentiating both sides with respect to x we get,

$$2Ax+2By\frac{dy}{dx}=0$$

$$\therefore \quad \frac{dy}{dx}=-\frac{A}{B}\frac{x}{y} \quad \dots \quad \dots \quad \text{(i)}$$

Differentiating both sides with respect to x we get

$$\frac{d^2y}{dx^2}=-\frac{A}{B}\left(\frac{y-x\frac{dy}{dx}}{y^2}\right)$$

$$\text{or,} \quad \frac{d^2y}{dx^2}=\frac{y}{x}\;\frac{dy}{dx}\left(\frac{y-x\frac{dy}{dx}}{y^2}\right)$$

$$\text{or,} \quad xy\frac{d^2y}{dx^2}=y\frac{dy}{dx}-x\left(\frac{dy}{dx}\right)^2$$

$$\text{or,} \quad x\left\{\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right\}=y\frac{dy}{dx}$$

Which is the required eliminant.

(ii) $y=Ae^x+Be^{-x}+x^2$ ... ... (1)

Differentiating both sides with respcct to x we get,

$$\frac{dy}{dx}=Ae^x-Be^{-x}+2x \quad \dots \quad \dots \quad (2)$$

Differentiating both sides of equation $-(2)$ with respect to $x$we get

$$\frac{d^2y}{dx^2}=Ae^x+Be^{-x}+2$$

$=y-x^2+2.$ [As from (1) $Ae^x+Be^{-x}=y-x^2$]

So, the required eliminant is

$$\frac{d^2y}{dx^2}-y=2-x^2.$$

Ex-4. Eliminate the two constants α and β from the equation $y=\alpha e^{\beta+x}$. Justisfy why the eliminant is of the first order though two constants α and β are eliminated.

$$y=\alpha e^{\beta+x}$$

$$\therefore \quad \frac{dy}{dx}=\alpha\frac{d}{dx}\left(e^{\beta+x}\right)=\alpha.e^{\beta+x}=y$$

So the required eliminant is $\frac{dy}{dx}=y$ which is of the first order.

Note here the two constants α and β are not mutually independent as the equation $y=\alpha e^{\beta+x}$ can be writhen in the form $y=\alpha.e^{\beta}.e^{x}$ $=\gamma e^{x}$ [taking $\gamma=\alpha.e^{\beta}$ as e^{β} is a constant] which contains only one constant γ.

Ex. 5. What is the differential equation of the family of straight lines passing through the origin ?

The equation $y=mx$ represents for different values of m different straight lines through the origin. So, the m—eliminant of the equation $y=mx$ will give the differential equation of the family.

From $y=mx$ we get $\frac{dy}{dx}=m$ [differentiating both sides with respect to x]

$\therefore \quad y=x\frac{dy}{dx}$ is the differential equation of the family.

Ex. 6. Find the differential equation of the family of circles which touch the $x-$axis at the origin.

The equation of a circle touching the $x-$axis at the origin is

$$x^2+(y-a)^2=a^2 \quad \text{or,} \quad x^2+y^2-2ay=0 \quad \dots \dots \quad (1)$$

For different values of a we shall get different circles of the family. The a-eliminant of the equation (1) will give the differential equation of the family.

Differentiating both sides of equation (1) with respect to x we get

$$2x+2y\frac{dy}{dx}-2a\frac{dy}{dx}=0 \quad \text{or, } 2a=\frac{2x+2y\dfrac{dy}{dx}}{\dfrac{dy}{dx}}$$

Hence putting,

$$2a=\frac{2x+2y\dfrac{dy}{dx}}{\dfrac{dy}{dx}}$$

in the equation (1) we get

$$x^2+y^2-\frac{2x+2y\dfrac{dy}{dx}}{\dfrac{dy}{dx}}y=0$$

or, $(x^2-y^2)\dfrac{dy}{dx}=2xy$, which is the required equation of the family.

Exercise 1

1. Find the order and degree of the following differential equations.

(i) $\dfrac{dy}{dx}=\sin x$ (ii) $x^2\left(\dfrac{dy}{dx}\right)^2-2xy\dfrac{dy}{dx}+2y^2-x^2=0$

(iii) $\dfrac{d^2y}{dx^2}-a\left(\dfrac{dy}{dx}\right)^2=0$ (iv) $\dfrac{d^2y}{dx^2}-7\dfrac{dy}{dx}+12y=0.$

(v) $y-\dfrac{dy}{dx}=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,(x^2+y^2).$

2, Form differential equations from the following relations.

(i) $xy=c^2$ (ii) $y=Ae^{mx}+Be^{-mx}$

(iii) $y=A\sin mx+B\cos mx.$

(iv) $y=a+bx^2$ (v) $r=a+b\cos\theta.$

3. Eliminate from the following relations the arbitrary constants.

(i) $y=a\cos 2x$ (ii) $y^2=4ax$ (iii) $ay^2=(x-a)^2$

(iv) $y+1=x+ce^{-x}$ (v) $y^2=2cx+c^2$.

4. Eliminate A and B from the following relations.

(i) $y=A+B\cos x$ (ii) $y=A+Bx$

(iii) $y=(Ax+B)e^{-2x}$ (iv) $y=A\sin x+B\cos x$

(v) $y=A\sec x+B\tan x$

5. Show that the differential equation of the family of circles whose centre is at the origin is $xdx+ydy=0$.

6. Show that the a, b, c eliminant of the equation,

$$ax+by+c=0\ (a, b)\neq(0, 0) \text{ is } \frac{d^2y}{dx^2}=0.$$

Give reasons why the equation is of the second order though the number of arbitrary constants is 3.

7. Find the differential equation of the family of circles which touch the $y-$axis at the origin.

8. Show that the A, B eliminant of the equation,

$$v=\frac{A}{r}+B \text{ is } \frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}=0. \qquad \text{[C. U.]}$$

9. Show that the differential equation of the family of curves represented by the relation $y=a\sin x+b\cos x+x\sin x$ is,

$$2xy\frac{dy}{dx}=y^2-x^2$$

10. Show that the differential equation the family of curves represented by the relation $y=e^x(a\cos x+b\sin x)$ is

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$$

CHAPTER TWO

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND OF FIRST DEGREE

§ 2·1 Solution of differential equations of the first order and of the first degree by the method of separation of variables.

You know that all equations are not solvable. So also all differential equations are not solvable. At the primary stage conditions of existence of solution of differential equations are not necessary. The methods of solution of solvable equations are also different. In this chapter we shall discuss only one type of solvable equations. They are Equations of the first order and of first degree which can be solved by the method of separation of variables. If a differential equation of the first order and degree can be expressed in the form $f_1(x)dx+f_2(y)dy=0$, where $f_1(x)$ and $f_2(y)$ are functions of only x and only y respectively, then we say that in the equation variables have been separated. In a differential equation variables are generally separated (i) by inspection and (ii) by substitution.

§ 2·2. The method of separation of variables.

Example Solve : $xdx+ydy=0$.

In this equation the variables are already separated. So integrating we get, $\int xdx+\int ydy=0$

or, $\frac{x^2}{2}+c_1+\frac{y^2}{2}+c_2=0$

or, $\frac{x^2}{2}+\frac{y^2}{2}=-c_1-c_2$

or, $\frac{x^2}{2}+\frac{y^2}{2}=c$. $[-c_1-c_2=c]$

or, $x^2+y^2=2c$

or, $x^2+y^2=a^2$ $[\,a^2=2c\,]$

§ 2·3. Solution of differential equations by substitution.

When the variables cannot be separated by inspection, then in certain cases the dependent variables are substituted by a new variable so that the new variable and the independent variable can be seperated. The solution in terms of the new variable is then expressed in terms of the old variable.

Ex. Solve : $(x+y)^2 \dfrac{dy}{dx}=a^2$

Let, $x+y=z$

$\therefore \quad 1+\dfrac{dy}{dx}=\dfrac{dz}{dx}$, or, $\dfrac{dy}{dx}=\dfrac{dz}{dx}-1$

$\therefore$ The given equation is $z^2\left(\dfrac{dz}{dx}-1\right)=a^2$,

or, $z^2\dfrac{dz}{dx}-z^2=a^2$, or, $z^2\dfrac{dz}{dx}=z^2+a^2$.

or, $\dfrac{z^2dz}{z^2+a^2}=dx$, or, $\displaystyle\int\frac{z^2dz}{z^2+a^2}=\int dx$

or, $\displaystyle\int\frac{z^2+a^2-a^2}{z^2+a^2}dz=\int dx$,

or, $\displaystyle\int dz-a^2\int\frac{dz}{z^2+a^2}=\int dx$, or, $z-a\tan^{-1}\dfrac{z}{a}=x-c$

or, $(x+y)-a\tan^{-1}\left(\dfrac{x+y}{a}\right)=x-c$

or, $y=a\tan^{-1}\left(\dfrac{x+y}{a}\right)-c$

or, $\dfrac{y+c}{a}=\tan^{-1}\left(\dfrac{x+y}{a}\right)$ or, $\dfrac{x+y}{a}=\tan\left(\dfrac{y+c}{a}\right)$.

§ 2·4. Solution of homogeneous equations of the first order and first degree.

In the previous article we have solved a differential equation by substitution. But we have not indicated any rule for substitution. In the present article and the next we shall discuss the method of substitution in certain special cases. In the present article we shall discuss solution of homogeneous differential equations of the first order and first degree.

Homogeneous equations. If in an equation of the form $f_1(x, y)dx+f_2(x, y)dy=0$, the power of each term of $f_1(x, y)$ and $f_2(x, y)$ be the same, the equation is said to be a homogeneous equation of the first order and first degree.

In the equation $(x^2+y^2)dy-xy\ dx=0$, the power of each of x^2, y^2 and xy is 2 and so the equation is homogeneous.

The equation $\dfrac{dy}{dx}=\dfrac{y^2}{x^2}-\dfrac{y}{x}$ can be expressed in the form $x^2dy=(y^2-xy)dx$ and the power of each of x^2, y^2, xy is 2 and so the equation is homogeneous. The equation $(x^3+y^3)dy=x^2y\ dx$ is also a homogeneous equation.

To solve homogeneous equations, put $y=vx$. In the new equation in v and x after substitution, the variables will be easily separated and the solution obtained in terms of v and x should be expressed in terms y and x.

Example. Solve : $2\dfrac{dy}{dx}=\dfrac{y}{x}+\dfrac{y^2}{x^2}$.

The equation is homogeneous. Hence for the determination of solution,

Let $y=vx$, $\therefore$ $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$. and $\dfrac{y}{x}=v$, $\dfrac{y^2}{x^2}=v^2$.

Hence the given equation reduces to

$$2\left(v+x\frac{dv}{dx}\right)=v+v^2, \text{ or, } 2v+2x\frac{dv}{dx}=v+v^2,$$

$$\text{or, } 2x\frac{dv}{dx}=v^2-v$$

$$\text{or, } \frac{2dv}{v^2-v}=\frac{dx}{x}, \text{ or, } \int\frac{2dv}{v^2-v}=\int\frac{dx}{x} \quad ...(1)$$

$$\text{Now, } \int\frac{2dv}{v^2-v}=\int\frac{2dv}{v(v-1)}=2\int\left(\frac{1}{v-1}-\frac{1}{v}\right)dv$$

$$=2\log(v-1)-2\log v=2\log\frac{v-1}{v}.$$

$\therefore$ From (1), $2\log\dfrac{v-1}{v}=\log x+\log c$

or, $\log\left(\frac{v-1}{v}\right)^2=\log cx$, or, $\left(\frac{\frac{y}{x}-1}{\frac{y}{x}}\right)^2=cx$

or, $\frac{(y-x)^2}{y^2}=cx$, or, $(y-x)^2=cxy^2$.

§ 2·5 Solution of equations of the form

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\quad\left(\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\right)$$

Equations of the above form can be solved by reducing them to homogeneous equations. To reduce the equation in the homogeneous form,

put $x=x'+h$ and $y=y'+k$, where h and k are to be so selected that the new equation after substitution does not contain any term independent of x' and y'.

If $x=x'+h$ and $y=y'+k$, then

$a_1x+b_1y+c_1$ and $a_2x+b_2y+c_2$

reduce respectively to the forms

$a_1x'+b_1y'+a_1h+b_1k+c_1$ and $a_2x'+b_2y'+a_2h+b_2k+c_2$

As in the new form there will be no term independent of x' and y' ;

so $a_1h+b_1k+c_1=0\cdots(1)$
and $a_2h+b_2k+c_2=0\cdots(2)$

as $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ or, $a_1b_2-a_2b_1\neq0$

So the equations (1) and (2) can be solved simultaneously and solving we get,

$$\frac{h}{b_1c_2-b_2c_1}=\frac{k}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

or, $h=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$ and $k=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$

For these values of h and k, $a_1x+b_1y+c_1$ and $a_2x+b_2y+c_2$ will reduce to the forms $a_1x'+b_1y'$ and $a_2x'+b_2y'$.

Again as $x=x'+h$ and $y=y'+k$.

$\therefore \quad dx=dx'$ and $dy=dy'$.

Hence the equation reduces to the form

$$\frac{dy'}{dx'}=\frac{a_1x'+b_1y'}{a_2x'+b_2y'}$$

This equation is homogeneous in x' and y' and can be solved by the method of § 2·4 by the sustitution $y'=vx'$, but the solution will be in terms of x' and y'. Hence putting $x'=x-h$ and $y'=y-k$, the general solution should be expressed in terms of x and y.

Example. Solve :

$$(2x+3y-5)\frac{dy}{dx}+(3x+2y-5)=0.$$ [C. U. 1962]

or, $\dfrac{dy}{dx}=-\dfrac{3x+2y-5}{2x+3y-5}$.

Now let, $x=x'+h$ and $y=y'+k$.

$\therefore \quad dx=dx'$ and $dy=dy'$ So, the equation reduces to

$$\frac{dy'}{dx'}=-\frac{3x'+2y'+3h+2k-5}{2x'+3y'+2h+3k-5}$$

Now select h and k so that

$3h+2k-5=0 \cdots (1)$

and $2h+3k-5=0 \cdots (2)$

Solving equations (1) and (2)

$$\frac{h}{-10+15}=\frac{k}{-10+15}=\frac{1}{9-4}$$

or, $h=\frac{5}{5}=1$ and $k=\frac{5}{5}=1$

$\therefore$ Let $x=x'+1$ and $y=y'+1$ and the equation reduces to

$$\frac{dy'}{dx'}=-\frac{3x'+2y'}{2x'+3y'}$$

Now this is a homogeneous equation. To solve it

let, $y'=vx'$; $\therefore \quad \dfrac{dy'}{dx'}=v+x'\dfrac{dv}{dx'}$

$$\therefore \quad v+x'\frac{dv}{dx'}=-\frac{3x'+2vx'}{2x'+3vx'}=-\frac{3+2v}{2+3v}$$

or, $$x'\frac{dv}{dx'}=-\frac{3+2v}{2+3v}-v=-\frac{3+4v+3v^2}{2+3v}$$

or, $$\frac{3v+2}{3v^2+4v+3}dv=-\frac{dx'}{x'}$$

Now integrating both sides we get,

$$\frac{1}{2}\int\frac{du}{u}=-\int\frac{dx'}{x'} \qquad \left[\text{ Taking } u=3v^2+4v+3, \right]$$

or, $\log u^{\frac{1}{2}}=-\log x'+\log c$

or, $\log(3v^2+4v+3)^{\frac{1}{2}}+\log x'=\log c,$

or, $\log(3v^2+4v+3)^{\frac{1}{2}}x'=\log c$

or, $(3v^2+4v+3)x'^2=c^2=A$

or, $\left(3\frac{y'^2}{x'^2}+4\frac{y'}{x'}+3\right)x'^2=A \quad \left[\because \quad y'=vx', \quad \therefore \quad v=\frac{y'}{x'}\right]$

or, $3y'^2+4x'y'+3x'^2=A$

or, $3(y-1)^2+4(x-1)(y-1)+3(x-1)^2=A$

$[\because \quad x=x'+1, \quad \therefore \quad x'=x-1;$

and $y=y'+1, \quad \therefore \quad y'=y-1]$

and this is the required solution.

§ 2·6. Solution of equations of the form $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ when $\frac{a_1}{a_2}=\frac{b_1}{b_2}$

If $\frac{a_1}{a_2}=\frac{b_1}{b_2}$, the equation $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ cannot be solved by the method of the preceding aricle. For, as $\frac{a_1}{a_2}=\frac{b_1}{b_2}$

or, $a_1b_2-a_2b_1=0$, so the values of h and k cannot be determined. In this case follow the method as shown below.

Method : Let $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{1}{m}$.

The equation reduces to

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{m(a_1x+b_1y)+c_2} \quad \cdots(2)$$

Now, let $a_1x+b_1y=z$.

$$\therefore \quad \frac{dz}{dx}=a_1+b_1\frac{dy}{dx} \text{ or, } \frac{dy}{dx}=\frac{1}{b_1}\left(\frac{dz}{dx}-a_1\right)$$

Hence the equation-(2) will reduce to the form.

$$\frac{1}{b_1}\left(\frac{dz}{dx}-a_1\right)=\frac{z+c_1}{mz+c_2}$$

or, $$\frac{dz}{dx}=a_1+\frac{b_1(z+c_1)}{mz+c_2}=\frac{z(a_1m+b_1)+c_2a_1+b_1c_1}{mz+c_2}$$

or, $$\frac{(mz+c_2)dz}{z(a_1m+b_1)+c_2a_1+b_1c_1}=dx.$$

Now the variables are separated and the equation can be solved by integrating both sides.

Example. Find the general solution :

$$(2x+4y+3)\frac{dy}{dx}=2y+x+1.$$

Given equation is $(2x+4y+3)\dfrac{dy}{dx}=2y+x+1$,

or, $$\frac{dy}{dx}=\frac{x+2y+1}{2x+4y+3}$$

or, $$\frac{dy}{dx}=\frac{x+2y+1}{2(x+2y)+3} \qquad \cdots(1)$$

Now let, $x+2y=v$ $\therefore \quad 1+2\dfrac{dy}{dx}=\dfrac{dv}{dx}$

or, $$\frac{dy}{dx}=\frac{1}{2}\left(\frac{dv}{dx}-1\right),$$

from (1), $$\frac{1}{2}\left(\frac{dv}{dx}-1\right)=\frac{v+1}{2v+3}$$

or, $$\frac{dv}{dx}-1=\frac{2v+2}{2v+3}, \text{ or, } \frac{dv}{dx}=\frac{2v+2}{2v+3}+1=\frac{4v+5}{2v+3}$$

or, $$\frac{(2v+3)dv}{4v+5}=dx, \quad \text{or, } \left\{\frac{1}{2}\cdot\frac{4v+5+1}{4v+5}\right\}dv=dx$$

or, $$\left\{\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{4v+5}\right\}dv=dx,$$

or, $$\int\frac{1}{2}dv+\frac{1}{2}\int\frac{dv}{4v+5}=\int dx.$$

or, $\frac{1}{2}v+\frac{1}{8}\log(4v+5)=x+c'$

or, $\frac{1}{2}(x+2y)+\frac{1}{8}\log(4x+8y+5)=x+c'$

or, $3x+8y+\log(4x+8y+5)=c.$

Examples 2

Ex. **1.** Find the general solution of

(i) $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

(ii) $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dx=0$ [Joint Entrance, 1982]

(iii) $x\sqrt{y^2-1}\,dx-y\sqrt{x^2-1}\,dx=0$ [H. S. 1979]

(iv) $y(1+x)dx+x(1+y)dy=0$ [H. S. '78, 1984]

(i) $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$ or, $\frac{dy}{\sqrt{1-y^2}}+\frac{dx}{\sqrt{1-x^2}}=0$

or, $\int\frac{dy}{\sqrt{1-y^2}}+\int\frac{dx}{\sqrt{1-x^2}}=c'$ or, $\sin^{-1}y+\sin^{-1}x=\sin^{-1}c$

or, $\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})=\sin^{-1}c.$

$\therefore\ x\sqrt{1-y^2}+y\sqrt{1-x^2}=c$

and this is the required general solution.

(ii) $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dx=0$

or, $\frac{\sec^2 x}{\tan x}dx+\frac{\sec^2 y}{\tan y}dy=0$

or, $\int\frac{\sec^2 x}{\tan x}dx+\int\frac{\sec^2 y}{\tan y}\,dy=c'$ ··· (i)

Now $\int\frac{\sec^2 x\,dx}{\tan x}=\int\frac{du}{u}$

[$\tan x=u$ (say) $\therefore\ \sec^2 x\,dx=du$]

$=\log u=\log\tan x$

similarly $\int\frac{\sec^2 y}{\tan y}=\log\tan y.$

$\therefore$ From (i) we get

$\log\tan x+\log\tan y=\log c\ [c'=\log c]$

or, $\log(\tan x \tan y)=\log c$

or, $\tan x \tan y = c.$

(iii) $x\sqrt{y^2-1}\,dx - y\sqrt{x^2-1}\,dy=0$

or, $\dfrac{x\,dx}{\sqrt{x^2-1}}=\dfrac{y\,dy}{\sqrt{y^2-1}}$

or, $\displaystyle\int\frac{x\,dx}{\sqrt{x^2-1}}=\int\frac{y\,dy}{\sqrt{y^2-1}}+c$

Now let $x^2-1=u^2$

$\therefore\ 2x\,dx=2u\,du$ or, $x\,dx=u\,du$

$\therefore\ \displaystyle\int\frac{x\,dx}{\sqrt{x^2-1}}=\int\frac{u\,du}{u}=\int du=u=\sqrt{x^2-1}$

similarly $\displaystyle\int\frac{y\,dy}{\sqrt{y^2-1}}=\sqrt{y^2-1}$

Hence the required general solution is $\sqrt{x^2-1}=\sqrt{y^2-1}+c.$

(iv) $y(1+x))dx+x(1+y)dy=0$

or, $\dfrac{1+x}{x}dx+\dfrac{1+y}{y}dy=0$ or, $\dfrac{1}{x}dx+dx+\dfrac{1}{y}dy+dy=0$

or $\displaystyle\int\frac{1}{x}dx+\int dx+\int\frac{1}{y}dy+\int dy=c$ or, $\log x+x+\log y+y=c$

or, $\log(xy)+x+y=c.$

Ex 2. Find the general solutions :—

(i) $y(1+x^2)dy=x(1+y^2)dx$ [H. S. 1980 ; 1981]

(ii) $x\cos^2 y\,dx-y\cos^2 x\,dy=0$

[H S. 1985 ; Join Entrance, 1980]

(iii) $\sin x\dfrac{dy}{dx}+y=y^2$ [H. S. 1987]

(iv) $\sqrt{1-y^2}\,dx+y^2\sqrt{1-x^2}\,dy=0.$ [H. S. 1984]

(i) $y(1+x^2)dy=x(1+y^2)dx$

or, $\dfrac{y\,dy}{1+y^2}=\dfrac{x\,dx}{1+x^2}$ or, $\displaystyle\int\frac{y\,dy}{1+y^2}=\int\frac{x\,dx}{1+x^2}+c'$

Now let $1+y^2=u$ $\therefore\ 2y\,dy=du$ or, $y\,dy=\dfrac{du}{2}$

$\therefore\ \displaystyle\int\frac{y\,dy}{1+y^2}=\frac{1}{2}\int\frac{du}{u}=\frac{1}{2}\log u=\tfrac{1}{2}\log(1+y^2)$

similarly $\int \frac{x\,dx}{1+x^2}=\frac{1}{2}\log(1+x^2)$

$\therefore\quad \frac{1}{2}\log(1+y^2)=\frac{1}{2}\log(1+x^2)+\frac{1}{2}\log c \qquad [c'=\frac{1}{2}\log c]$

or, $\log\frac{1+y^2}{1+x^2}=\log c$ or, $\frac{1+y^2}{1+x^2}=c.$ or, $1+y^2=c(1+x^2)$.

(ii) $x\cos^2 y\,dy-y\cos^2 x\,dx=0.$ or, $x\cos^2 y\,dy=y\cos^2 x\,dy$

or, $\frac{x}{\cos^2 x}dx=\frac{y}{\cos^2 y}dy$ or, $x\sec^2 x\,dx=y\sec^2 y\,dy$

or, $\int x\sec^2 x dx=\int y\sec^2 y dy$

or, $x\int\sec^2 x\,dx-\int\left\{\frac{d}{dx}(x)\int\sec^2 x\,dx\right\}dx$

$=y\int\sec^2 y\,dy-\int\left\{\frac{d}{dy}(y)\int\sec^2 y\,dy\right\}dy+c$

or, $x\tan x-\int 1.\tan x=y\tan y-\int 1.\tan y\,dy+c$

or, $x\tan x-\int\tan x\,dx=y\tan y-\int\tan y\,dy+c$

or, $x\tan x-\log|\sec x|=y\tan y-\log|\sec y|+c$

or, $x\tan x-y\tan y=\log|\sec x|-\log|\sec y|+c$

(iii) $\sin x\frac{dy}{dx}+y=y^2$

or, $\sin x\frac{dy}{dx}=y^2-y$ or, $\frac{dy}{y(y-1)}=\frac{dx}{\sin x}$

or, $\left\{\frac{1}{y-1}-\frac{1}{y}\right\}dy=\operatorname{cosec} x\,dx$

or, $\int\frac{1}{y-1}\,dy-\int\frac{1}{y}\,dy=\int\operatorname{cosec} x\,dx$

or, $\log(y-1)-\log y=\log|\tan\frac{x}{2}|+\log c'$

or, $\log\frac{y-1}{y}=\log(c'|\tan\frac{x}{2}|)$

or, $\frac{y-1}{y}=c'|\tan\frac{x}{2}|$ or, $(y-1)^2=c^2y^2\tan^2\frac{x}{2}.$

(iv) $\sqrt{1-y^2}\,dx+y^2\sqrt{1-x^2}\,dy=0.$

or, $\frac{dx}{\sqrt{1-x^2}}+\frac{y^2dy}{\sqrt{1-y^2}}=0$

or, $\dfrac{dx}{\sqrt{1-x^2}}-\dfrac{1-y^2-1}{\sqrt{1-y^2}}dy=0$

or, $\dfrac{dx}{\sqrt{1-x^2}}-\sqrt{1-y^2}\,dy+\dfrac{dy}{\sqrt{1-y^2}}=0$

or, $\displaystyle\int\frac{dx}{\sqrt{1-x^2}}+\int\frac{dy}{\sqrt{1-y^2}}=\int\sqrt{1-y^2}\,dy$

or, $\sin^{-1}x+\sin^{-1}y=\dfrac{y\sqrt{1-y^2}}{2}+\frac{1}{2}\sin^{-1}y+c$

or, $2\sin^{-1}x+\sin^{-1}y=y\sqrt{1-y^2}+c$

Ex. 3. Find the general solutions of

(i) $e^{x-y}dx+e^{y-x}dy=0.$

[Joint Entrance, 1984]

(ii) $\log\dfrac{dy}{dx}=3x-5y$

[H. S. 1986]

(iii) $\dfrac{dy}{dx}=\dfrac{y^2-y-2}{x^2+2x-3}$

(iv) $\dfrac{dy}{dx}+\dfrac{y^2+y+1}{x^2+x+1}=0$

(i) $e^{x-y}dx+e^{y-x}dy=0$ or, $\dfrac{e^x}{e^y}dx+\dfrac{e^y}{e^x}dy=0$

or, $e^{2x}dx+e^{2y}dy=0$ or, $\int e^{2x}dx+\int e^{2y}dy=\frac{c}{2}$

or, $\dfrac{e^{2x}}{2}+\dfrac{e^{2y}}{2}=\dfrac{c}{2}$ or, $e^{2x}+e^{2y}=c$

(ii) $\log\dfrac{dy}{dx}=3x-5y$

or, $\dfrac{dy}{dx}=e^{3x-5y}=\dfrac{e^{3x}}{e^{5y}}$ or, $e^{3x}dx=e^{5y}dy$

or, $\int e^{3x}dx=\int e^{5y}dy+c$ or, $\dfrac{e^{3x}}{3}=\dfrac{e^{5y}}{5}+c.$

(iii) $\dfrac{dy}{dx}=\dfrac{y^2-y-2}{x^2+2x-3}$

or, $\dfrac{dy}{(y-2)(y+1)}=\dfrac{dx}{(x+3)(x-1)}$

or, $\dfrac{1}{3}\left\{\dfrac{1}{y-2}-\dfrac{1}{y+1}\right\}dy=\dfrac{1}{4}\left\{\dfrac{1}{x-1}-\dfrac{1}{x+3}\right\}dx$

or, $\frac{1}{3}\left\{\int\frac{dy}{y-2}-\int\frac{dy}{y+1}\right\}=\frac{1}{4}\left\{\int\frac{dx}{x-1}-\int\frac{dx}{x+3}\right\}$

or, $\frac{1}{3}\log\frac{y-2}{y+1}=\frac{1}{4}\log\frac{x-1}{x+3}+\log c'$

or, $4\log\frac{y-2}{y+1}=3\log\frac{x-1}{x+3}+\log c$

or, $\log\left(\frac{y-2}{y+1}\right)^4=\log\left\{c\left(\frac{x-1}{x+3}\right)^3\right\}$

or, $\left(\frac{y-2}{y+1}\right)^4=c\left(\frac{x-1}{x+3}\right)^3.$

(iv) $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$

or, $\frac{dy}{y^2+y+1}+\frac{dx}{x^2+x+1}=0$

or, $\int\frac{dy}{y^2+y+1}+\int\frac{dx}{x^2+x+1}=c'$

or, $\int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=c'$

or, $\frac{2}{\sqrt{3}}\tan^{-1}\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{2c}{\sqrt{3}}$

or, $\tan^{-1}\frac{2y+1}{\sqrt{3}}+\tan^{-1}+\frac{2x+1}{\sqrt{3}}=c.$

Ex. 4. Find the particular solutions of the following equations :

(i) $\frac{dy}{dx}=-2y$ (when $x=0$ then $y=2$) [H.S. 1978]

(ii) $\tan x\frac{dy}{dx}=1+y^2$ (when $x=\frac{\pi}{2}$, $y=1$) [H.S. 1980]

(iii) $\tan x\frac{dy}{dx}=\tan y$ $\left(\text{when } x=\frac{\pi}{6},\ y=\frac{\pi}{3}\right)$ [H.S. 1981]

(iv) $\tan x\,dy-\tan y\,dx=0$

given $y=\frac{\pi}{2}$ when $x=\frac{\pi}{4}$ [H. S. 1982]

(v) $\frac{dy}{dx}=y\sec x.$ $y=1$ when $x=\frac{\pi}{6}$ [H. S. 1983]

(i) $\frac{dy}{dx}=-2y$ বা, $\frac{dy}{y}=-2dx$

or, $\int\frac{dy}{y}=-2\int dx$ or, $\log y=-2x+c.$

when $x=0$, then $y=2$. $\therefore$ $\log 2=c$

$\therefore$ The required solution is $\log y=-2x+\log 2$

or, $\log\frac{y}{2}=-2x$ or, $\frac{y}{2}=e^{-2x}$ or, $y=2e^{-2x}$

(ii) $\tan x\frac{dy}{dx}=1+y^2$

or, $\int\frac{dy}{1+y^2}=\int\cot x\,dx$ or, $\tan^{-1}y=\log\sin x+\log c$

Now when $x=\frac{\pi}{2}$, then $y=1$, $\therefore$ $\tan^{-1}1=\log\sin\frac{\pi}{2}+\log c$

or, $\frac{\pi}{4}=\log 1+\log c$ $\therefore$ $\log c=\frac{\pi}{4}$ [$\because$ $\log 1=0$]

$\therefore$ $\tan^{-1}y=\log\sin x+\frac{\pi}{4}.$

which is the required solution.

(iii) $\tan x\frac{dy}{dx}=\tan y$

or, $\cot y\,dy=\cot x\,dx$ or, $\int\cot y\,dy=\int\cot x\,dx$

or, $\log\sin y=\log\sin x+\log c=\log(c\sin x)$

or, $\sin y=c\sin x$

Now when $x=\frac{\pi}{6}$, then $y=\frac{\pi}{3}$.

$\therefore$ $\sin\frac{\pi}{3}=c\sin\frac{\pi}{6}$ or, $\frac{\sqrt{3}}{2}=c.\frac{1}{2}$ $\therefore$ $c=\sqrt{3}$

$\sin y=\sqrt{3}\sin x$. which is the required solution.

(iv) $\tan x\,dy-\tan y\,dx=0$

By (iii) above, $\sin y=c\sin x$. is the general solution of the equation.

Now $y=\frac{\pi}{2}$, when $x=\frac{\pi}{4}$.

$\therefore$ $\sin\frac{\pi}{2}=c\sin\frac{\pi}{4}$ or, $1=c.\frac{1}{\sqrt{2}}$ or, $c=\sqrt{2}$

Hence the required solution is $y=\sqrt{2}\sin x$.

(v) $\dfrac{dy}{dx}=y \sec x$

ro, $\dfrac{dy}{y}=\sec x\, dx$ or, $\displaystyle\int\frac{dy}{y}=\int \sec x\ dx$

or, $\log y=\log(\sec x+\tan x)+\log c=\log\{c(\sec x+\tan x)\}$

or, $y=c(\sec x+\tan x)$.

Now $y=1$ when $x=\frac{\pi}{6}$,

$$\therefore\quad 1=c\left(\sec\frac{\pi}{6}+\tan\frac{\pi}{6}\right)=c\left(\frac{2}{\sqrt3}+\frac{1}{\sqrt3}\right)$$

$$=c\left(\frac{3}{\sqrt3}\right)=\sqrt3 c,\quad\therefore\quad c=\frac{1}{\sqrt3}.$$

So, the required solution is $y=\dfrac{1}{\sqrt3}(\sec x+\tan x)$

or, $\sqrt3 y=\sec x+\tan x$.

Ex. 5. Find the particular solutions

(i) $\dfrac{dy}{dx}=\dfrac{3x^2+1}{4y+2}$ ($y=1$ when $x=1$) [H.S. 1980]

(ii) $\dfrac{dy}{dx}=\dfrac{1+y^2}{xy}$ ($y=0$ when $x=1$) [H. S. 1985]

(iii) $(1-x^2)\dfrac{dy}{dx}=2y$ ($y=1$, when $x=2$) [H. S 1981]

(iv) $\log\dfrac{dy}{dx}=4x-2y\quad 2$; given $y=1$ when $x=1$ [H. S. 1983]

(v) $(e^x+1)y\,dy=(y^2+1)\,e^x dx$; given $y=0$ when $x=0$ [H. S. 1987]

(vi) $a^2y\dfrac{dy}{dx}+b^2x=0$; given $y=\dfrac{b}{\sqrt2}$ when $x=\dfrac{a}{\sqrt2}$ [H. S. 1988]

(vii) $v\dfrac{dv}{dx}+n^2x=0$; given $v=u$, when $x=a$. H. S. 1988]

(i) $\dfrac{dy}{dx}=\dfrac{3x^2+1}{4y+2}$ or, $(3x^2+1)dx=(4y+2)dy$

or, $\int(3x^2+1)dx=\int(4y+2)dy$ or, $x^3+x=2y^2+2y+c$.

Now $y=1$ when $x=1$.

$\therefore\quad 1+1=2+2+c \quad \therefore\quad c=-2$

$\therefore\quad x^3+x=2y^2+2y-2$ is the required solution.

(ii) $\dfrac{dy}{dx}=\dfrac{1+y^2}{xy}$ or, $\dfrac{ydy}{1+y^2}=\dfrac{dx}{x}$

or, $\displaystyle\int\frac{ydy}{1+y^2}=\int\frac{dx}{x}$ or, $\frac{1}{2}\log(1+y^2)=\log x+\log c'$

$\log(1+y^2)^{\frac{1}{2}}=\log(c'x)$

or, $(1+y^2)^{\frac{1}{2}}=c'x$ or, $1+y^2=cx^2$ $[\,c'^2=c\,]$

Now when $x=1$, then $y=0$.

$\therefore\quad 1=c.1$ or, $c=1$

$\therefore\quad 1+y^2=x^2$ or, $x^2-y^2=1$ is the required solution.

(iii) $(1-x^2)\dfrac{dy}{dx}=2y$

or, $\dfrac{dy}{2y}=\dfrac{dx}{1-x^2}$ or, $\displaystyle\frac{1}{2}\int\frac{dy}{y}=\int\frac{dx}{1-x^2}$

or, $\frac{1}{2}\log y=\frac{1}{2}\log\left|\dfrac{1+x}{1-x}\right|+\frac{1}{2}\log c$

or, $\log y=\log\left|\dfrac{1+x}{1-x}\right|+\log c=\log\left|c\left(\dfrac{1+x}{1-x}\right)\right|$

$\therefore\quad y=c\left|\dfrac{1+x}{1-x}\right|$

Now $y=1$, when $x=2$, $\therefore\quad 1=c\left|\dfrac{1+2}{1-2}\right|=3c \quad\therefore\quad c=\frac{1}{3}$.

$\therefore\quad y=\dfrac{1}{3}\left(\dfrac{x+1}{x-1}\right)$ is the required solution

(iv) $\log\dfrac{dy}{dx}=4x-2y-2$

or, $\dfrac{dy}{dx}=e^{4x-2y-2}=\dfrac{e^{4x}}{e^{2y+2}}$

or, $e^{4x}dx=e^{2(y+1)}dy$ or, $\int e^{4x}dx=\int e^{2(y+1)}dy$

or, $\dfrac{e^{4x}}{4}=\dfrac{e^{2(y+1)}}{2}+\dfrac{c}{4}$ or, $e^{4x}=2e^{2(y+1)}+c$.

Now $y=1$ when $x=1$,

$\therefore \quad e^4=2e^4+c$ or, $c=-e^4$.

$\therefore \quad e^{4x}=2e^{2(y+1)}-e^4$ is the required solution.

(v) $(e^x+1)y\,dy=(y^2+1)e^x\,dx$

or $\dfrac{e^x dx}{e^x+1}=\dfrac{y\,dy}{y^2+1}$ or, $\displaystyle\int\frac{e^x dx}{e^x+1}=\int\frac{y\,dy}{y^2+1}$

or, $\log(e^x+1)=\frac{1}{2}\log(y^2+1)+\log c'$

[$e^x+1=u$ or, $e^x dx=dy$ and $y^2+1=v$ or, $y\,dy=\frac{1}{2}dv$]

or, $\log\dfrac{e^x+1}{c'}=\log\,(y^2+1)^{\frac{1}{2}}$ or, $\dfrac{e^x+1}{c'}=(y^2+1)^{\frac{1}{2}}$

or, $\dfrac{(e^x+1)^2}{c}=y^2+1$ $[(c'^2=c$

or, $(e^x+1)^2=c(y^2+1)$.

Now when $x=0$, then $y=0$

$\therefore \quad (1+1)^2=c\,(1)^2$ [$\because \; e^0=1$]

or, $c=4$.

$\therefore \quad (e^x+1)^2=4(y^2+1)$ is the required solution.

(vi) $a^2y\,\dfrac{dy}{dx}+b^2x=0$

or, $a^2y\,dy+b^2x\,dx=0$ or, $\displaystyle\int a^2y\,dy+\int b^2x\,dx=\frac{c}{2}$

or, $\dfrac{a^2y^2}{2}+\dfrac{b^2x^2}{2}=\dfrac{c}{2}$ or, $a^2y^2+b^2x^2=c$.

Now $y=\dfrac{b}{\surd 2}$, when $x=\dfrac{a}{\surd 2}$

$\therefore \quad \dfrac{a^2b^2}{2}+\dfrac{b^2a^2}{2}=c$ or, $c=a^2b^2$

So, $a^2y^2+b^2x^2=a^2b^2$

or, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is the required solution.

(vii) $v\,\dfrac{dv}{dx}+n^2x=0$

or, $v\,dv+n^2x\,dx=0$ or, $\displaystyle\int v\,dv+\int n^2x\,dx=\frac{c}{2}$

or, $\dfrac{v^2}{2}+\dfrac{n^2x^2}{2}=\dfrac{c}{2}$

or, $v^2+n^2x^2=c$. Now given that $v=u$ when $x=a$

$\therefore\ u^2+n^2a^2=c.$

$\therefore\ v^2+n^2x^2=u^2+n^2a^2$ is the required solution.

Ex-6. 6. Solve :

(i) $\dfrac{dy}{dx}=(x+y)^2$, given that $y=1$ when $x=0$

[Joint Entrance, 1978 ; H.S. '84]

(ii) $(x+y)^2\dfrac{dy}{dx}=a^2$ [Joint Entrance, 980]

(iii) $(x-y)^2\dfrac{dy}{dx}=1$ [Joint Entrance, 1 83]

(iv) $\dfrac{dy}{dx}=\sqrt{y-x}$ [H. S. '78 ; Joint Entrance, '79, '81]

(i) Let $x+y=z$

$\therefore\ 1+\dfrac{dy}{dx}=\dfrac{dz}{dx}$ or, $\dfrac{dy}{dx}=\dfrac{dz}{dx}-1$

$\therefore\ \dfrac{dz}{dx}-1=z^2$ or, $\dfrac{dz}{dx}=1+z^2$

or, $\dfrac{dz}{1+z^2}=dx$ or, $\displaystyle\int\frac{dz}{1+z^2}=\int dx$

or, $\tan^{-1}z=x+c$

or, $\tan^{-1}(x+y)=x+c$ and this is the general solution.

Now $x=0$, when $y=1$

$\therefore\ \tan^{-1}1=0+c$ or, $c=\dfrac{\pi}{4}$. $\therefore\ \tan^{-1}(x+y)=x+\dfrac{\pi}{4}$

or, $x+y=\tan\left(x+\dfrac{\pi}{4}\right)$

(ii) Let $x+y=z$ $\therefore\ 1+\dfrac{dy}{dx}=\dfrac{dz}{dx}$ or, $\dfrac{dy}{dx}=\dfrac{dz}{dx}-1$

$\therefore\ z^2\left(\dfrac{dz}{dx}-1\right)=a^2$ or, $\dfrac{dz}{dx}=\dfrac{a^2}{z^2}+1=\dfrac{a^2+z^2}{z^2}$

or, $\dfrac{z^2dz}{z^2+a^2}=dx$ or, $\dfrac{z^2+a^2-a^2}{z^2+a^2}dz=dx$

or, $\displaystyle\int dz-a^2\int\frac{dz}{z^2+a^2}=\int dx$

or, $z-\dfrac{a^2}{a}\tan^{-1}\dfrac{z}{a}=x+c$

or, $x+y-a\tan^{-1}\dfrac{x+y}{a}=x+c$

or, $y=a\tan^{-1}\left(\dfrac{x+y}{a}\right)+c.$

(iii) Let $x-y=z$ $\quad\therefore\quad 1-\dfrac{dy}{dx}=\dfrac{dz}{dx}$

or, $\dfrac{dy}{dx}=1-\dfrac{dz}{dx}.$

$\therefore\quad z^2\left(1-\dfrac{dz}{dx}\right)=1$ or, $1-\dfrac{dz}{dx}=\dfrac{1}{z^2}$

or, $\dfrac{dz}{dx}=1-\dfrac{1}{z^2}=\dfrac{z^2-1}{z^2}$

or, $\dfrac{z^2dz}{z^2-1}=dx$ or, $\int\left(\dfrac{z^2-1+1}{z^2-1}\right)dz=\int dx.$

or, $\int dz+\int\dfrac{dz}{z^2-1}=\int dx$ or, $z+\frac{1}{2}\log\left|\dfrac{z-1}{z+1}\right|=x+c$

or, $x-y+\frac{1}{2}\log\left|\dfrac{x-y-1}{x-y+1}\right|=x+c$

or, $y=\frac{1}{2}\log\left|\dfrac{x-y-1}{x-y+1}\right|+c$

(iv) Let $y-x=z^2$

$\therefore\quad \dfrac{dy}{dx}-1=2z\dfrac{dz}{dx}$ or, $\dfrac{dy}{dx}=1+2z\dfrac{dz}{dx}$

$\therefore\quad 1+2z\dfrac{dz}{dx}=z$ or, $2z\dfrac{dz}{dx}=z-1$ or, $2\dfrac{zdz}{z-1}=dx$

or, $2\int\left\{\dfrac{z-1+1}{z-1}\right\}dz=\int dx$ or, $2\int dz+\int\dfrac{dz}{z-1}=\int dx$

or, $2z+2\log(z-1)=x+c$

or, $2\sqrt{y-x}+2\log[\sqrt{y-x}-1]=x+c.$

Ex. 7. Solve :

(i) $\dfrac{dy}{dx}=\sin(x+y)$

[H. S. '82]

(ii) $\dfrac{dy}{dx}=\tan^2(x+y)$

[H. S. '88]

(iii) $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)$ [Joint Entrance, '88]

Let $x+y=z$ $\therefore$ $1+\frac{dy}{dx}=\frac{dz}{dx}$ or, $\frac{dy}{dx}=\frac{dz}{dx}-1$

$\therefore$ (i) $\frac{dz}{dx}-1=\sin z$ or, $\frac{dz}{dx}=1+\sin z$

or, $\frac{1-\sin z}{(1+\sin z)(1-\sin z)}=dx$ or, $\int\frac{1-\sin z}{\cos^2 z}\,dz=\int dx$

or, $\int\sec^2 z\,dz-\int\sec z\tan z\,dz=x+c$

or, $\tan z-\sec z=x+c$

or, $\tan(x+y)-\sec(x+y)=x+c$

(ii) $\frac{dz}{dx}-1=\tan^2 z$ or, $\frac{dz}{dx}=1+\tan^2 z=\sec^2 z$

or, $\cos^2 z\,dz=dx$ or, $\frac{1}{2}\int(1+\cos 2z)\,dz=\int dx$

or, $\frac{1}{2}\left(z+\frac{\sin 2z}{2}\right)=x+c'$ or, $\frac{1}{2}(x+y)+\frac{1}{4}\sin 2(x+y)=x+c'$

or, $\frac{1}{2}(y-x)+\frac{1}{4}\sin 2(x+y)=c'$ or, $\sin 2(x+y)+2(y-x)=c.$

(iii) $\frac{dz}{dx}-1=\sin z+\cos z$

or, $\frac{dz}{dx}=1+\sin z+\cos z$ or, $\frac{dz}{1+\cos z+\sin z}=dx$

or, $\frac{dz}{2\cos^2\frac{z}{2}+2\sin\frac{z}{2}\cos\frac{z}{2}}=dx$ or, $\frac{\frac{1}{2}\sec^2\frac{z}{2}\,dz}{1+\tan\frac{z}{2}}=dx$

or, $\int\frac{dv}{v}=\int dx$

[$1+\tan\frac{z}{2}=v$ (say) $\therefore$ $\frac{1}{2}\,\mathrm{secc}^2\frac{z}{2}\,dz=dv.$]

or, $\log v=x+\log c$ or, $\log\frac{1+\tan\frac{z}{2}}{c}=x$

or, $1+\tan\left(\frac{x+y}{2}\right)=ce^x.$

Ex. 8. Solve:

(i) $\frac{dy}{dx}=e^{x-y}+1$ [H. S. 1981]

(ii) $\dfrac{dy}{dx}+1=e^{x-y}$ [Joint Entrance, 1986]

(iii) $\dfrac{dy}{dx}=1+e^{2x-y}$ when $=x=y=2$ [H. S. 1986]

(i) Let $x-y=z$

$\therefore\ 1-\dfrac{dy}{dx}=\dfrac{dz}{dx}$ or, $\dfrac{dy}{dx}=1-\dfrac{dz}{dx}$

$\therefore\ 1-\dfrac{dz}{dx}=e^{z}+1$ or, $-\dfrac{dz}{dx}=e^{z}$

or, $-e^{-z}\,dz=dx$ or, $-\int e^{-z}\,dz=\int dx$

or, $e^{-z}=x+c$ or, $e^{-(x-y)}=x+c$

or, $e^{y-x}=x+c.$

(ii) Let $x-y=z$ $\therefore\ 1-\dfrac{dy}{dx}=\dfrac{dz}{dx}$

or, $\dfrac{dy}{dx}=1-\dfrac{dz}{dx}$ $\therefore\ 1-\dfrac{dz}{dx}+1=e^{z}$

or, $\dfrac{dz}{dx}=2-e^{z}$. or, $\dfrac{dz}{2-e^{z}}=dx.$ or, $\dfrac{e^{-z}dz}{2e^{-z}-1}=dx$

Let $2e^{-z}-1=u$ $\therefore\ -2e^{-z}dz=du$ or, $e^{-z}dz=-\dfrac{du}{2}$

$\therefore\ -\dfrac{du}{2u}=dx$ or, $-\dfrac{1}{2}\int\dfrac{du}{u}=\int dx$

or, $-\frac{1}{2}\log u=x+c$ or, $-\frac{1}{2}\log(2e^{-z}-1)=x-c'$

or, $\log(2e^{-z}-1)=-2x+2c'$

or, $2e^{-z}-1=e^{-2x+2c'}=e^{-2x}e^{2c'}$ or, $(2e^{-z}-1)e^{2x}=c$

or, $(2e^{y-x}-1)e^{2x}=c$ or, $e^{x}(2e^{y}-e^{x})=c.$

(iii) Let $2x-y=z$ $\therefore\ 2-\dfrac{dy}{dx}=\dfrac{dz}{dx}$ $\therefore\ \dfrac{dy}{dx}=2-\dfrac{dz}{dx}.$

$\therefore\ 2-\dfrac{dz}{dx}=1+e^{z}$ or, $\dfrac{dz}{dx}=1-e^{z}$

or, $\dfrac{dz}{1-e^{z}}=dx$ or, $\dfrac{e^{-z}}{e^{-z}-1}dz=dx$

or, $-\frac{du}{u}=dx$ $\quad$ [$u=e^{-z}-1$ (say) $\therefore$ $e^{-z}dz=-du$]

or, $-\log u=x-\log c$ $\quad$ or, $\log\frac{u}{c}=-x$

or, $\frac{u}{c}=e^{-x}$ or, $e^{-z}-1=ce^{-x}$ or, $e^{y-2x}-1=ce^{-x}$

or, $e^{y}-e^{2x}=ce^{x}$ [Multiplying both sides by e^{2x}]

Now $x=y=2$ $\therefore$ $e^2-e^4=ce^2$ $\therefore$ $c=1-e^2$

$\therefore$ Required solution is $e^{y}-e^{2x}=(1-e^2)e^{x}$

Ex. 9. Solve :

(i) $(x^2+y^2)dx-2xy\,dy=0$

[given $y=0$ when $x=1$] [H. S. 1982]

(ii) $(x^2+y^2)\,dy-xy\,dx=0$; $x=0$, $y=1$

[H.S. 1985 ; Joint Entrance, 1979]

Let $y=vx$ $\therefore$ $\frac{dy}{dx}=v+x\frac{dv}{dx}$.

(i) $(x^2+y^2)dx-2xydy=0$

or, $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

or, $v+x\frac{dv}{dx}=\frac{x^2+v^2x^2}{2vx^2}=\frac{1+v^2}{2v}$ $\quad$ or, $x\frac{dv}{dx}=\frac{1+v^2}{2v}-v=\frac{1-v^2}{2v}$

or, $\frac{2v\,dv}{1-v^2}=\frac{dx}{x}$ $\quad$ or, $\int\frac{2v\,dv}{1-v^2}=\int\frac{dx}{x}$

or, $-\log(i-v^2)=\log x-\log c$

or, $\log(1-v^2)+\log x=\log c$ $\quad$ or, $\log\{(1-v^2).x\}=\log c$

or, $(1-v^2).x=c$ or, $\left(1-\frac{y^2}{x^2}\right)x=c$

or, $x^2-y^2=cx$. Now $y=0$ when $x=1$, $\therefore$ $c=1$

$\therefore$ $x^2-y^2=x$. is the required solution.

(ii) $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$ $\quad$ or, $v+x\frac{dv}{dx}=\frac{vx^2}{x^2+v^2x^2}=\frac{v}{1+v^2}$

or, $\frac{x\,dv}{dx}=\frac{v}{1+v^2}-v=-\frac{v^3}{1+v^2}$ ∴ $\frac{1+v^2}{v^3}dv+\frac{dx}{x}=0$

or, $\int\frac{1+v^2}{v^3}+\int\frac{dx}{x}=\log c$

or, $-\frac{1}{2}\cdot\frac{1}{v^2}+\log v+\log x=\log c$

or, $\log\frac{vx}{c}=\frac{x^2}{2y^2}$. or, $\frac{y}{c}=\frac{x^2}{2y^2}$

or, $y=ce^{\frac{-x^2}{2y^2}}$. when $x=0$, then $y=1$

∴ $1=c$ ∴ $y=e^{\frac{x^2}{2v^2}}$.

Ex. 10. Solve :

(i) $(x+y)\frac{dy}{dx}=x-2y$ [H.S. 1980]

(ii) $(2x+y)dy=(x-2y)dx$ [H.S. 1988]

(iii) $(x^2-xy)dy=(xy+y^2)dx$ [H.S. 1986]

(iv) $x(x-y)dy+y^2dx=0$ [Joint Entrance, 1985]

Let $y=vx$ ∴ $\frac{dy}{dx}=v+x\frac{dv}{dx}$

(i) $(x+y)\frac{dy}{dx}=x-2y$

or, $\frac{dy}{dx}=\frac{x-2y}{x+y}$ or, $v+x\frac{dv}{dx}=\frac{x-2vx}{x+vx}=\frac{1-2v}{1+v}$

or, $x\frac{dv}{dx}=\frac{1-2v}{1+v}-v=\frac{1-3v-v^2}{1+v}$ or, $\frac{1+v}{1-3v-v^2}dv=\frac{dx}{x}$

or, $\int\frac{dx}{x}+\int\frac{-1-v}{1-3v-v^2}dv=c$

or, $\int\frac{dx}{x}+\frac{1}{2}\int\frac{-3-2v+1}{1-3v-v^2}dv=c$

or, $\int\frac{dx}{x}+\frac{1}{2}\int\frac{-3-2v}{1-3v-v^2}dv+\frac{1}{2}\int\frac{dv}{1-3v-v^2}=c$

or, $\int\frac{dx}{x}+\frac{1}{2}\int\frac{dz}{z}-\frac{1}{2}\int\frac{dv}{v^2+3v-1}=c$

where $1-3v-v^2=z$ $\therefore$ $(-3-2v)dv=dz$

or, $\int\frac{dx}{x}+\frac{1}{2}\int\frac{dz}{z}-\frac{1}{2}\int\frac{dv}{(v+\frac{3}{2})^2-\frac{13}{4}}=c$

or, $\log x+\frac{1}{2}\log z-\frac{1}{2}\cdot\frac{1}{2.\frac{\sqrt{13}}{2}}\log\frac{v+\frac{3}{2}-\frac{\sqrt{13}}{2}}{v+\frac{3}{2}+\frac{\sqrt{13}}{2}}=c$

or, $\log x+\log(1-3v-v^2)^{\frac{1}{2}}-\frac{1}{2\sqrt{13}}\log\frac{\frac{y}{x}+\frac{3-\sqrt{13}}{2}}{\frac{y}{x}+\frac{3+\sqrt{13}}{2}}=c$

or, $\log x+\log\left(1-\frac{3y}{x}-\frac{y^2}{x^2}\right)^{\frac{1}{2}}-\frac{1}{2\sqrt{13}}\log\frac{2y+(3-\sqrt{13})x}{2y+(3+\sqrt{13})x}=c$

or, $\log x+\log\frac{(x^2-3xy-y^2)^{\frac{1}{2}}}{x}-\frac{1}{2\sqrt{13}}\log\frac{2y+(3-\sqrt{13})x}{2y+(3+\sqrt{13})x}=c$

or, $\frac{1}{2}\log(x^2-3xy-y^2)-\frac{1}{2\sqrt{13}}\log\frac{2y+(3-\sqrt{13})x}{2y+(3+\sqrt{13})x}=c$

(ii) $(2x+y)dy=(x-2y)dx$

or, $\frac{dy}{dx}=\frac{x-2y}{2x+y}$

or, $v+x\frac{dv}{dx}=\frac{x-2vx}{2x+vx}=\frac{1-2v}{2+v}$ or, $x\frac{dv}{dx}=\frac{1-2v}{2+v}-v=\frac{1-4v-v^2}{2+v}$

or, $\frac{(2+v)dv}{v^2+4v-1}+\frac{dx}{x}=0$ or, $\frac{1}{2}\int\frac{(4+2v)\,dv}{v^2+4v-1}+\int\frac{dx}{x}=\log c'$

or, $\frac{1}{2}\log(v^2+4v-1)+\log x=\log c'$

or, $\log\left(\frac{y^2}{x^2}+\frac{4y}{x}-1\right)^{\frac{1}{2}}x=\log c'$

or, $(y^2+4xy-x^2)^{\frac{1}{2}}=c'$ or, $y^2+4xy-x^2=c.$

(iii) $\frac{dy}{dx}=\frac{xy+y^2}{x^2-xy}$

or, $v+x\frac{dv}{dx}=\frac{x^2v+x^2v^2}{x^2-x^2v}=\frac{v+v^2}{1-v}$ or, $x\frac{dv}{dx}=\frac{v+v^2}{1-v}-v=\frac{2v^2}{1-v}$

or, $\frac{1-v}{v^2}dv=2\frac{dx}{x}$ or, $\int\frac{dv}{v^2}-\int\frac{dv}{v}=2\int\frac{dx}{x}$

or, $-\frac{1}{v}-\log v=2\log x-\log c$

or, $-\frac{1}{v}=\log\frac{x^2v}{c}=\log\frac{x^2.\frac{y}{x}}{c}$

or, $-\frac{x}{y}=\log\frac{xy}{c}$ $\therefore$ $\frac{xy}{c}=e^{-\frac{x}{y}}$ or, $xy=ce^{-\frac{x}{y}}$

(iv) $x(x-y)\,dy+y^2dx=0$

or, $\frac{dy}{dx}+\frac{y^2}{x(x-y)}=0$

or, $v+x\frac{dv}{dx}+\frac{v^2x^2}{x^2-vx^2}=0$ or, $x\frac{dv}{dx}+\frac{v^2}{1-v}+v=0$

or, $x\frac{dv}{dx}+\frac{v}{1-v}=0$ or, $\frac{1-v}{v}dv+\frac{dx}{x}=0$

or, $\int\frac{dx}{x}=\int\frac{v-1}{v}dv$ or, $\log x=v-\log v+\log c$

or, $\log\frac{vx}{c}=v$ বা, $\log\frac{y}{c}=\frac{y}{x}$

or, $\frac{y}{c}=e^{\frac{y}{x}}$ or, $y=c.e^{\frac{y}{x}}$.

Ex- 11. Solve :

(i) $\frac{dy}{dx}=\frac{x+y+1}{3x+3y+1}$ [H. S. 1978]

(ii) $(6x+9y-7)\,dx=(2x+3y-6)\,dy$ [H. S. 1979]

(iii) $(2x-2y+5)\,dy=(x-y+3)\,dx$ [H. S. 1983]

(iv) $\frac{dy}{dx}=\frac{x-y+1}{2x-2y+3}$ [H. S. 1985]

(v) $\frac{dy}{dx}=\frac{x+y}{2x+2y+3}$ [H. S. 1987]

(vi) $\frac{dy}{dx}=\frac{x+y}{x+y-2}$

(i) Let $x+y=z$

$\therefore$ $1+\frac{dy}{dx}=\frac{dz}{dx}$ or, $\frac{dy}{dx}=\frac{dz}{dx}-1$

$\therefore$ $\frac{dz}{dx}-1=\frac{z+1}{3z+1}$ or, $\frac{dz}{dx}=\frac{z+1}{3z+1}+1=\frac{4z+2}{3z+1}$

or, $\frac{3z+1}{4z+2}dz=dx$ or, $\frac{3}{4}\int\frac{z+\frac{1}{3}}{z+\frac{1}{2}}dz=\int dx$

or, $\frac{3}{4}\int\frac{z+\frac{1}{2}-\frac{1}{6}}{z+\frac{1}{2}}dz=\int dx$ or, $\frac{3}{4}\int dz-\frac{1}{8}\int\frac{dz}{z+\frac{1}{2}}=\int dx$

or, $\frac{3}{4}(x+y)-\frac{1}{8}\log(z+\frac{1}{2})=x+c'$

or, $6(x+y)-\log(x+y+\frac{1}{2})=8x+8c'$

or, $6y-2x=\log(2x+2y+1)+8c'-\log 2$

or, $6y-2x=\log(2x+2y+1)+c$

(ii) $(6x+9y-7)dx=(2x+3y-6)dy$

or, $\frac{dy}{dx}=\frac{6x+9y-7}{2x+3y-6}$

Let $2x+3y=z$ $\therefore$ $2+3\frac{dy}{dx}=\frac{dz}{dx}$ or, $\frac{dy}{dx}=\frac{1}{3}\left(\frac{dz}{dx}-2\right)$

$\therefore$ $\frac{1}{3}\left(\frac{dz}{dx}-2\right)=\frac{3z-7}{z-6}$ or, $\frac{dz}{dx}-2=\frac{9z-21}{z-6}$

or, $\frac{dz}{dx}=\frac{9z-21}{z-6}+2=\frac{11z-33}{z-6}=11\left(\frac{z-3}{z-6}\right)$

o, $\frac{z-6}{z-3}dz=11dx$ or, $\int\frac{z-3-3}{z-3}dz=11\int dx$

or, $z-3\log(z-3)=11x+c$

or, $2x+3y-11x=3\log(2x+3y-3)+c$

or, $3y-9x=3\log(2x+3y-3)+c$

or, $y-3x=\log(2x+3y-3)+k.$

(iii) $(2x-2y+5)dy=(x-y+3)dx$

or, $\dfrac{dy}{dx}=\dfrac{x-y+3}{2x-2y+5}$

Let $x-y=z$ $\quad\therefore\ 1-\dfrac{dy}{dx}=\dfrac{dz}{dx}$ $\quad$ or, $\dfrac{dy}{dx}=1-\dfrac{dz}{dx}$

$\therefore\ 1-\dfrac{dz}{dx}=\dfrac{z+3}{2z+5}$ বা, $\dfrac{dz}{dx}=1-\dfrac{z+3}{2z+5}=\dfrac{z+2}{2z+5}$

$\therefore\ \dfrac{2z+5}{z+2}dz=dx$ বা, $\displaystyle\int\left(\frac{2z+4}{z+2}+\frac{1}{z+2}\right)dz=\int dx$

or, $2z+\log(z+2)=x+c$

or, $2x-2y+\log(x-y+2)=x+c$

or, $x-2y+\log(x-y+2)=c.$

(iv) Let $x-y=z$ $\quad\therefore\ 1-\dfrac{dy}{dx}=\dfrac{dz}{dx}$ $\quad$ or, $\dfrac{dy}{dx}=1-\dfrac{dz}{dx}$

Now, $\dfrac{dy}{dx}=\dfrac{x-y+1}{2x-2y+3}$ $\quad$ or, $1-\dfrac{dz}{dx}=\dfrac{z+1}{2z+3}$

or, $\dfrac{dz}{dx}=1-\dfrac{z+1}{2z+3}=\dfrac{z+2}{2z+3}$ $\quad$ or, $\dfrac{2z+3}{z+2}dz=dx$

or, $\displaystyle\int\frac{2z+4-1}{z+2}dz=\int dx$ $\quad$ or, $\displaystyle 2\int dz-\int\frac{dz}{z+2}=\int dx$

or, $2z-\log(z+2)=x+c$ or, $2(x-y)-\log(x-y+2)=x+c$

or, $x-2y=\log(x-y+2)+c.$

(v) Let $x+y=z$ $\quad\therefore\ 1+\dfrac{dy}{dx}=\dfrac{dz}{dx}$ $\quad$ or, $\dfrac{dy}{dx}=\dfrac{dz}{dx}-1$

Now $\dfrac{dy}{dx}=\dfrac{x+y}{2x+2y+3}$ $\quad$ or, $\dfrac{dz}{dx}-1=\dfrac{z}{2z+3}$

or, $\dfrac{dz}{dx}=\dfrac{z}{2z+3}+1=\dfrac{3z+3}{2z+3}=\dfrac{3}{2}\left(\dfrac{z+1}{z+\frac{3}{2}}\right)$

or, $\frac{2}{3}\left(\frac{z+\frac{3}{2}}{z+1}\right)dz=dx$ or, $\frac{2}{3}\left(\frac{z+1+\frac{1}{2}}{z+1}\right)dz=dx$

or, $\frac{2}{3}\int\left(1+\frac{1}{2(z+1)}\right)dz=\int dx$ or, $\frac{2}{3}z+\frac{1}{3}\log(z+1)=x+c'$

or, $2z+\log(z+1)=3x+c$

or, $2x+2y+\log(x+y+1)=3x+c$

or, $2y-x+\log(x+y+1)=c.$

(vi) $\frac{dy}{dx}=\frac{x+y}{x+y-2}$

Let $x+y=z$ $\therefore$ $1+\frac{dy}{dx}=\frac{dz}{dx}$ or, $\frac{dy}{dx}=\frac{dz}{dx}-1$

$\therefore$ $\frac{dz}{dx}-1=\frac{z}{z-2}$ or, $\frac{dz}{dx}=\frac{z}{z-2}+1=\frac{2z-2}{z-2}$

or, $\frac{z-2}{2z-2}dz=dx$ or, $\frac{1}{2}\left(\frac{z-2}{z-1}\right)dz=dx$

or, $\frac{1}{2}\int\left(1-\frac{1}{z-1}\right)dz=\int dx$

or, $\frac{1}{2}z-\frac{1}{2}\log(z-1)=x+c'$ or, $z-\log(z-1)=2x+c$

or, $x+y-\log(x+y-1)=2x+c$ or, $y-x=\log(x+y-1)+c.$

Ex. 12. Solve :

(i) $x\frac{dy}{dx}=y+x\tan\left(\frac{y}{x}\right)$ [H.S. 1981 ; 1983]

(ii) $\left\{x+y\cos\left(\frac{y}{x}\right)\right\}dx=x\cos\left(\frac{y}{x}\right)dy.$ [Joint Entrance, 1987]

(iii) $(1+e^{\frac{x}{y}})\,dx+e^{\frac{x}{y}}(1-\frac{x}{y})dy=0$

(i) $x\frac{dy}{dx}=y+x\tan\left(\frac{y}{x}\right)$ or, $\frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)$

Let $y=vx$ $\therefore$ $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ $v+x\frac{dv}{dx}=v+\tan v$ Let $x\frac{dv}{dx}=\tan v$

or, $\int \cot v \, dv = \int \frac{dx}{x}$; or, $\log \sin v = \log x + \log c$

or, $\log\left(\sin \frac{y}{x}\right) = \log(cx)$ or, $\sin \frac{y}{x} = cx.$

(ii) $\left\{x + y \cos\left(\frac{y}{x}\right)\right\} dx = x \cos\left(\frac{y}{x}\right) dy$

or, $$\frac{dy}{dx} = \frac{x + y\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)} = \frac{1 + \frac{y}{x}\cos\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right)}$$

Let $y = vx$ $\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{1 + v\cos v}{\cos v}$ or, $x\frac{dv}{dx} = \frac{1 + v\cos v}{\cos v} - v = \frac{1}{\cos v}$

or, $\cos v \, dv = \frac{dx}{x}$ or, $\int \cos v \, dv = \int \frac{dx}{x}$

or, $\sin v = \log x + c$ or, $\sin\left(\frac{y}{x}\right) = \log x + c.$

(iii) $(1 + e^{\frac{x}{v}}) \, dx + e^{\frac{x}{v}}\left(1 - \frac{x}{y}\right) dy = 0$

or, $$\frac{dx}{dy} \frac{e^{\frac{x}{v}}\left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{v}}} = 0$$

Let $x = vy$ or, $\frac{d}{dy}(x) = \frac{d}{dy}(vy)$ or, $\frac{dx}{dy} = v + y\frac{dv}{dy}$

$\therefore v + y\frac{dv}{dy} + \frac{e^v(1 - v)}{1 + e^v} = 0$ or, $y\frac{dv}{dy} + \frac{e^v - e^v v}{1 + e^v} + v = 0$

or, $y\frac{dv}{dy} + \frac{e^v + v}{1 + e^v} = 0$ or, $\int \frac{1 + e^v}{v + e^v} dv + \int \frac{dy}{y} = 0$

or, $\log(v + e^v) + \log y = \log c$ [$\because d(v + e^v) = 1 + e^v$]

or, $\log(v + e^v)y = \log c$ or, $\left(\frac{x}{y} + e^{\frac{x}{v}}\right)y = c$

or, $x + ye^{\frac{x}{v}} = c.$

Note : Try to understand why we have put $x = vy$ in this sum.

Ex. 13. Solve :

$x\frac{dy}{dx}+2y=\sqrt{1+x^2}$ given $y=1$, when $x=1$.

[Joint Entrance 1978]

$$x\frac{dy}{dx}+2y=\sqrt{1+x^2}$$

or, $x^2\frac{dy}{dx}+2xy=x\sqrt{1+x^2}$ [Multiplying both sides by x]

or, $\frac{d}{dx}(x^2y)=x\sqrt{1+x^2}$

or, $d(x^2y)=x\sqrt{1+x^2}\,dx$

or, $\int d(x^2y)=\int x\sqrt{1+x^2}\,dx$

$=\frac{1}{2}\int\sqrt{u}\,dx$ where $1+x^2=u$

$\therefore\ 2x\,dx=du.$

or, $x^2y=\frac{1}{3}u^{\frac{3}{2}}+c$

or, $x^2y=\frac{1}{3}(1+x^2)^{\frac{3}{2}}+c$

Now when $x=1$, then $y=1$.

$\therefore\ 1=\frac{2^{\frac{3}{2}}}{3}+c$

$\therefore\ c=1-\frac{2\sqrt{2}}{3}$

So the required solution is

$$3x^2y=(1+x^2)^{\frac{3}{2}}+3-2\sqrt{2}.$$

14. Solve : $\frac{dy}{dx}=\frac{2x+9y-20}{6x+2y-10}$.

Let $x=x'+h$, $y=y'+k$, $\therefore\ dx=dx'$, $dy=dy'$

$$\therefore\ \frac{dy'}{dx'}=\frac{2(x'+h)+9(y'+k)-20}{6(x'+h)+2(y'+k)-10}$$

$$=\frac{2x'+9y'+(2h+9k-20)}{6x'+2y'+(6h+2k-10)}$$

Now if $2h+9k-20=0$ and $6h+2k-10=0$, then $h=1$ and $k=2$ and for these values of h and k, the equation takes the form,

$$\frac{dy'}{dx'}=\frac{2x'+9y'}{6x'+2y'}.$$

Now, let $y'=vx'$ $\therefore$ $\frac{dy'}{dx'}=v+x'\frac{dv}{dx'}$

$\therefore$ $v+x'\frac{dv}{dx'}==\frac{2+9v}{6+2v}$ or, $x'\frac{dv}{dx'}=\frac{2+9v}{6+2v}-v$

or, $x'\frac{dv}{dx'}=\frac{2+3v-2v^2}{6+2v}$ or, $-\frac{(2v+6)dv}{2v^2-3v-2}=\frac{dx'}{x'}$

or, $\frac{(2v+6)}{(2v+1)(v-2)}dv+\frac{dx'}{x'}=0.$

বা, $\frac{2}{v-2}dv-\frac{2}{2v+1}dv+\frac{dx'}{x'}=0$

or, integrating both sides

or, $2\log(v-2)-\log(2v+1)+\log x'=\log c$

or, $\log\frac{x'(v-2)^2}{2v+1}=\log c$, or, $x'(v-2)^2=c(2v+1)$

or, $x'\left(\frac{y'}{x'}-2\right)^2=c\left(2\frac{y'}{x'}+1\right)$ or, $(y'-2x')^2=c(x'+2y')$

or, $[(y-2)-2(x-1)]^2=c[(x-1)+2(y-2)]$

or, $(y-2x)^2=c(x+2y-5)$.

Ex. 15. Find the general solution of the following equations.

(i) $(x+y)(dx-dy)=dx+dy$

(ii) $x\,dx+y\,dy+\frac{x\,dy-y\,dx}{x^2+y^2}=0$ [H. S. 1986]

(iii) $dx-dy+y\,dx+x\,dy=0$ [H. S. 1979]

(iv) $y\,dx+x\,dy=xy(dy-dx)$ [Joint Entrance 1983]

(i) $(x+y)(dx-dy)=dx+dy$

or, $(x+y)(dx-dy)=d(x+y)$

$\therefore \quad dx-dy=\dfrac{d(x+y)}{x+y}$, or, $\quad x-y=\log(x+y)-\log c.$

[Integrating]

or, $\quad x-y=\log\dfrac{x+y}{c}$

or, $\quad \dfrac{x+y}{c}=e^{x-y}$, or, $\quad x+y=ce^{x-y}$.

(ii) $\quad xdx+ydy+\dfrac{xdy-ydx}{x^2+y^2}=0$

or, $\quad x\,dx+y\,dy+\dfrac{\dfrac{x\,dy-y\,dx}{x^2}}{\dfrac{x^2+y^2}{x^2}}=\dfrac{d\left(\dfrac{y}{x}\right)}{1+\left(\dfrac{y}{x}\right)^2}$

or, $x\,dx+y\;\;dy+\dfrac{d\left(\dfrac{y}{x}\right)}{1+\left(\dfrac{y}{x}\right)^2}=0,$

or, $\quad \dfrac{x^2}{2}+\dfrac{y^2}{2}+\tan^{-1}\dfrac{y}{x}=\dfrac{c}{2}$

or $\quad x^2+y^2+2\tan^{-1}\dfrac{y}{x}=c$

and this is the required general solution.

(iii) $\quad dx-dy+y\,dx+x\,dy=0$

or, $\quad dx-dy+d(xy)=0$

or, $\quad x-y+xy=c$ (integrating)

and this is the required general solution.

(iv) $\quad y\,dx+x\,dy=xy(dy-dx)$

or, $\quad d(xy)=xy(dy-dx)$

or, $\quad \dfrac{d(xy)}{xy}=dy-dx$

or, $\quad \displaystyle\int\frac{d(xy)}{xy}=\int dy-\int dx$

or $\log(xy)=y-x+c$

and this is the required general solution.

Ex. 16. (i) Find the equation of the curve in which the portion of the tangent included between the co-ordinate axes is bisected at the point of contact. [Joint Entrance 1980]

(ii) A curve passes through the potnt (5, 3) and at any point (x, y) on it, the product of its slope and the ordinate is equal to the abscissa. Find the equation of the curve and identify it.

[Joint Entrance 1983]

(iii) Prove that the equation of a curve whose slope at any point (x, y) is $\left(-\frac{x+y}{x}\right)$ and which passes through the point (2, 1) is $x^2+2xy=8$. [Joint Entrance 1982]

(iv) Find the equation to the curve through (1,0) for which the slope at any point (x, y) is $\frac{x^2+y^2}{2xy}$

(i) Let the tangent at any point (x, y) of the curve intersect the axes of co-ordinates at the points $A(\alpha, 0)$ and $B(0, \beta)$. Then the middle point of AB is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$. By question $\frac{\alpha}{2}=x$, $\frac{\beta}{2}=y$.

or, $\alpha=2x$, $\beta=2y$.

Now the gradient of the tangent AB is

$$\frac{0-\beta}{\alpha-0}=-\frac{\beta}{\alpha}=-\frac{2y}{2x}=-\frac{y}{x}.$$

Again the gradient of the tangent at the point (x, y) is $\frac{dy}{dx}$.

$\therefore \frac{dy}{dx}=-\frac{y}{x}$ or, $xdy+ydx=0$ or, $d(xy)=0$.

$\therefore xy=c^2$ (integrating) which is the equation of the curve and it represents a rectangular hyperbola.

(ii) The slope of a curve at any point (x, y) of it is $\frac{dy}{dx}$

So, by question $y\frac{dy}{dx}=x$

or, $xdx=ydy$ or, $\frac{x^2}{2}=\frac{y^2}{2}+c.$

The curve passes through the point (5. 3)

$\therefore\ \frac{25}{2}=\frac{9}{2}+c \qquad \therefore\ c=8.$

So, the equation of the curve is

$$\frac{x^2}{2}=\frac{y^2}{2}+8 \quad \text{or} \quad x^2-y^2=16.$$

(iii) The gradient of a curve at any point (x, y) of it is $\frac{dy}{dx}$

So, by question $\frac{dy}{dx}=\left(-\frac{x+y}{x}\right)$

or, $xdy=-xdx-ydx$

or, $xdy+ydx=-xdx$

or, $d(xy)=-xdx$

or, $xy=-\frac{x^2}{2}+c$ (Integrating)

The curve passes through the point (2, 1)

$\therefore\ 2.1=-\frac{4}{2}+c \qquad$ or, $c=4$

$\therefore$ The equation of the curve is

$$xy=-\frac{x^2}{2}+4 \qquad \text{or,} \quad x^2+2xy=8$$

(iv) The gradient of a curve at any point of it is $\frac{dy}{dx}$

So by question $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

Hence the equation of the curve is

$x^2-y^2=x.$ [Sec Ex. 9(i)]

Ex. 17. A ball of ice melts. The rate at which ice melts is proportional to the amount of ice at the instant. If half the quantity of ice melts in 30 minutes, show that after 90 minutes from the initial, the amount of ice left is $\frac{1}{8}$th of the original.

[State Council W. Bengal 1987]

Let at any time t minutes after the initial v denote the amount of ice at that instant.

Let V denote the original volume of ice.

Now $\frac{dv}{dt}$ denotes the rate of melting.

So, by question $-\frac{dv}{dt} \propto v$ or, $\frac{dv}{dt} = -kv$

or, $\frac{dv}{v} = -kdt$ or, $\log v = -kt + \log c$ [Integrating]

or, $\log \frac{v}{c} = -kt$ or, $\frac{v}{c} = e^{-kt}$

at $t=0$, $v=V$

$\therefore \frac{V}{c} = e^0 = 1$ or, $c = V$.

$\therefore \frac{v}{V} = e^{-kt}$

By question when $t=30$, then $v = \frac{1}{2}V$

$\therefore \frac{\frac{1}{2}V}{V} = e^{-30k}$ or, $e^{-30k} = \frac{1}{2}$

So, when $t=90$,

$$\frac{v}{V} = e^{-90.k} = \left(e^{-30k}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$\therefore v = \frac{1}{8}V$.

So 90 minutes after the initial the amount of ice is $\frac{1}{8}$th of the original.

Exercise—2

Find the general solutions of the following differential equations :

1. $\frac{dy}{dx} = \frac{x+1}{y+1}$.
2. $\frac{dy}{dx} = \frac{x^2+x+1}{y^2+y+1}$.
3. $x\,dx - y\,dy = 0$.
4. $y\,dx - x\,dy = xy\,dx$.
5. $\log\left(\frac{dy}{dx}\right) = ax + by$.
6. $\frac{dx}{x} - \frac{y\,dy}{1+y^2} = 0$.

7. $dr + r\tan\theta\, d\theta = 0.$ 8. $\dfrac{dy}{dx} = \dfrac{x(1+y^2)}{y(1+x^2)}$

9. $\sec^2 x \tan^2 y\, dx + \sec^2 y \tan^2 x\, dy = 0.$

10. $x\sqrt{1-y^2}\, dx + y\sqrt{1-x^2}\, dy = 0$

11. $x^2(y-1)\, dx + y^2(x-1)\, dy = 0$

12. $x\cos^2 y\, dx - y\cos^2 x\, dy = 0.$

13. $(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0.$

14. $x^2(x\, dx + y\, dy) + 2y(x\, dy - y\, dx) = 0.$

15. A curve $y = f(x)$ is such that $\dfrac{dy}{dx} = 5e^x$. If $y = 6$ when $x = 0$, then find $f(x)$.

16. Prove that a particular solution of the equation

$\dfrac{dy}{dx} = \dfrac{2}{y}$ is $y^2 = 4ax$.

17. Prove that the foci of the curve through the point $(2, \sqrt{3})$ having differential equation $(1+y^2)dx - xy\, dy = 0$ are $(\pm\sqrt{2}, 0)$.

18. If $y = \frac{\pi}{4}$, when $x = 0$, then find the particular solution of the equation $\cos y\, dx + (1 + 2e^{-x})\sin y\, dy = 0.$

Find the general solutions of the following differential equations :—

19. $\dfrac{dy}{dx} = \sin(x+y)$ 20. $\dfrac{dy}{dx} = \cos(x-y)$

21. $\dfrac{dy}{dx} = (y-x)^2.$ 22. $\dfrac{dy}{dx} = \sec(x+y)$

23. $\dfrac{dy}{dx} + 1 = e^{x+y}$ 24. $\cos^{-1}\left(\dfrac{dy}{dx}\right) = x + y.$

25. $\dfrac{dy}{dx} = (x+y)$ 26. $(x+y+1)\dfrac{dy}{dx} = 1.$ 27. $\dfrac{dy}{dx} = \sqrt{x+y}.$

28. $\dfrac{dy}{dx} = f(ax+by+c).$ 29. $\dfrac{dy}{dx} = \dfrac{y^3}{x^3}$ 30. $(x^2+y^2)\, dx = xy\, dy.$

31. $x\dfrac{dy}{dx}+\dfrac{y^2}{x}=y.$ 32. $\dfrac{dy}{dx}=\dfrac{2y-x}{2x-y}.$

33. $xy^2\,dy=(x^3+y^3)dx.$

34. $x^2y\,dx-(x^3+y^3)dy=0.$

35. $\dfrac{dy}{dx}=\dfrac{3x+2y}{2x-3y}$ 36. $\dfrac{dy}{dx}=\dfrac{y(x-2y)}{x(x-3y)}$ 37. $\dfrac{dy}{dx}=\dfrac{y(y+x)}{x(y-x)}$

38. $y^2dx+(x^2+xy)dy=0.$ 39. $1+\dfrac{dy}{dx}=e^{x+y}.$

40. $\dfrac{dy}{dx}=\dfrac{2x-y+1}{x-2y+1}.$ 41. $\dfrac{dy}{dx}=\dfrac{4x-5y+3}{5x-6y+2}.$

42. $(6x-5y+4)dy+(y-2x+1)dx=0.$

43. $\dfrac{dy}{dx}=\dfrac{6x-2y-7}{3x-y+4}$

44. $(2x+4y+3)dy=(2y+x+1)dx$

45. $(x-3y+4)dy+(7y-5x)dx=0.$ [C. U. 1963]

46. $(x+y+1)dx-(2x+2y+1)dy=0$

47. $\dfrac{dy}{dx}=\dfrac{6x-4y+3}{3x-2y+1}.$ 48. $\dfrac{dy}{dx}=\dfrac{x+y+1}{x+y-1}$

49. $\dfrac{dy}{dx}=\dfrac{3x+4y+5}{-4x+5y+6}.$

50. $RI+L\dfrac{dI}{dt}=0$ and $I==I_0$ when $t=0$

[Rand L are constants]

51. $r\dfrac{dp}{dr}+2p=2A.$

52. $\dfrac{dy}{dx}=\dfrac{x+1}{y+1}$ $\left[x=1 \text{ when } y=1\right]$

53. $\dfrac{dy}{dx}=\dfrac{x^2+x+1}{y^2+y+1}$ $\left[x=0 \text{ when } y=0.\right]$

54. $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$

$\left[y=\dfrac{\pi}{6} \text{ when } x=\dfrac{\pi}{3}\right]$

55. $\dfrac{dy}{du}=\dfrac{y^2+y+1}{x^2+x+1}$ $[x=1 \text{ when } y=1]$

56. $x^2(y-1)dx+y^2(x-1)dy=0$

[when $x=2$, then $y=2$]

57. Show that if $\dfrac{dy}{dx}=\dfrac{3x^2+2y}{2x-3y}$ and $y=1$, when $x=1$, then

$$3\log\frac{(x^2+y^2)}{2}=4\tan^{-1}\frac{y}{x}-11$$

58. Show that the solution of the equation $y^2dx+(x^2+xy)dy=0$ ($y=1$ when $x=2$) is $2xy^2=x+2y$.

59. Show that the curve for which the normal at every point passes through a fixed point is a circle.

60. A curve passes through the origin and the gradient at any point (x, y) is $1-\dfrac{x^2}{3}$. Find the equation of the curve and also the area included by the curve, the x-axis and the ordinates $x=1$ and $x=2$.

CHAPTER THREE

LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

§3·1. In this chapter we shall discuss about the general and particular solutions of second order differential equations of the form $\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0$ where the coefficients p_1, p_2 are constants. As the highest order differential coefficient present in this equation is of the second order, so it is an equation of the second order. Since the highest power of the highest order derivative $\frac{d^2y}{dx^2}$ in the equation is one, so the equation is of the first degree. Note that in this from $\frac{dy}{dx}$ is also in the first degree. Before we proceed to discuss the method of solutions of these equations we must learn the Theorem discussed in the next section.

§ 3·2. **Theorem :** If $y=f_1(x)$ and $y=f_2(x)$ be two solutions of a linear differerential equation of the second order of the form $\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0$ {p_1, p_2 constants) then $y=c_1f_1(x)+c_2f_2(x)$ where c_1 and c_2 are two arbitrary constants is also a solution of the equation.

Proof. Let $y=f_1(x)$ and $y=f_2(x)$, be two solutions of the differential equation

$$\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0\cdots(1)$$

As $f_1(x)$ is a solution of the equation $-(1)$

$$\text{so } \frac{d^2}{dx^2}\{f_1(x)\}+p_1\frac{d}{dx}\{f_1(x)\}+p_2\{f_1(x)\}=0\cdots\cdots(2)$$

$$\text{Now, } \frac{d^2}{dx^2}\{c_1f_1(x)\}+p_1\frac{d}{dx}\{c_1f_1(x)\}+p_1\{c_1f_1(x)\}$$

$$=c_1\frac{d^2}{dx^2}\{f_1(x)\}+c_1p_1\frac{d}{dx}\{f_1(x)\}+c_1p_2\{f_1(x)\}$$

$$=c_1\left[\frac{d^2}{dx^2}\{f_1(x)\}+p_1\frac{d}{dx}\{f_1(x)\}+p_1\{f_1(x)\}\right]$$

$$=c_1.0 \quad [\text{ by } (2)\,\}$$

$$=0.$$

So, $y=c_1f_1(x)$ is a solution of the equation – (1)

Symilarly $y=c_2f_2(x)$ is also a solution of the equation – (1)

Now let $y=c_1f_1(x)+c_2f_2(x)$.

$$\therefore \quad \frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y$$

$$=\frac{d^2}{dx^2}[\{c_1f_1(x)\}+\{c_2f_2(x)\}]+p_1\frac{d}{dx}[\{c_1f_1(x)\}+c_2f_2(x)] \\ +p_2\{c_1f_1(x)+c_2f_2(x)\}$$

$$=\frac{d^2}{dx^2}\{c_1f_1(x)\}+\frac{d^2}{dx^2}\{c_2f_2(x)\} \\ +p_1\frac{d}{dx}\{c_1f_1(x)\}+p_1\frac{d}{dx}\{c_2f_2(x)\} \\ +p_1\{c_1f_1(x)\}+p_2\{c_2f_2(x)\}$$

$$=\frac{d^2}{dx^2}\{c_1f_1(x)\}+p_1\frac{d}{dx}\{c_1f_1(x)\}+p_2f_1(x) \\ +\frac{d^2}{dx^2}\{c_2f_2(x)\}+p_1\frac{d}{dx}\{c_2f_2(x)\}+p_2f_2(x)$$

$$=0+0=0$$

[As $c_1f_1(x)$ and $c_2f_2(x)$ are solutions of the equation – (1).]

Hence $c_1f_1(x)+c_2f_2(x)$ is a solution of given differential equation – (1)

Note. In §1·4 we have already proved the theorem for the equation $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$ taking $f_1(x)=e^{2x}$ and $f_2(x)=e^{3x}$.

§3·3. General solution of a linear differential equation of the second order.

Let $\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0\cdots(1)$

CHAPTER THREE

LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

§3·1. In this chapter we shall discuss about the general and particular solutions of second order differential equations of the form $\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0$ where the coefficients p_1, p_2 are constants. As the highest order differential coefficient present in this equation is of the second order, so it is an equation of the second order. Since the highest power of the highest order derivative $\frac{d^2y}{dx^2}$ in the equation is one, so the equation is of the first degree. Note that in this from $\frac{dy}{dx}$ is also in the first degree. Befere we proceed to discuss the method of solutions of these equations we must learn the Theorem discussed in the next section.

§ 3·2. **Theorem :** If $y=f_1(x)$ and $y=f_2(x)$ be two solutions of a linear differerential equation of the second order of the form $\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0$ $\{p_1, p_2$ constants) then $y=c_1f_1(x)+c_2f_2(x)$ where c_1 and c_2 are two arbitrary constants is also a solution of the equation.

Proof. Let $y=f_1(x)$ and $y=f_2(x)$, be two solutions of the differential equation

$$\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0\cdots(1)$$

As $f_1(x)$ is a solution of the equation $-(1)$

$$\text{so } \frac{d^2}{dx^2}\{f_1(x)\}+p_1\frac{d}{dx}\{f_1(x)\}+p_2\}f_1(x)\}=0\cdots\cdots(2)$$

$$\text{Now, } \frac{d^2}{dx^2}\{c_1f_1(x)\}+p_1\frac{d}{dx}\{c_1f_1(x)\}+p_1\{c_1f_1(x)\}$$

$$=c_1\frac{d^2}{dx^2}\{f_1(x)\}+c_1p_1\frac{d}{dx}\{f_1(x)\}+c_1p_2\{f_1(x)\}$$

$$=c_1\left[\frac{d^2}{dx^2}\{f_1(x)\}+p_1\frac{d}{dx}\{f_1(x)\}+p_1\{f_1(x)\}\right]$$

$$=c_1.0 \quad [\text{ by (2) }\}$$

$$=0.$$

So, $y=c_1f_1(x)$ is a solution of the equation – (1)

Symilarly $y=c_2f_2(x)$ is also a solution of the equation – (1)

Now let $y=c_1f_1(x)+c_2f_2(x)$.

$$\therefore \quad \frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y$$

$$=\frac{d^2}{dx^2}[\{c_1f_1(x)\}+\{c_2f_2(x)\}]+p_1\frac{d}{dx}[\{c_1f_1(x)\}+c_2f_2(x)]$$
$$+p_2\{c_1f_1(x)+c_2f_2(x)\}$$

$$=\frac{d^2}{dx^2}\{c_1f_1(x)\}+\frac{d^2}{dx^2}\{c_2f_2(x)\}$$
$$+p_1\frac{d}{dx}\{c_1f_1(x)\}+p_1\frac{d}{dx}\{c_2f_2(x)\}$$
$$+p_1\{c_1f_1(x)\}+p_2\{c_2f_2(x)\}$$

$$=\frac{d^2}{dx^2}\{c_1f_1(x)\}+p_1\frac{d}{dx}\{c_1f_1(x)\}+p_2f_1(x)$$
$$+\frac{d^2}{dx^2}\{c_2f_2(x)\}+p_1\frac{d}{dx}\{c_2f_2(x)\}+p_2f_2(x)$$
$$=0+0=0$$

[As $c_1f_1(x)$ and $c_2f_2(x)$ are solutions of the equation – (1).]

Hence $c_1f_1(x)+c_2f_2(x)$ is a solution of given differential equation – (1)

Note. In §1·4 we have already proved the theorem for the equation $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$ taking $f_1(x)=e^{2x}$ and $f_2(x)=e^{3x}$.

§3·3. General solution of a linear differential equation of the second order.

Let $\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0\cdots(1)$

be a given differential equation.

Let $y=e^{mx}$ be a solution of the equation.

As $y=e^{mx}$, $\therefore$ $\frac{dy}{dx}=me^{mx}$ and $\frac{d^2y}{dx^2}=m^2e^{mx}$

So from equation-(1) we get

$$m^2e^{mx}+p_1me^{mx}+p_2e^{mx}=0.$$

or, $m^2+p_1m+p_2=0\cdots(2)$ $[\because e^{mx}\neq 0]$

This equation-(2) is called the Auxiliary equation of the equation $-$ (1).

The equation-(2) is a quadratic equation in m and so will give two and only two values of m. Now three cases may arise.

The roots of the equation-(2) may be

(i) real and distinct

(ii) real and equal.

(iii) Imaginary.

Let us discuss these cases one by one.

(i) Let α and β $(\alpha\neq\beta)$ be the two real roots of equation-(2). So $y=e^{\alpha x}$ and $y=e^{\beta x}$ are two solutions of the equation.

Hence $y=c_1e^{\alpha x}+c_2e^{\beta x}$ is also a solution of the equation

[See §3·2]

Now the solution $y=c_1e^{\alpha x}+c_2e^{\beta x}$ contains two independent arbitrary constants. Hence this solution is the general solution of the equation.

(ii) Let the roots of the auxiliary equation-(2) be real and equal and be α, α. So by the theorem proved in §3·2, $y=c_1e^{\alpha x}+c_2e^{\alpha x}$ where c_1 and c_2 are two arbitrary constants is also a solution of the equation. But these constants c_1 and c_2 are not independent as

$c_1 e^{\alpha x}+c_2 e^{\alpha x}$ can be written as $(c_1+c_2)\, e^{\alpha x}=c e^{\alpha x}$. So the solution $y=c_1 e^{\alpha x}+c_2 e^{\alpha x}$ is not the general solution of the equation.

In this case the general solution will be $y=(c_1+c_2 x)\, e^{\alpha x}$ where c_1 and c_2 are two arbitrary constants. That in this case $y=(c_1+c_2 x)\, e^{\alpha x}$ is the general solution of the equation will be proved in the next section.

(iii) Finally let the roots of the auxiliary equation be imaginary and be $\alpha \pm i\beta$.

[We know that imaginary roots of quadratic equations with real coefficients occur in conjugate pairs]

In this case $y=e^{\alpha x}\cos \beta x$ and $y=e^{\alpha x}\sin \beta x$ will be two solutions and the general solution will be $y=e^{\alpha x}\,(\,\mathrm{A}\cos \beta x+\mathrm{B}\sin \beta x\,)$ where A and B are two arbitrary constants. But its proof is beyond our sclope we shall verify it *by* examples in the Examples set 3.

Note 1. The auxiliary equation of a given second order differential equation can be written atonce by putting m^2, m and 1 for $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y respectively. The students are advised to verify it.

2. The symbols D and D^2 are frequently used to denote the operators $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ respectively. They are called symbolic operators. In higher classes you shall find that use of symbolic operators are very much helpful.

Let us once again consider the differential equation.

$$\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2 y=0\cdots(1)$$

As $D^2=\frac{d^2}{dx^2}$, so D^2y is $\frac{d^2}{dx^2}(y)=\frac{d^2y}{dx^2}$

$D=\frac{d}{dx}$, so $Dy=\frac{d}{dx}\,(y)=\frac{dy}{dx}$

So $(D^2+p_1D+p_2)y=D^2y+p_1Dy+p_2y$

$$=\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y.$$

Hence equation-(1) can be written as $(D^2+p_1D+p_2)y=0$...(2).

Yo can atonce see that putting $D=m$, in $D^2+p_1D+p_2=0$ equation-(2) we get the equation $m^2+p_1m+p^2=0$ which is the auxiliarly equation of the given differential equation-(1).

§3·4. Theorem. If the roots of the auxiliary equation of the differential equation $\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0$ be real and equal, then the general solution of the equation is $y=(c_1+c_2x)e^{\alpha x}$ where α, α are the equal roots and c_1 and c_2 are two arbitrary constants.

Proof : We have seen in §3·3 that the auxiliary equation of the differential equation

$$\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0 \text{.......(1) is}$$

$$m^2+p_1m+p_2=0\text{......(2)}$$

It is given that the roots of the equation-(2) are real and equal and are α, α.

$\therefore\ \alpha+\alpha=-p_1$ or, $p_1=-2\alpha$ and $p_2=\alpha^2$.

So the equation-(1) can be written as

$$\frac{d^2y}{dx^2}-2\alpha\frac{dy}{dx}+\alpha^2y=0\text{.......(3).}$$

Let $y=e^{\alpha x}.V$ where V is a function of x be a solution of equation-(3).

As $y=e^{\alpha x}.V$

$$\therefore\ \frac{dy}{dx}=\alpha.e^{\alpha x}V+e^{\alpha x}\frac{dV}{dx}=e^{\alpha x}\left(\alpha V+\frac{dV}{dx}\right)$$

and $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left\{e^{\alpha x}\left(\alpha V+\frac{dV}{dx}\right)\right\}$$

$$=\alpha e^{\alpha x}\left(\alpha V+\frac{dV}{dx}\right)+e^{\alpha x}\left(\alpha\frac{dV}{dx}+\frac{d^2V}{dx^2}\right)$$

$$=e^{\alpha x}\left(\alpha^2V+\alpha\frac{dV}{dx}+\alpha\frac{dV}{dx}+\frac{d^2V}{dx^2}\right)$$

$$=e^{\alpha x}\left(\frac{d^2V}{dx^2}+2\alpha\frac{dV}{dx}+\alpha^2V\right).$$

Putting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation-(3)

we obtain.

$$e^{\alpha x}\left(\frac{d^2V}{dx^2}+2\alpha\frac{dV}{dx}+\alpha^2V\right)-2\alpha e^{\alpha x}\left(\alpha V+\frac{dV}{dx}\right)+\alpha^2e^{\alpha x}.V=0$$

or, $e^{\alpha x}\left(\frac{d^2V}{dx^2}+2\alpha\frac{dV}{dx}+\alpha^2V-2\alpha^2V-2\alpha\frac{dV}{dx}+\alpha^2V\right)=0$

or, $e^{\alpha x}\frac{d^2V}{dx^2}=0$

or, $\frac{d^2V}{dx^2}=0$ [as $e^{\alpha x}\neq0$]

$\therefore$ $\frac{d}{dx}\left(\frac{dV}{dx}\right)=0$ or, $d\left(\frac{dV}{dx}\right)=0$

$\therefore$ $\frac{dV}{dx}=c_2$ [Integrating, where c_2 is an arbitrary constant of integration]

or, $dV=c_2dx$, or, $\int dV=\int c_2dx$

or, $V=c_2x+c_1$ where c_1 is an arbitrary constant of integration.

Hence $y=e^{\alpha x}(c_1+c_2x)$ is a solution of the differential equation. The two arbitrary constants c_1 and c_2 are here independent and so the solution $y=e^{\alpha x}(c_1+c_2x)$ is the general solution of the equation-(1).

§ 3·5 Gyst.

1. The auxiliary equation of the differential equation

$$\frac{d^2y}{dx^2}+p_1\frac{dy}{dx}+p_2y=0 \text{ is}$$

$m^2+p_1m+p_2=0$ which can be written atonce by putting m^2, m and 1 for, $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y respectively.

2. If the roots of the auxiliary equation be

(i) real and distinct α, β (say), then the general solution of the equation is $y=c_1e^{\alpha x}+c_2e^{\beta x}$.

(ii) real and equal α, α (say), then the general solution of the equation is $y=(c_1+c_2x)\,e^{\alpha x}$.

(iii) Imaginary, $\alpha\pm i\beta$ (say), then the general solution is $y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

Note : This case (iii) cannot be proved at this stage. We shall verify them.

Examples 3.

Example 1. Find the general solution of the following equations.

(i) $\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=0$ (ii) $\frac{d^2y}{dx^2}-3\frac{dy}{dx}-5y=0$

(iii) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$ (iv) $2\frac{d^2y}{dx^2}-5\frac{dy}{dx}+2y=0$

(v) $\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby=0$. (vi) $\frac{d^2y}{dx^2}-4y=0$.

(i) Let $y=e^{mx}$ be a solution of the given equation.

As $y=e^{mx}$ $\therefore$ $\frac{dy}{dx}=me^{mx}$, $\frac{d^2y}{dx^2}=m^2e^{mx}$

So from the given equation we get

$m^2e^{mx}-me^{mx}-2e^{mx}=0$

or, $e^{mx}(m^2-m-2)=0$

or, $m^2-m-2=0$ [as $e^{mx}\neq 0$]

or, $(m-2)(m+1)$ $\therefore$ $m=2,-1$.

Hence the required general solution of the given equation is

$y=c_1e^{2x}+c_2e^{-x}$ where c_1 and c_2 are two arbitrary constants.

(ii) Evidently the auxiliary equation of the given equation is

$$m^2-3m-5=0$$

[We refrain from repeating the process of determination of the auxiliary equation in every sum]

$$\therefore \quad m=\frac{3\pm\sqrt{9+20}}{2}=\frac{3\pm\sqrt{29}}{2}$$

Hence the required general solution is

$$y=c_1e^{\frac{3+\sqrt{29}}{2}x}+c_2e^{\frac{-3+\sqrt{29}}{2}x}.$$

(iii) Evidently the auxiliary equation is

$m^2+2m=0$ or, $m(m+2)=0$ $\therefore$ $m=0,-2$.

Hence the general solution of the given equation is

$$y=c_1+c_2e^{-2x}.$$

[Corresponding to $m=0$, we get $c_1e^{0.x}=c_1.e^0=c_1$]

(iv) Evidently the auxiliary equation is

$2m^2-5m+2=0$ or, $(2m-1)(m-2)=0$

$\therefore$ $m=2$, or, $\frac{1}{2}$.

Hence the required general solution is

$$y=c_1e^{2x}+c_2e^{\frac{1}{2}x}.$$

(v) The auxiliary equation is

$m^2-(a+b)m+ab=0$ or, $(m-a)(m-b)=0$.

$\therefore$ $m=a,b$.

Hence the general solution of the equation is

$$y=c_1e^{ax}+c_2e^{bx}$$

(vi) The auxiliary equation is

$$m^2-4=0 \quad \text{or,} \quad m=\pm 2.$$

So, the general solution is $y=c_1e^{2x}+c_2e^{-2x}$,

Ex. 2. Find the general solution of the following differential equations.

(i) $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=0$ (ii) $\dfrac{d^2x}{dt^2}-4\dfrac{dx}{dt}+4x=0$

(iii) $(D^2+6D+9)y=0$ $\left[D\equiv \dfrac{d}{dx}\right]$

(iv) $(D+1)^2)y=0$ $\left[D\equiv \dfrac{d}{dx}\right]$

(i) The auxiliary equation is

$$m^2-2m+1=0 \quad \text{or,} \quad (m-1)^2=0 \quad \therefore \quad m=1,\,1$$

So, the general solution is $y=(c_1+c_2x)e^x$.

[Here the roots are real and equal]

(ii) Let $x=e^{mt}$, be a solution of the equation.

$$\therefore \quad \frac{dx}{dt}=me^{mt}, \quad \frac{d^2x}{dt^2}=m^2e^{mt}$$

So from the given equation we get

$$m^2e^{mt}-4me^{mt}+4e^{mt}=0.$$

or, $e^{mt}(m^2-4m+4)=0$

or, $m^2-4m+4=0$ [$\because$ $e^{mt}\neq 0$]

or, $(m-2)^2=0$ $\therefore$ $m=2,\,2$

Hence the required general solution is $x=(c_1+c_2t)e^{2t}$.

[Note : Here x is the dependent variable and t is independent variable. So we take $x=e^{mt}$]

(iii) The auxiliary equation is

$m^2+6m+9=0$ or, $(m+3)^2=0 \therefore m=-3, -3$

Hence the general solution is $y=(c_1+c_2x)e^{-3x}$

(iv) The auxiliary equation is

$(m+1)^2=0 \quad \therefore \quad m=-1, -1.$

Hence the general solution is $y=(c_1+c_2x)e^{-x}$.

Ex. 3. Solve :

(i) $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+3y=0 \quad \left[\text{given } y=5,\ \dfrac{dy}{dx}=9 \text{ when } x=0\right]$

(ii) $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-2y=0 \quad \left[y=3 \text{ and } \dfrac{dy}{dx}=0, \text{ when } x=0\right]$

(iii) $\dfrac{d^2x}{dt^2}-3\dfrac{dx}{dt}+2x=0$ when $t=0$ then $x=2$ and $\dfrac{dx}{dt}=0$.

(i) The auxiliary equation is $m^2-4m+3=0$

or, $(m-1)(m-3)=0 \quad \therefore \quad m=1,3.$

So, the general solution is $y=c_1e^x+c_2e^{3x}$ ... (i)

$\therefore \quad \dfrac{dy}{dx}=c_1e^x+3c_2e^{3x}$ ... (ii)

[differentiating both sides of (i)]

Now $y=5, \dfrac{dy}{dx}=9$ when $x=0$.

$\therefore$ From (i) we get $5=c_1+c_2$ ......(iii)

From (ii) we get $9=c_1+3c_2$ ......(iv)

Solving (iii) and (iv) we get $c_1=3, c_2=2$

Hence the particular solution is $y=3e^x+2_2e^{3x}$.

(ii) The auxiliary equation is

$m^2+m-2=0$, or, $(m+2)(m-1)=0$

$\therefore \quad m=1, -2.$

Hence the general solutions is

$y=c_1e^x+c_2e^{-2x}$ .... (i)

differentiating both sides of (i) with respect to x we get

$$\frac{dy}{dx}=c_1e^x-2c_2e^{-2x}\ldots\text{(ii)}$$

Now when $x=0$, then $y=3$, $\frac{dy}{dx}=0$

So, from equations-(i) and (ii) we get

$3=c_1+c_2\cdots$(iii) and $0=c_1-2c_2\cdots$(iv)

From (iv) we get $c_1=2c_2$.

From (iii) $3=3c_2$ $\quad\therefore c_2=1.$ $\quad\therefore c_1=2,$

Hence the required solution is $y=2e^x+e^{-2x}$

(iii) The auxiliary equation is

$m^2-3m+2=0$ or, $(m-1)(m-2)=0$ $\therefore$ $m=1, 2.$

Hence the general solution is

$$x=c_1e^t+c_2e^{2t}\ldots\text{(i)}$$

$$\therefore \frac{dx}{dt}=c_1e^t+2c_2e^{2t}\ldots\text{(ii)}$$

Now when $t=0$, then $x=2$ and $\frac{dx}{dt}=0$.

$\therefore$ From (i) and (ii) we get

$2=c_1+c_2\cdots$(iii) and $0=c_1+2c_2\cdots$(iv)

From (iv) $c_1=-2c_2$

$\therefore$ From (iii), $2=-2c_2+c_2=-c_2$, or, $c_2=-2$.

$\therefore$ $c_1=-2c_2=4$.

Hence the required solution is $x=4e^t-2e^{2t}$.

Ex. 4. Solve : $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$.

$y=2$, $\frac{dy}{dx}=0$ when $x=0$.

The auxiliary equation of the given equation is

$m^2-4m+4=0$ or, $(m-2)^2=0$, $\therefore$ $m=2, 2.$

So the general solution is

$$y=(c_1+c_2x)e^{2x}\cdots\text{(i)}.$$

$$\therefore\quad \frac{dy}{dx}=(c_1+c_2x)2.e^{2x}+c_2e^{x}\cdots\text{(ii)}$$

Now when $x=0$, then $y=2$, $\frac{dy}{dx}=0$

$\therefore$ From (i) $2=c_1$

From (ii) $0=2c_1+c_2$ or, $c_2=-2c_1=-4$.

So, the required solution is $y=(2-4x)e^{2x}=2(1-2x)e^{2x}$.

Ex. 5. Show that $y=\cos x$ and $y=\sin x$ are solutions of the equation $\frac{d^2y}{dx^2}+y=0$ and hence find the general solution of the equation.

Let $y=\cos x$, $\therefore\quad \frac{dy}{dx}=-\sin x,\ \frac{d^2y}{dx^2}=-\cos x.$

$$\therefore\quad \frac{d^2y}{dx^2}+y=-\cos x+\cos x=0.$$

So $y=\cos x$ satisfies the equation i.e. is a solution of the equation.

Next let $y=\sin x$ $\therefore\quad \frac{dy}{dx}=\cos x,\ \frac{d^2y}{dx^2}=-\sin x.$

$$\therefore\quad \frac{d^2y}{dx^2}+y=-\sin x+\sin x=0.$$

So $y=\sin x$ is also a solution of the equation. Thus we have obtained two solutions $y=\cos x$ and $y=\sin x$ of the equation. Hence $y=A\cos x+B\sin x$, where A and B are two arbitrary constants is also a solution of the equation. Here the two constants A and B are independent. Hence $y=A\cos x+B\sin x$ is the general solution of the equation.

Ex. 6. Show that $y=e^{3x}.\cos 2x$ and $y=e^{3x}\sin 2x$, are two solutions of the equation.

$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=0$ and hence find the general solution of the equation.

Let $y=e^{3x}\cos 2x$.

$$\therefore\ \frac{dy}{dx}=3e^{3x}\cos 2x-2e^{3x}\sin 2x$$

$$=e^{3x}(3\cos 2x-2\sin 2x)$$

$$\therefore\ \frac{d^2y}{dx^2}=3e^{3x}(3\cos 2x-2\sin 2x)$$

$$+e^{3x}(-6\sin 2x-4\cos 2x)$$

$$=e^{3x}(9\cos 2x-6\sin 2x-6\sin 2x-4\cos 2x)$$

$$=e^{3x}(5\cos 2x-12\sin 2x)$$

$$\therefore\ \frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y$$

$$=e^{3x}(5\cos 2x-12\sin 2x)-6e^{3x}(3\cos 2x-2\sin 2x)$$

$$+13e^{3x}\cos 2x$$

$$=e^{3x}(5\cos 2x-12\sin 2x-18\cos 2x+12\sin 2x+13\cos 2x)$$

$$=e^{3x}.0=0.$$

So $e^{3x}\cos 2x$ is a solution of the equation.

Next let $y=e^{3x}\sin 2x$.

$$\therefore\ \frac{dy}{dx}=3e^{3x}\sin 2x+2e^{3x}\cos 2x=e^{3x}(3\sin 2x+2\cos 2x)$$

$$\therefore\ \frac{d^2y}{dx^2}=3e^{3x}(3\sin 2x+2\cos 2x)+e^{3x}(6\cos 2x-4\sin 2x)$$

$$=e^{3x}(12\cos 2x+5\sin 2x)$$

$$\therefore\ \frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y$$

$$=e^{3x}(12\cos 2x+5\sin 2x)-6e^{3x}(3\sin 2x+2\cos 2x)$$

$$+13e^{3x}\sin 2x$$

$$=e^{3x}(12\cos 2x+5\sin 2x-18\sin 2x-12\cos 2x+13\sin 2x).$$

$$=e^{3x}.0=0.$$

So $e^{3x}\sin 2x$ is another solution of the equation.

Hence $y=A.e^{3x}\cos 2x+B.e^{3x}\sin 2x$.

$=e^{3x}(A\cos 2x+B\sin 2x)$ is a solution of the equation. Also A and B are two independent arbitrary constants so ; it is the general solution of the equation.

Ex. 7. (i) Show that $y=\cos x$ and $y=\sin x$ are two solutions of the equation $\frac{d^2y}{dx^2}+y=0$ and hence find the particular solution which satisfies the conditions

when $x=0$, then $y=4$ and when $x=\frac{\pi}{2}$, then $y=0$.

(ii) Show that $x=\cos nt$ and $x=\sin nt$ are two solutions of the equation $\frac{d^2x}{dt^2}+n^2x=0$ and hence find the particular solution which satisfies the condition $x=a$, $\frac{dx}{dt}=0$ when $t=0$.

(iii) verify that $y=e^{3x}\cos 4x$ and $y=e^{3x}\sin 4x$ are two solutions of the equation.

$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+25\,y=0$ and hence find the particular solution of the equation satisfying the condition,

$$y=1,\quad \frac{dy}{dx}=7 \text{ when } x=0$$

(i) If $y=\cos x$ $\quad\therefore\quad \frac{dy}{dx}=-\sin x,\ \frac{d^2y}{dx^2}=-\cos x$

$\therefore\ \frac{d^2y}{dx^2}+y=-\cos x+\cos x=0$. So $y=\cos x$ is a solution of the equation.

Again if $y=\sin x$, $\frac{dy}{dx}=\cos x$, $\frac{d^2y}{dx^2}=-\sin x$.

$$\therefore\ \frac{d^2y}{dx^2}+y=-\sin x+\sin x=0$$

So, $y=\sin x$ is another solution of the equation.

$\therefore\ y=A\cos x+B\sin x$, where A and B are arbitrary constants, is the general solution of the equation.

Now when $x=0$, then $y=4$.

$\therefore\ 4=A\cos 0+B\sin 0=A$.

Again when $x=\frac{\pi}{2}$, then $y=0$

$\therefore\ 0=A\cos\frac{\pi}{2}+B\sin\frac{\pi}{2}$ or, $0=B$

$\therefore$ The required solution of the equation is $y=4\cos x$.

(ii) If $x=\cos nt$, $\frac{dx}{dt}=-n\sin nt$, $\frac{d^2x}{dt^2}=-n^2\cos nt$

$$\therefore\ \frac{d^2x}{dt^2}+n^2x=-n^2\cos nt+n^2\cos nt=0.$$

So $x=\cos nt$ is a solution of the equation.

Again when $x=\sin nt$, $\frac{dx}{dt}=n\cos nt$

and $\frac{d^2x}{dt^2}=-n^2\sin nt$

$$\therefore\ \frac{d^2x}{dt^2}+n^2x=-n^2\sin nt+n^2\sin nt=0$$

$\therefore\ x=\sin nt$ is another solution of the equation. Hence the general solution of the equation is $x=A\cos nt+B\sin nt$, where A and B are arbitrary constants of integration.

Now $x=A\cos nt+B\sin nt$.......(i)

$\therefore\ \frac{dx}{dt}=-An\sin nt+Bn\cos nt$...(ii)

It is given that when $t=0$, then $x=a$ and $\frac{dx}{dt}=0$.

$\therefore$ From (i) and (ii) we get,

$a=A\cos 0+B\sin 0$ $\therefore\ A=a$

and $0=-A\sin 0+B\cos 0$ or, $0=B$.

So, the required solution is $x=a\cos nt$.

Note : Let us consider the general solution $x=A\cos nt+B\sin nt$ of the equation. If we put $A=a\cos\epsilon$ and $B=-a\sin\epsilon$ where ϵ is a constant,

the general solution can be written as

$x=a\cos\epsilon\cos nt-a\sin\epsilon\sin nt$

$=a\cos(nt+t)$.

Here a and ϵ are two arbitrary constants.

(iii) Let $y=e^{3x}\cos 4x \quad \therefore \quad \dfrac{dy}{dx}=3e^{3x}\cos 4x-4e^{3x}\sin 4x$

$=e^{3x}(3\cos 4x-4\sin 4x)$

$\dfrac{d^2y}{dx^2}=3e^{3x}(3\cos 4x-4\sin 4x)+e^{3x}(-12\sin 4x-16\cos 4x)$

$=e^{3x}(9\cos 4x-12\sin 4x-12\ 4\sin x-16\cos 4x)$

$=-e^{3x}(7\cos 4x+24\sin 4x)$

$\therefore \quad \dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+25y$

$=-e^{3x}(7\cos 4x+24\sin 4x)-6e^{3x}(3\cos 4x-4\sin 4x)$
$+25e^{3x}\cos 4x$

$=e^{3x}(-7\cos 4x-24\sin x-18\cos 4x+24\sin 4x+25\cos 4x)$

$=e^{3x}.0=0.$

So $e^{3x}\cos 4x$ is a solution of the equation.

Next let $y=e^{3x}\sin 4x$.

$\therefore \quad \dfrac{dy}{dx}=3e^{3x}\sin 4x+4e^{3x}\cos 4x=e^{3x}(3\sin 4x+4\cos 4x)$

$\therefore \quad \dfrac{d^2y}{dx^2}=3e^{3x}(3\sin 4x+4\cos 4x)+e^{3x}(12\cos 4x-16\sin 4x)$

$=e^{3x}(9\sin 4x+12\cos 4x+12\cos 4x-16\sin 4x)$

$=e^{3x}(-7\sin 4x+24\cos 4x)$

$\therefore \quad \dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+25y$

$=e^{3x}(-7\sin 4x+24\cos 4x)-6e^{3x}(3\sin 4x+4\cos 4x)$
$+25e^{3x}\sin 4x$

$=e^{3x}(-7\sin 4x+24\cos 4x-18\sin 4x-24\cos 4x+25\sin 4x)$

$=e^{3x}.0=0$

$\therefore$ $e^{3x}\sin 4x$ is another solution of the equation.

$\therefore$ $y = Ae^{3x}\cos 4x + Be^{3x}\sin 4x$

or, $y = e^{3x}(A\cos 4x + B\sin 4x)$ ....(i)

where A and B are arbitrary constants is the general solution of the equation.

From (i) differentiating with respect to x we get

$$\frac{dy}{dx} = 3e^{3}\ (A\cos 4x + B\sin 4x)$$

$$+e^{3x}(-4A\sin 4x + 4B\cos 4x)$$

or, $\dfrac{dy}{dx} = e^{3x}\{(3A+4B)\cos 4x + (3B-4A)\sin 4x\}$ ......(ii)

Now, $y = 1$, $\dfrac{dy}{dx} = 7$ when $x = 0$

From (i) and (ii) we get,

$1 = e^{0}(A\cos 0 + B\sin 0) = A$

From (ii) we get

$7 = e^{0}\{(3A+4B)\cos 0 + (3B-4A)\sin 0\}$

$= 3A + 4B.$

But $A = 1$ $\therefore$ $4 = 4B$ or, $B = 1$.

Hence the required solution of the equation is

$$y = e^{3x}(\cos 4x + \sin 4x)$$

Ex. 8. If $p^2 = \mu - \frac{1}{4}k^2$, verify that $x = e^{-\frac{1}{2}kt}\cos pt$ and $x = e^{-\frac{1}{2}kt}\sin pt$ are both solution of the equation

$\dfrac{d^2x}{dt^2} + k\dfrac{dx}{dt} + \mu x = 0$ and hence find the general solution of the equation.

If $x = e^{-\frac{1}{2}kt}\cos pt$

$$\frac{dx}{dt} = -\tfrac{1}{2}ke^{-\frac{1}{2}kt}\cos pt - e^{-\frac{1}{2}kt}p\sin pt$$

$$= -e^{-\frac{1}{2}kt}\left(\tfrac{1}{2}k\cos pt + p\sin pt\right)$$

$$\frac{d^2x}{dt^2}=\frac{1}{2}ke^{-\frac{1}{2}kt}\left(\tfrac{k}{2}\cos pt+p\sin pt\right)$$

$$-e^{-\frac{1}{2}kt}\left(-\frac{pk}{2}\sin pt+p^2\cos pt\right)$$

$$=e^{-\frac{1}{2}kt}\left\{\left(\frac{k^2}{4}-p^2\right)\cos pt+kp\sin pt\right\}$$

$$\therefore\quad \frac{d^2x}{dt^2}+k\frac{dx}{dt}+\mu x$$

$$=e^{-\frac{1}{2}kt}\left\{\left(\frac{k^2}{4}-p^2\right)\cos pt+kp\sin pt\right\}$$

$$-ke^{-\frac{1}{2}kt}\{\tfrac{1}{2}k\cos pt+p\sin pt\}+\mu e^{-\frac{1}{2}kt}\cos pt$$

$$=e^{-\frac{1}{2}kt}\left\{\left(\frac{k^2}{4}-p^2-\tfrac{1}{2}k^2+\mu\right)\cos pt+(kp-kp)\sin pt\right\}$$

$$=e^{-\frac{1}{2}k}\left[\left\{\mu-\left(p^2+\frac{k^2}{4}\right)\right\}\cos pt\right]=e^{-\frac{1}{2}kt}(\mu-\mu)\cos pt$$

$$\left[\because\ p^2=\mu-\tfrac{1}{4}k^2,\ p^2+\frac{k^2}{4}=\mu\right]=e^{-\frac{1}{2}kt},\ 0.\cos pt=0.$$

So $x=e^{-\frac{1}{2}kt}\cos pt$ is a solution of the equation.

Again let $x=e^{-\frac{1}{2}kt}\sin pt$

$$\therefore\quad \frac{dx}{dt}=-\frac{1}{2}ke^{-\frac{1}{2}kt}\sin pt+e^{-\frac{1}{2}kt}p\cos pt$$

$$=e^{-\frac{1}{2}kt}(p\cos pt-\tfrac{1}{2}k\sin pt)$$

$$\frac{d^2x}{dt^2}=-\frac{1}{2}ke^{-\frac{1}{2}kt}(p\cos pt-\tfrac{1}{2}k\sin pt)$$

$$+e^{-\frac{1}{2}kt}(-p^2\sin pt-\tfrac{1}{2}kp\cos pt)$$

$$=-e^{-\frac{1}{2}kt}\{(\tfrac{1}{2}kp+\tfrac{1}{2}kp)\cos pt+(p^2-\tfrac{1}{4}k^2)\sin pt\}$$

$$\therefore\quad \frac{d^2x}{dt^2}+k\frac{dx}{dt}+\mu x$$

$$= e^{-\frac{1}{2}kt}\{-(kp)\cos pt + (\tfrac{1}{4}k^2 - p^2)\sin pt\}$$

$$+ e^{-\frac{1}{2}kt}(kp\cos pt - \tfrac{1}{2}k^2\sin pt) + \mu e^{-\frac{1}{2}kt}\sin pt$$

$$= e^{-\frac{1}{2}kt}\{(-kp+kp)\cos pt + (\tfrac{1}{4}k^2 - p^2 - \tfrac{1}{2}k^2 + \mu)\sin pt\}$$

$$= e^{-\frac{1}{2}kt}\{0.\cos pt + \{\mu - (p^2 + \tfrac{1}{4}k^2)\sin pt\}$$

$$= e^{-\frac{1}{2}kt}(\mu - \mu)\sin pt = 0.$$

So, $x = e^{-\frac{1}{2}kt}\sin pt$ is another solution of the equation. Hence $x = Ae^{-\frac{1}{2}kt}\cos pt + Be^{-\frac{1}{2}kt}\sin pt$ is a solution of the equn. Clearly the two arbitrary constants of integration A and B are independent.

Hence $x = Ae^{-\frac{1}{2}kt}\cos pt + Be^{-\frac{1}{2}kt}\sin pt$

or, $x = e^{-\frac{1}{2}kt}(A\cos pt + B\sin pt)$ is the general solution of the equatian.

Exercise 3

Determine the general solution of the following differential equations (1 – 16) :—

1. $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y=0.$
2. $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0.$
3. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}=0.$
4. $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}+2y=0$
5. $3\frac{d^2x}{dt^2}+10\frac{dx}{dt}+3x=0.$
6. $(D^2-11D+24)y=0.$
7. $(D^2-1)y=0.$
8. $\frac{d^2x}{dt^2}-(a+1)\frac{dx}{dt}+ax=0.$
9. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-8y=0$
10. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-6y=0.$
11. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0.$
12. $4\frac{d^2x}{dt^2}+4\frac{dx}{dt}+x=0.$

13. $\frac{d^2s}{dt^2}-6\frac{ds}{dt}+9s=0.$ 14. $\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+a^2y=0.$

15. $(D+1)^2.y=0.$ 16. $a^2\frac{d^2y}{dx^2}-2ab\frac{dy}{dx}+b^2y=0.$

17. Show that $y=e^x \cos x$ and $y=e^x \sin x$ are two solutions of the equation

$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$ and from this determine the general solution of the equation.

18. Show that $y=\cos 2x$ and $y=\sin 2x$ are two solutions of the equation $\frac{d^2y}{dx^2}+4y=0$ and hence find the general solution of the equation.

19 Verify that $y=e^{-2x} \cos 6x$ and $y=e^{-2x} \sin 6x$ are two solutions of the differential equation.

$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+40y=0$ and hence write the general solution of the equation.

20. Verify that $x=e^{3t} \cos 5t$ and $x=e^{3t} \sin 5t$ are two solutions of the differential equation $\frac{d^2x}{dt^2}-6\frac{dx}{dt}+34x=0$. Also determine the general solution of the equation.

21. Show that $y=\cos 3x$ and $y=\sin 3x$ are two solutions of the differential equation $\frac{d^2y}{dx^2}+9y=0$ and hence show that the general solution of the equation can be written as $y=a \cos (3x+\alpha)$ where a and α are two arbitrary constants.

22. Verify that two solutions of the equation $\frac{d^2x}{dt^2}+\mu^2x=0$ are $x=\cos \mu t$ and $x=\sin \mu t$ and from it show that the general solution of the equation can be expressed in the form $x=a \cos (\mu t+\epsilon)$ where a and ϵ are two arbitrary constants.

Find the particular solutions of the following differential equations. (Ex 23—30).

23. $\frac{d^2y}{dx^2}-7\frac{dy}{dx}+12y=0$, $y=5, \frac{dy}{dx}=18$ when $x=0$.

24. $2\frac{d^2y}{dx^2}-5\frac{dy}{dx}=0$ $y=\frac{dy}{dx}=5$ when $x=0$

25. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=0$ $y=1,\ \frac{dy}{dx}=0$ when $x=0$

26. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+4y=0$ $y=3,\ \frac{dy}{dx}=0$ when $x=0$

27. $(D+2)^2y=0$, $y=\frac{dy}{dx}=2$ when $x=0$.

28. $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$, $x=4, \frac{dx}{dt}=-2$ when $t=0$.

29. Verify that $y=e^{-2x}\cos 3x$ and $y=e^{-2x}\sin 3x$ are two solutions of the equation.

$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$ and hence find the particular solution of the equation which will satisfy the condition

$y=2, \frac{dy}{dx}=-1$ when $x=0$.

30. Verify that $x=e^{-t}\cos 2t$ and $x=e^{-t}\sin 2t$ satisfy the equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=0$. Also determine the solution of the equation which satisfy the condition $x=4, \frac{dx}{dt}=-2$ when $t=0$

31. Solve the equation.

$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{c}i=0$ where $R^2c=4L$

32. Show $x=\cos kt$ and $x=\sin kt$ both satisfy the equation $\frac{d^2x}{dt^2}=-k^2x$ and then provethat is $x=0, \frac{dx}{dt}=a$ when $t=0$ then $x=\frac{a}{k}\sin kt$.

33. Verify that $\theta=\cos\sqrt{\frac{g}{l}}t$ and $\theta=\sin\sqrt{\frac{g}{l}}t$ are two solutions of the equation $l\frac{d^2\theta}{dt^2}+g\theta=0$, Hence show that if $\theta=\alpha$, $\frac{d\theta}{dt}=0$,

then $\theta=\alpha\cos\left(\sqrt{\frac{g}{l}}t\right)$

ANSWERS

Exercise 1

1 (i) First order and First degree

(ii) First order and second degree

(iii) Second order and first degree

(iv) Second order and first degree

(v) First order and second degree

2. (i) $x\frac{dy}{dx}+y=0$ (ii) $\frac{d^2y}{dx^2}=m^2y$

(iii) $\frac{d^2y}{dx^2}+m^2y=0$. (iv) $x\frac{d^2y}{dx^2}=\frac{dy}{dx}$.

(v) $\frac{d^2r}{d\theta^2}-\frac{dr}{d\theta}\cot\theta=0$

3. (i) $\frac{dy}{dx}+2y\tan 2x=0$ (ii) $y=2x\frac{dy}{dx}$.

(iii) $x\left(\frac{dy}{dx}\right)^2=1+\frac{dy}{dx}$.

(iv) $\frac{dy}{dx}+y=x$. (v) $y=2x\frac{dy}{dx}+y\left(\frac{dy}{dx}\right)^2$.

4. (i) $\frac{d^2y}{dx^2}=\frac{dy}{dx}\cot x$. (ii) $\frac{d^2y}{dx^2}=0$

(iii) $\frac{d^2y}{dx^2}+4y\frac{dy}{dx}+4y=0.$ (iv) $\frac{d^2y}{dx^2}=y.$

(v) $\frac{d^2y}{dx^2}-\tan x\frac{dy}{dx}-y\sec^2 x=0.$

7. $2xy\frac{dy}{dx}=y^2-x^2.$

Exercise 2

1. $\frac{1}{2}(y^2-x^2)+(y-x)=c$ 2. $\frac{y^3}{3}+\frac{y^2}{2}+y=x^3+\frac{x^2}{2}+x+c$

3. $x^2-y^2=a^2$ 4. $ye^x=cx$ 5. $ae^{-by}+be^{ax}=A$

6. $x=c\sqrt{1+y^2}$ 7. $r=c\cos\theta$ 8. $1+x^2=c(1+y^2)$

9. $\cot x+\cot y=c.$ 10. $\sqrt{1-x^2}+\sqrt{1-y^2}=c.$

11. $(x+1)^2+(y+1)^2+2\log(x-1)(y-1)=c.$

12. $x\tan x-\log\sec x=y\tan y-\log\sec y+c.$

13. $\log\frac{x}{y}-\frac{x+y}{xy}=c.$ 14. $(x^2+y^2)(x+2)^2=cx^2$

15. $y=5e^x+1$ 18. $(e^x+2)\sec y=3\sqrt{2}.$

19. $x=\tan(x+y)+\sec(x+y)+c.$ 20. $-\cot\left(\frac{x+y}{2}\right)=x+c.$

21. $(y-x-1)^2=(y-x+1)^2e^2e^{4x}.$ 22. $y=\tan\left(\frac{x+y}{2}\right)+c.$

23. $-e^{-x-y}=x+c$ 24. $\tan\left(\frac{x+y}{2}\right)=x+c$

25. $(x+y+1)^2=c^2e^{2x}$ 26. $(x+y+2)^2=c^2e^{2y}$

27. $2\{\sqrt{x+y}-\log(1+\sqrt{x+y})\}=x+c.$

28. $x=\int\frac{dz}{a+bf(z)}+c$, where $z=ax+by+c.$

29. $x^2-y^2=cx^2y^2$ 30. $\log x=\frac{y^2}{2x^2}+c.$

31. $cx=e^{\frac{x}{y}}$ 32. $(x+y)^3=c(y-x)$

33. $y^3=x^3\log cx^3$ 34. $y=ce^{\frac{x^3}{3y^3}}.$

35. $3 \log (x^2+y^2)=4 \tan^{-1} \frac{y}{x}+c$

36. $y^3 e^x = cx^2$ 37. $xy = ce^{\frac{y}{x}}$

38. $x^2 y^4 = c(x+2y)^2$ 39. $e^{-x-y}+x=c$

40. $x^2+y^2-xy+x-y=c$

41. $2x^2-5xy+3y^2+3x-2y=c$

42. $(5y-2x-3)^4=c(4y-4x-3)$

43. $2x-y-15 \log (3x-y+19)=c$

44. $\log (4x+8y+5)=4x-8y+c$

45. $(3y-5x+10)^2\ c=(y-x+1).$

46. $6y-3x=\log (3x+3y+2)+A.$

47. $2x-y=\log(3x-2y+3)+c$

48. $y-x=\log (x+y)+c$

49. $3x^2+8xy-5y^2+10x-12y+c=0.$

50. $\frac{I}{I_0}=e^{-\frac{R}{L}t}$ 51. $p=A+cr^{-2}.$ 52. $y^2-x^2=2(x-y).$

53. $\frac{y^3}{3}+\frac{y^2}{2}+y=\frac{x^3}{3}+\frac{x^2}{2}+x.$ 54. $\tan x \tan y=1$

55. $\tan^{-1}\frac{2x+1}{\sqrt{3}}+\tan^{-1} \frac{2y+1}{\sqrt{3}}=\frac{2\pi}{3}$

56. $(x+1)^2+(y+1)^2+2 \log (x-1)(y-1)=18$

Exercise 3

1. $y=c_1 e^{-3x}+c_2 e^{-4x}$ 2. $y=c_1 e^{2x}+c_2 e^{3x}$

3. $y=c_1+c_2 e^{-3x}.$ 4. $y=c_1 e^{-2x}+c_2 e^{-\frac{1}{2}x}$

5. $x=c_1 e^{-3t}+c_2 e^{-\frac{1}{3}t}.$ 6. $y=c_1 e^{3x}+c_2 e^{8x}.$

7. $y=c_1 e^x+c_2 e^{-x}.$ 8. $x=c_1 e^t+c_2 e^{at}.$

9. $y=c_1e^{2x}+c_2e^{-4x}$.

10. $y=c_1e^{(-1+\sqrt{7})x}+c_2e^{(-1-\sqrt{7})x}$

11. $y=(c_1+c_2x)e^{-x}$ 12. $x=(c_1+c_2t)e^{-\frac{1}{2}t}$

13. $s=(c_1+c_2t)e^{3t}$ 14. $y=(c_1+c_2x)e^{ax}$

15. $y=(c_1+c_2x)e^{-x}$ 16. $y=(c_1+cx)e^{\frac{b}{a}x}$

17. $y=Ae^{x}\cos x+Be^{x}\sin x$.

18. $y=A\cos 2x+B\sin 2x$. 19. $y=e^{-2x}(A\cos 6x+B\sin 6x)$

20. $x=e^{3t}(A\cos 5t+B\sin 5t)$.

23. $y=2e^{3x}+3e^{4x}$. 24. $y=3+2e^{\frac{5}{2}x}$.

25. $y=2e^{-x}-e^{-2x}$. 26. $x=4e^{-t}-e^{-4t}$

27. $y=(2+6x)e^{-2x}$. 28. $x=(2t+4)e^{-t}$

29. $y=e^{-2x}(2\cos 3x+\sin 3x)$

30. $x=e^{-t}(4\cos 2t+\sin 2t)$

31. $i=e^{-\frac{Rt}{2L}}(c_1+c_2t)$.

APPLICATION OF CALCULUS

CHAPTER ONE

SIGNIFICANCE OF DERIVATIVE

§ *1.1. Average rate of change of a function :*

Example 1. Let the variable y be a function of x. and $y=x^2$. Let us discuss the increase in the value of y as the value of x increases from 2.

The following table shows the values of y corresponding to values of x very close to 2.

x	2	2·10	2·01	2·002	2·0003
$y=x^2$	4	4·41	4·0401	4·008004	4·00012009

From the above table we find that as x increases from 2 by 0·1, y increases from 4 by ·41. So when x increases by 1, y increases by $\frac{\cdot 41}{\cdot 1}=4{\cdot}1$. Here we say that as x increases from 2 to 2·1, the average rate of increase in the value of y is 4·1. Similarly as the value of x increases from 2 to 2·01, 2·002, 2·0003 the average rate of change in the value of y are

$$\frac{\cdot 0401}{\cdot 01}=4{\cdot}01,\ \frac{\cdot 008004}{\cdot 002}=4{\cdot}002, \frac{\cdot 00012009}{\cdot 003}=4{\cdot}0003$$

So, when the value of x is 2, then the average rate of increase of y is not the same for different increases of values of x. In the above table we have considered actual increases in the value of x. Similarly by preparing table corresponding to decreases in the values of x from 2 we can see that the average rates of decrease in the values of y will also be different in different cases.

Note : If the values of x increases then the values of y may increase or decrease. In case of the function $y=x^2$, as the values of x increase, the values of y also increase and as the values of x decrease, the values of y also decrease. But in many cases corresponding to increase or decrease in values of x, the values of y respectively decrease or increase.

In the above example we have seen that the average rates of increase in the values of y for different increases in the values of x from 2 are different. So, none of them can be taken as the **instantaneous rate** or **rate** of increase of y with respect to x when $x=2$. But determination of this instantaneous rate is very important in mathematics and different branches of science. This importance is illustrated by another example.

Example 2. Average velocity of a particle moving along a straight line at a particular instant.

Let a particle moving along a straight line AB comes to the position P at time t units of time after start and the direction of motion be $\overrightarrow{AB}$ (i.e., from A to B). Let the distance of P from a fixed point A of the straight line be x i.e., $AP=x$. Let after a subsequent time Δt, the particle reaches a point Q of the straight line and $AQ=x+\Delta x$. $\therefore$ $PQ=\Delta x$. Even if the time Δt be very small, it cannot be assumed that the particle has traversed the distance $PQ=\Delta x$ uniformly.

So $\frac{\Delta x}{\Delta t}$ cannot be taken as the velocity of the particle at P; $\frac{\Delta x}{\Delta t}$ is the average velocity of the particle during the time Δt. In this case also to determine the velocity of the particle at P, the instantaneous rate of change of x at P with respect to time 't' must be known. [The ancient Hindu Mathematicians used the term "Tatkalya gati".]

Note: In the above discussion 'Δt' may be positive or negative. If Δt be negative, then the particle will reach the position Q before it reaches the position P and Δx will be negative. If 'Δt' be negative, then actually we consider decrease in the value of 't' or 'x'; but in this case also we use the term 'increment'; this increment is negative increment.

§ *1.2. Significance of derivative* :

Let x and y be two variables and they are connected by the relation $y=f(x)$. So if there is change in the value of x, then the value of y will also change. Let the value of x changes to

$x+\Delta x$. Then the value of y will change to $y+\Delta y=f(x+\Delta x)$ and Δx and $\Delta y=f(x+\Delta x)-f(x)$ are the corresponding changes in x and y respectively. Hence $\frac{\Delta y}{\Delta x}$ is the average rate of change of y with respect to x. In this case if $|\Delta x|$ be less then any positive number however small i.e., if $\Delta x\to 0$ (Δx cannot be zero, for then $\frac{\Delta y}{\Delta x}$ will become undefined), then we can assume that y changes uniformly corresponding to the small change Δx of x. In this case we can say that $\frac{\Delta y}{\Delta x}$ is the instantaneous rate of change of y with respect to x.

So, $\underset{\Delta x\to 0}{Lt}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}$, the derivative of y with respect to x, is the instantaneous rate of change of y with respect to x.

Example 1. When $x=-1$, find the rate of change of $y=x^3$ with respect to x.

Let the value of x changes from -1 to $-1+\Delta x$; so the changed value of y is

$$y+\Delta y=(-1+\Delta x)^3=-1+3\Delta x-3(\Delta x)^2+(\Delta x)^3$$

$$\therefore \quad \Delta y=(-1+\Delta x)^3-(-1)^3=-1+3\Delta x-3(\Delta x)^2+(\Delta x)^3-1$$

$$=3\Delta x-3(\Delta x)^2+(\Delta x)^3$$

$$\therefore \quad \frac{\Delta y}{\Delta x}=\frac{3\Delta x-3(\Delta x)^2+(\Delta x)^3}{\Delta x}=3-3\Delta x+(\Delta x)^2$$

So, when $x=-1$, the rate of change of y with respect to x is

$$\underset{\Delta x\to 0}{Lt}\frac{\Delta y}{\Delta x}=\underset{\Delta x\to 0}{Lt}\{3-3\Delta x+(\Delta x)^2\}=3.$$

Again note that $\frac{dy}{dx}=\frac{d}{dx}(x^3)=3x^2$.

So, $\left(\frac{dy}{dx}\right)_{x=-1}=\left(3x^2\right)_{x=-1}=3\times 1=3.$

So, the rate of change of y with respect to x is 3 when $x=-1$ and is equal to $\frac{dy}{dx}$ when $x=-1$.

§ *1.3. Velocity and acceleration of a particle moving along a straight line at time 't' units after start.*

See Example 2 of § 1.1.

In this case, the position of the particle at time 't' after start is at P; at that instant i.e., at P the average rate of displacement i.e., average velocity of the particle is $\frac{\Delta x}{\Delta t}$. So the velocity of the particle at that instant is $\underset{\Delta t \to 0}{Lt} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$.

Again if the velocity of the particle at P be $v = \frac{dx}{dt}$, then the acceleration of the particle at that instant is the rate of change of v with respect to $t = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

Again $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$.

Note 1. Use one of the expressions $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$ or $v\frac{dv}{dx}$ to denote acceleration of a particle according to convenience or necessity.

2. At first Newton and afterwards the British Mathematicians used for sometimes the notations $\dot{x}$ and $\ddot{x}$ instead of $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ to denote derivatives of x with respect 't'.

3. Displacement is frequently denoted by s instead of x. In terms of S, evidently, the expressions for velocity and acceleration at time 't' are $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ respectively.

4. $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ represent velocity and acceleration in the direction in which x is positive.

5. Negative acceleration is generally called retardation.

§ 1.4. *Geometrical interpretation of $\frac{dy}{dx}$ or derivative.*

Let the two variables x and y be connected by the relation $y=f(x)$. Draw the graph of $y=f(x)$ with respect to a suitably chosen pair of mutually orthogonal straight lines OX and OY as cartesian axes of co-ordinates, of course with a conveniently chosen scale (Take OX and OY as the axis of x and axis of y respectively as usual). Let the graph be the curve PQ. $P(x, y)$ is a given point of the graph. Take a point Q on the graph very close to P. Join PQ; PQ is then a chord of the graph. If $(x+\Delta x,\ y+\Delta y)$ be the co-ordinates of Q, then the gradient of the chord PQ is $\tan\theta=\frac{y+\Delta y-y}{x+\Delta x-x}=\frac{\Delta y}{\Delta x}$. ($\theta$ being the angle which the straight line PQ makes with the positive direction of the x-axis). Let now the point Q gradually approach the point P along the graph. In the limiting position as the point Q practically coincides with P, the chord PQ becomes the tangent to the curve at P. Let the tangent intersect the x-axis at a point T, making an angle ψ with the positive direction of the x-axis. So, when $Q\to P$ then $\theta\to\psi$ or $\tan\theta\to\tan\psi$. Again when $Q\to P$, then $\Delta x\to 0$.

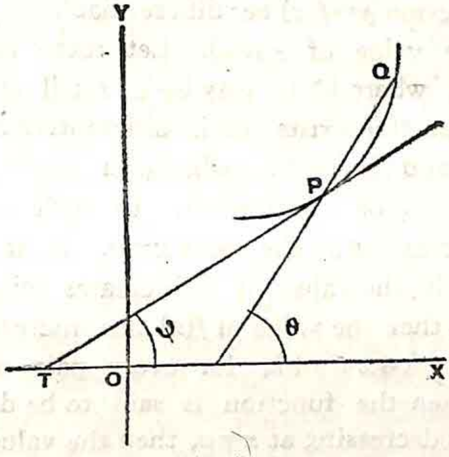


Fig. 1·1

$$\text{So } \tan\psi=\underset{\Delta x\to 0}{Lt}\tan\theta=\underset{\Delta x\to 0}{Lt}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}.$$

We know that the gradient of a straight line is the tangent of the angle which the straight line makes with the positive direction of the x-axis. So when $\frac{dy}{dx}$ exists, it is the gradient $\tan\psi$ of the tangent to the curve $y=f(x)$ at $x=P$.

Note. 1. The gradient or slope of the tangent to the curve $y=f(x)$ at the point $P(\alpha,\beta)$ is $\left(\frac{dy}{dx}\right)_{(\alpha,\beta)}$.

2. $\left(\frac{dy}{dx}\right)_{(\alpha,\beta)}$ is also said to be the gradient of the curve at the point $P(\alpha,\beta)$.

3. If $\frac{dy}{dx}=0$ at any point, then the tangent to the curve at the point is parallel to the x-axis ; if $\frac{dy}{dx}$ is undefined at a point of the curve, then the tangent to the curve at the point is parallel to the y-axis.

§ 1·5. *Significance of the sign of derivatie.*

Let the function $y=f(x)$ be differentiable at the point $x=a$ (i.e., when the value of x is a). Let there exists an interval $a-h \leqslant x \leqslant a+h$, where $h>0$, may be as small as is required, so that the function $f(x)$ exists and is differentiable at every point of the interval and $f(x_1)>f(x_2)$ when ever $x_1>x_2$ for every pair of points x_1 and x_2 of the interval. In such cases the function is said to be increasing at the point $x=a$. So if $f(x)$ is increasing at $x=a$, then if the value of x increases within the interval $a-h \leqslant x \leqslant a+h$, then the value of $f(x)$ also increases. Again if in the interval $a-h \leqslant x \leqslant a+h$, for every pair of points $x_1>x_2$, $f(x_1)<f(x_2)$, then the function is said to be decreasing at $x=a$. So when $f(x)$ is decreasing at $x=a$, then the value of $f(x)$ decreases as x increases within the interval.

Theorem : When $x=a$, if a function $f(x)$ is differentiable and $f'(a)>0$, then the function is increasing at $x=a$ and the function is decreasing at $x=a$ if $f'(a)<0$.

Proof : First Let $\frac{dy}{dx}=f'(x)>0$ at $x=a$.

So if $|h|$ be sufficiently small, then $\frac{f(a+h)-f(a)}{h}$ is positive.
So when $h>0$, then $f(a+h)-f(a)>0$ i.e., $f(a+h)>f(a)$.
Also if $h<0$, then $f(a+h)-f(a)<0$, i.e., $f(a+h)<f(a)$.
So, when $x=a$, $f(x)$ is increasing.

Next let when $x=a$, then $\frac{dy}{dx}<0$.

So, if $|h|$ be sufficiently small then $\frac{f(a+h)-f(a)}{h}<0$.

$\therefore$ when $h>0$, then $f(a+h)-f(a)<0$ i.e., $f(a+h)<f(a)$ and when $h<0$, then $f(a+h)-f(a)>0$ i.e., $f(a+h)>f(a)$.

So when $x=a$, $f(x)$ is decreasing.

Geometrical discussion : In Fig 1·2 when $x=a$, the point $(a, 0)$ is the point P on the x-axis, and when $x=a_1, a_2, a_3$ or a_4, then the corresponding points on the x-axis are P_1, P_2, P_3, P_4 and $a_1<a_2<a<a_3<a_4$. At these points P_1, P_2, P, P_3, P_4, P_1Q_1, P_2Q_2. PQ, P_3Q_3, P_4Q_4 are drawn perpendiculars on the x-axis and they meet the curve $y=f(x)$ at the points Q_1, Q_2, Q, Q_3, Q_4 respectively. So $P_1Q_1 =f(a_1)$, $P_2Q_2=f(a_2)$, $PQ =f(a)$, $P_3Q_3=f(a_3)$ and $P_4Q_4=f(a_4)$. From the figure it is seen that $P_1Q_1 < P_2Q_2 < PQ < P_3Q_3 < P_4Q_4$. Similarly taking other points on the x-axis between P_1 and P_4, it will be found that as the abscissas of points on the curve between Q_1 and Q_4 increase, the corresponding ordinates also increase. So at the point P i.e., when $x=a$, the function $y=f(x)$ is increasing. Notice that in this case the tangent to the curve Q makes an acute angle ψ with the positive direction of the x-axis.

$$\therefore \tan\psi=\frac{dy}{dx}>0.$$

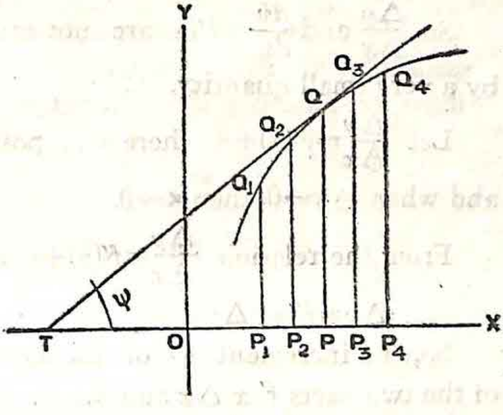


Fig. 1·2

Again in fig 1·3, we find that as the abscissas of points on the curve, very close to $Q\{a, f(a)\}$, increase, the corresponding ordinates decrease. So, in this case the function $y=f(x)$ is decreasing at $x=a$. Notice that here the tangent to the curve at the point Q makes an obtuse angle ψ. with the positive direction of the x-axis. So $\tan\psi=\dfrac{dy}{dx}$ is negative.

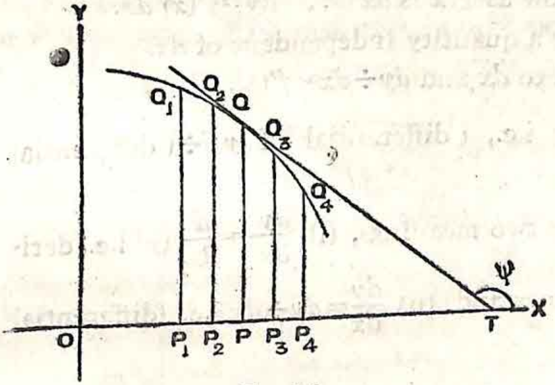


Fig. 1·3

§ 1·6. *Differential.*

Let the function $y=f(x)$ be differentiable at the point x and so $\underset{\Delta x\to 0}{Lt}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}$ exists.

So $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}=f'(x)$ are not equal in value but they differ by a very small quantity.

Let $\frac{\Delta y}{\Delta x}=f'(x)+\alpha$ where α is positive or negative but not 0 and when $\Delta x\to 0$, then $\alpha\to 0$.

From the relation $\frac{\Delta y}{\Delta x}=f'(x)+\alpha$ we get

$$\Delta y=f'(x)\,\Delta x+\alpha.\Delta x \quad \ldots \quad \ldots \quad \text{(i)}$$

So, the increment Δy of the dependent variable y is the sum of the two parts $f'(x)\Delta x$ and $\alpha.\Delta x$.

The part $f'(x)\Delta x$ is called the differential of y and is denoted by dy.

$\therefore\ dy=f'(x).\Delta x.$

Now if $y=x$, then $dx=1.\Delta x=\Delta x$ [as when $y=x$, then $\frac{dy}{dx}$ or $f'(x)$ is equal to 1.]

So, the differential of the independent variable is equal to its small increment Δx. Now as Δx is dx $\therefore\ dy=f'(x)\,dx.$

Note. 1. Here $f'(x)$ is a quantity independent of dx.

$\therefore$ dy is proportional to dx and $dy\div dx=f'(x)$.

So $\frac{dy}{dx}=f'(x)=dy\div dx$ i.e., (differential of y)$\div$(differential of x).

So, the symbol $\frac{dy}{dx}$ has two meanings, (i) $\frac{dy}{dx}=\frac{d}{dx}(y)$ i.e., derivative of y with respect to x and (ii) $\frac{dy}{dx}=dy\div dx$ i.e. (differential of y)$\div$(differential of x).

2. If u and v be two differentiable functions of x, then

$$d(u\pm v)=\frac{d}{dx}(u\pm v)\,dx.=\left(\frac{du}{dx}\pm\frac{dv}{dx}\right)dx$$
$$=\frac{du}{dx}\cdot dx\pm\frac{dv}{dx}\cdot dx=du\pm dv.$$

Similarly if can be shown that

$$d(uv)=udv+vdu \text{ and } d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}.$$

§ 1·7 *Geometrical representation of differential.*

In figures 1·4 and 1·5, the co-ordinates of the points P and Q are (x, y) and $(x+\Delta x, y+\Delta y)$ respectively. PL and QM are drawn perpendiculars on the x-axis and PN is drawn perpendicular QM on PN intersects MQ (in fig. 1·4) or MQ produced (in fig. 1·5) at the point N. Let the tangent to the curve $y=f(x)$ at P intersect. MN produced (fig. 1·4) or MN (fig. 1·5) at R.

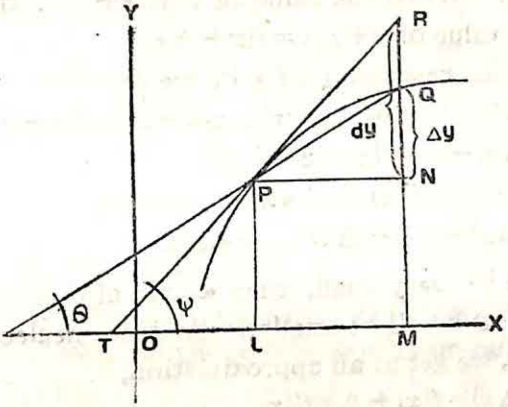


Fig. 1·4

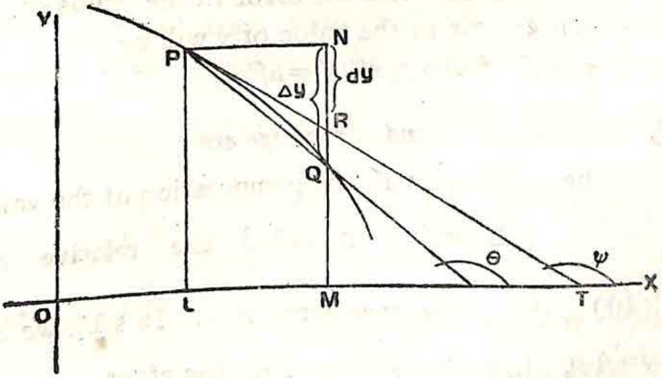


Fig. 1·5

Now $OL=x$, $PL=y$, $OM=x+\Delta x$, $QM=y+\Delta y$.

$\therefore\ PN=LM=\Delta x$

Now in fig. 1·4, $RN = PN \tan\psi = \Delta x\, f'(x)$
$[\because \angle RPN = \angle PTL = \psi] = f'(x)dx = dy.$
$\therefore$ dy = (ordinate of the point R) − (ordinate of the point N)
Again in fig. 1·5, $\angle RPN = \pi - \psi.$
$\therefore$ $RN = PN \tan(\pi - \psi) = -PN \tan\psi = -f'(x)\Delta x = -f'(x)dx.$
$\therefore$ $dy = -f'(x)dx = -RN$ = (ordinate of N) − (ordinate of R)

§ 1·8. *Calculation of approximate value and Error.*

Let $y = f(x)$. When the value of x is $x + \Delta x$, then we are to determine the value of $y + \Delta y = f(x + \Delta x)$.

Now from the relation (i) of § 1·6 we get

$$\Delta y = f'(x)\Delta x + \alpha.\Delta x \quad \text{where } \alpha \to 0 \text{ as } \Delta x \to 0$$

or, $(y + \Delta y) - y = f'(x)\Delta x + \alpha.\Delta x$
or, $y + \Delta y = y + f'(x)\Delta x + \alpha.\Delta x$
or, $f(x + \Delta x) = f(x) + \Delta x f'(x) + \alpha.\Delta x.$

Now if Δx be very small, then α will also be very small and consequently $\alpha.\Delta x$ will be smaller. So if we neglect this smaller quantity $\alpha.\Delta x$, we get as an approximation,

$$f(x + \Delta x) = f(x) + \Delta x . f'(x).$$

or, $f(x + h) = f(x) + hf'(x)$ (putting $\Delta x = h$.)

So for the increment Δx or h in the value of x, the corresponding value of y will be approximately

$$f(x + \Delta x) = f(x) + \Delta x f'(x) \quad \text{or,} \quad f(x + h) = f(x) + hf'(x)$$

In this case Δx may be called the error in the value of x and so the corresponding error in the value of y will be

$$\Delta y = f(x + \Delta x) - f(x) = \Delta x f'(x) = hf'(x).$$

§ 1·9, *Relative Error and percentage error.*

If Δx be the error in the computation of the value of x, then $\dfrac{\Delta x}{x} = \dfrac{dx}{x}$ $[\Delta x = dx]$ is called the relative error in x. $\dfrac{dx}{x} \times 100$ is the percentage error in x. In § 1·8 we have seen that if $y = f(x)$, then $dy = f'(x)dx$ i.e., the error in y due to an error $\Delta x = dx$ of x is dy. $\dfrac{dy}{y}$ and $\dfrac{dy}{y} \times 100$ are respectively the relative error and percentage error in the values of y.

EXAMPLES 1

Example 1. Rate of change of radius of a circle is $\frac{1}{\pi}$. Find the rate of changes of (i) circumferential length and (ii) area of the circle at the instant when the radius is 2 units. [H.S. 1985]

Let S, A and r denote the length of the circumference, area and radius of a circle in units, square units and units respectively.

$\therefore \; S = 2\pi r \cdots\cdots$(i) and $A = \pi r^2$.

So if 't' denotes time, then by question $\frac{dr}{dt} = \frac{1}{\pi}$

Now $\frac{dS}{dt} = 2\pi \frac{dr}{dt} = 2\pi . \frac{1}{\pi} = 2$

Also from (ii) we get when $r = 2$,

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi . 2 . \frac{1}{\pi} = 4.$

So when $r = 2$ units, then the rates of change of circumferencial length and area of the circle are 2 units/unit time and 4 square units/unit time.

Example 2. If the area of a circle increases at a uniform rate, show that the rate of increase of its circumference is inversely proportional to the rate of increase of the radius of the circle. [C. U.]

Let S, A, r and t denote the length of circumference, area, radius of the circle and time respectively.

$\therefore \; S = 2\pi r \cdots\cdots$(i) $A = \pi r^2 \cdots\cdots$(i)

From (ii), rate of increase of the area is $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

So, by question, $\frac{dA}{dt}$ is constant or $2\pi r \frac{dr}{dt} = k$ (say).

$$\text{or,} \quad \frac{dr}{dt} = \frac{k}{2\pi r}$$

Now rate of increase in the length of eircumference is from (ii) $\frac{dS}{dt} = 2\pi \frac{dr}{dt} = 2\pi \frac{k}{2\pi r} = \frac{k}{r} \quad \left[\because \; \frac{dr}{dt} = \frac{k}{2\pi r} \right]$

$\therefore$ The rate of change of the circumferential length is inversely proportional to the radius as k is a constant.

Example 3. The volume of a spherical balloon is increasing at the rate of 10 cubic centimetres per second. Find the rate of change of its curved surface at the instant when the radius is 16 cm.

[Tripura 1981, '87]

Let V, S and r denote volume, area of curved surface and radius of the balloon in cubic cenrimetre, square centimetre and centimetres respectively. Also let 't' denote time in second.

$\therefore$ By question rate of change of volume $\frac{dv}{dt}=10$ (given).

Now $v=\frac{4}{3}\pi r^3$ or, $\frac{dv}{dt}=4\pi r^2\frac{dr}{dt}$

$\therefore\ 4\pi r^2\frac{dr}{dt}=10 \quad \therefore\ \frac{dr}{dt}=\frac{10}{4\pi r^2}$

Again the curved surface $S=4\pi r^2$

$\therefore$ rate of change of the curved surface

$=\frac{dS}{dt}=8\pi r\frac{dr}{dt}=8\pi r\cdot\frac{10}{4\pi r^2}=\frac{20}{r}$

So when $r=16$, $\frac{dS}{dt}=\frac{20}{16}=\frac{5}{4}$

So the rate of imcrease of the curved surface is $\frac{5}{4}$ sq.cm./sec.

Example 4. The magnitude of an angle is increasing at a uniform rate. Prove that when the magnitude of the angle is 60°, then the rate of increase of the tangent of the angle is eight times the rate of increase of the sine of the angle.

Let θ and t respectively denote the magnitude of the angle in radians and time in second. So by question, $\frac{d\theta}{dt}$ = constant $=a$ (say).

$\therefore$ Now the rate of increase of sine of the angle$=\frac{d}{dt}(\sin\theta)$ $=\cos\theta\frac{d\theta}{dt}=a\cos\theta$ and rate of the increase of tangent of the angle is $\frac{d}{dt}(\tan\theta)=\sec^2\theta\frac{d\theta}{dt}=a\sec^2\theta$.

So, when $\theta=60°$, then

$$\frac{\text{rate of increase of tangent}}{\text{rate of increase of sine}}=\frac{a\sec^2 60°}{a\cos 60°}=\frac{4}{\frac{1}{2}}=8.$$

So, when $\theta=60^\circ$, the rate of increase of the tangant of the angle is 8 times the rate of increase of sine of the angle.

Example 5. The candle-power of an incandescent lamp and its voltage V are connected by the equation $C=5\times10^{-11}V^6$. Find the rate at which the candle-power increases with the voltage when $V=200$. [State council W. Bengal 1986]

Here the rate of change of C with respect to V are

$$\frac{dC}{dV}=30\times10^{-11}\,V^5.$$

So when $V=200$, its value$=30\times10^{-11}\times(200)^5$

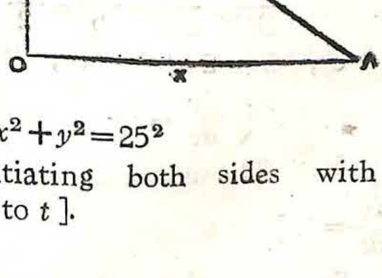
$$=3\times32\times10^{-11}\times10^{11}=96.$$

[$(200)^5=2^5\,.\,(100)^5=32\times10^{10}$]

Note : In the first four examples the rates of change were measured with respect to time. So, the students may think that rates of change are always measured with respect to time. But that it is not true has been illustrated in the ex. 5.]

Example 6. One end of a ladder 25 m long rests against a vertical wall and the other end on the horizontal ground. The horizontal end slides away from the wall at the rate 5 metres per second. When the height of the ladder is 24 metres, find with what velocity its upper end is coming down ?

Let at time t (second) the position of the ladder be AB. Let the perpendicular drawn from A on the vertical wall and the vertical line drawn on the wall through B intersect at O. Let $OA=x$ metre and $OB=y$ metre.



So, the velocities of the lower and upper ends are $\frac{dx}{dt}$ and $\frac{dy}{dt}$ respectively. So $\frac{dx}{dt}=5$ (given)

Again $AB=25\ m$ (given) $\quad\therefore\quad x^2+y^2=25^2$

$\therefore\ 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0.$ [differentiating both sides with respect to t].

$$\text{or,}\quad \frac{dy}{dt}=-\frac{x}{y}\,\frac{dx}{dt}.$$

Again when the height y of the ladder is 24 m,
Then $x=\sqrt{25^2-24^2}=7$.

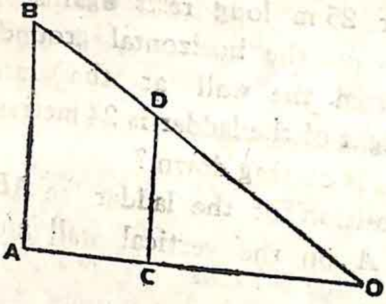
So, when $y=24$, the velocity of the upper end is

$$\frac{dy}{dt}=-\frac{5}{24}\cdot 7=-1\frac{11}{24}.$$

The negative sign of $\frac{dy}{dt}$ indicates that the velocity is in the direction opposite to that of y increasing. So, y is decreasing i.e., the ladder is coming downwards and its velocity then is $1\frac{11}{24}$ m / sec.

Example 7. A man 5 ft. tall is moving away from a lamp post $12\frac{1}{2}$ ft. high with a velocity of 3 miles per hour. Find the rate at which his shadow is lengthening. [Joint Entrance 1979]

Find also the rate at which the furthest extremity of the shadow is moving.



Let the source of light be placed at the top B of the lamp-post and CD be the position of the man at time t (hour). BD is joined and produced to intersect AC produced at O.

So CO is the shadow of the man and O is the furthest extremity of the shadow.

Let $AC=x$ and $CO=y$. $\therefore$ $AO=AC+CO=x+y$ (each in mile).

Now from the similar triangles ABO and CDO we find,

$$\frac{OA}{OC}=\frac{AB}{CD}=\frac{12\frac{1}{2}}{5}=\frac{5}{2} \quad \text{or} \quad \frac{x+y}{y}=\frac{5}{2}$$

or, $5y=2x+2y$ or, $3y=2x$ or, $y=\frac{2}{3}x$.

$$\therefore \quad \frac{dy}{dt}=\frac{2}{3}\frac{dx}{dt}=\frac{2}{3}\cdot 3=2.$$

[$\because$ The man is moving away with a velocity of 3 mph, so $\frac{dx}{dt}=3$.]

So, the shadow is lengthening at a rate of 2 miles per hour.

Again $\frac{d}{dt}(AO)=\frac{d}{dt}(x+y)=\frac{dx}{dt}+\frac{dy}{dt}=3+2=5$

So, the end O of the shadow is moving away at the rate of 5 miles per hour.

Example 8. A circular blot of ink grows uniformly at the rate of 2 sq. inches per second. Find the rate at which its radius is increasing after $2\frac{6}{11}$ seconds.

Let A, r and t respectively denote the area (in sq. inches), radius (in inches) of the circular blot and t time (in seconds).

So, $\frac{dA}{dt}=2$ (by question).

Now the area of the circular blot after $2\frac{6}{11}$ second is $2\times2\frac{6}{11}$ $=\frac{56}{11}$ square inches. $\therefore$ $\pi r^2=\frac{56}{11}$ square inches

$$\therefore\ r^2=\frac{56}{11\times\pi}=\frac{56}{11\times\frac{22}{7}}=\frac{56\times7}{11\times22}\quad\therefore\ r=\frac{14}{11}$$

Now $A=\pi r^2$ or, $\frac{dA}{dt}=2\pi r\frac{dr}{dt}=2\times\frac{22}{7}\times\frac{14}{11}\frac{dr}{dt}$

$$\therefore\ 2=2\times\frac{22}{7}\times\frac{14}{11}\times\frac{dr}{dt}\quad\therefore\ \frac{dr}{dt}=\frac{1}{4}=0\cdot25.$$

Hence radius of the blot is increasing at the rate of 0·25 inches.

Example 9. Water is poured into an inverted conical vessel of which the radius of the base is 6 metres and height 12 metres at the rate of 5·5 cubic centimetre per minute. At what rate is water level rising at the instant when the depth is 3·5 cm ?

[State Council W. Bengal 1987]

Let at time t (minute), the volume of water in the vessel be V c.c., the radius and height of the water level be r cm. and h cm. respectively.

So, from figure we get $OA=12$ metres, $AB=6$ m. $OC=h$ cm. and $CD=r$ cm. (O is the vertex of the cone.).

Now from the similar triangles OCD and OAB we get

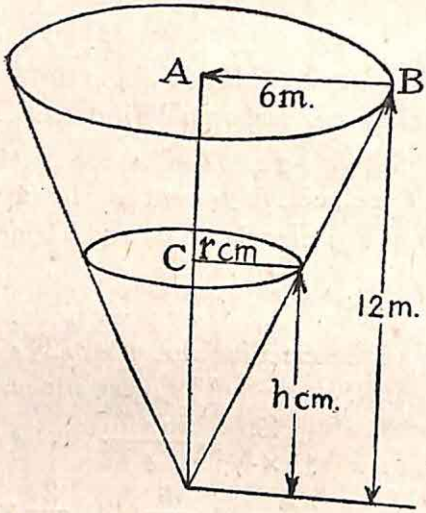
$$\frac{OC}{OA}=\frac{CD}{AB}\ \text{or,}\ \frac{OC}{CD}=\frac{OA}{AB}=\frac{12m}{6m}=2.$$

$\therefore$ $CD=\frac{1}{2}OC$ i.e., $r=\frac{1}{2}h$

$\therefore$ $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\frac{1}{4}h^2.h=\frac{1}{12}\pi h^3$

Differentiating both sides with respect to t we get

$$\frac{dv}{dt}=\frac{1}{4}\pi h^2\frac{dh}{dt}$$



Now for all values of t, $\frac{dv}{dt}=5{\cdot}5=\frac{11}{2}$

$\therefore$ When $h=3{\cdot}5=\frac{7}{2}$, $\frac{11}{2}=\frac{1}{4}\times\frac{22}{7}\times\frac{7\times7}{2\times2}\times\frac{dh}{dt}$ $\therefore$ $\frac{dh}{dt}=\frac{4}{7}$

Hence water level is rising at the rate of $\frac{4}{7}$ cm. when $h=3{\cdot}5$ cm.

Example 10. Air is discharged from a spherical balloon by reducing its diameter at the rate of 1 cm. per second. Find the rate at which air will be discharged when the radius of the balloon is 2 cm.

Let at time t (second), the volume of air in side the balloon and the radius of the balloon be v c.c. and r cm. respectively.

$\therefore$ By question, $\frac{dr}{dt}=-\frac{1}{2}$ [As the diameter is reduced at the rate of 1 cm / sec]

Now $v=\frac{4}{3}\pi r^3$ or, $\frac{dv}{dt}=4\pi r^2\frac{dr}{dt}$.

So, when $r=2$, $\frac{dv}{dt}=4\cdot\pi\cdot(2)^2\cdot\left(-\frac{1}{2}\right)=-8\pi$.

Hence when the radius of the balloon is 2 cm, water is discharged at the rate of 8π c.c. per second.

Note : As r is reducing, so $\frac{dr}{dt}$ is negative. Again as $\frac{dv}{dt}$ is found negative, so the volume of air is decreasing.

Example 11. The pressure P and volume V of a gas are connected by the relation $PV^{1\cdot4}=C$ (C is a constant) when P is 10 kg. / sq. metre and $V=3$ cubic metre, then V increases at the rate of 0·3 cubic metre / sec. Find the rate of change of P at that instant.

Let 't' denote time (in second).

Now $P \,.\, V^{1\cdot4}=C$ or, $\log (P \,.\, V^{1\cdot4})=\log C$

or, $\log P+1\cdot4 \log V=\log C$.

Differentiating both sides with respect to 't' we get

$$\frac{1}{P}\,\frac{dP}{dt}+1\cdot4\frac{1}{V}\,\frac{dV}{dt}=0.$$

$$\text{or, } \frac{1}{P}\,\frac{dP}{dt}=-1\cdot4\frac{1}{V}\,\frac{dV}{dt} \text{ or, } \frac{dP}{dt}=-1\cdot4\,\frac{P}{V}\,\frac{dV}{dt}.$$

So, when $P=10$, $V=3$, $\frac{dV}{dt}=0\cdot3$ (given),

$$\text{then } \frac{dP}{dt}=-1\cdot4\times\tfrac{10}{3}\times0\cdot3=-1\cdot4$$

Hence pressure P is decreasing at the rate of 1·4 kg. / square metre. (The negative sign indicates that the pressure is decreasing).

Example 12. While a train is travelling from rest to the next station, its distance x km from the start in t hours is given by $x=90t^2-45t^3$. Find its velocity and acceleration after 6 minutes. [H.S. 1981]

$$\left(\frac{dx}{dt}=\text{velocity and } \frac{d^2x}{dt^2}=\text{acceleration}\right).$$

6 mins $=\frac{1}{10}$ hour. $x=90t^2-45t^3$.

$$\therefore \frac{dx}{dt}=180t-135t^2 \,;\, \frac{d^2x}{dt^2}=180-270t$$

$$\text{Hence velocity after 6 mins}=\left(\frac{dx}{dt}\right)_{t=\frac{1}{10}}$$

$$=180\times\tfrac{1}{10}-135\times\frac{1}{10^2}=18-1\cdot35=16\cdot65 \text{ km/hr.}$$

Also acceleration after 6 mins. $=180-270\times\frac{1}{10}=153$ km/(hour)^2

Example 13. A train starts from A and moves in a straight path and stops at B. Its distance x km. from the start after

t minutes is given by $x=\frac{3}{4}t^2-\frac{1}{4}t^3$ and its velocity then is $\frac{dx}{dt}$. Find the distance between A and B. What is the maximum velocity of the train ? [H.S. 1986]

$$x=\frac{3}{4}t^2-\frac{1}{4}t^3. \quad \therefore \frac{dx}{dt}=\frac{3}{2}t-\frac{3}{4}t^2.$$

As the train stops at B, so its velocity at B is 0.

$\therefore\ 0=\frac{3}{2}t-\frac{3}{4}t^2=\frac{3}{4}t(2-t)\ \therefore\ t=0$ or 2.

$t=0$ corresponds to the time of start from A.

So the train takes 2 minutes to reach B.

$\therefore$ distance $AB=\frac{3}{4}.2^2-\frac{1}{4}.2^3=1$km.

Again velocity at any time t (in minutes) is

$$\frac{dx}{dt}=\frac{3}{2}t-\frac{3}{4}t^2=-\frac{3}{4}(t^2-2t)=-\frac{3}{4}(t^2-2t+1)+\frac{3}{4}=\frac{3}{4}-\frac{3}{4}(t-1)^2.$$

So, the velocity will be maximum when $\frac{3}{4}(t-1)^2=0$ i e., when $t=1$ and the maximum velocity is $\frac{3}{4}$ km/minute.

Example 14. If the distance traversed by a point moving in a straight line at time t is $s=\sqrt{t}$, show that at any time the retardation of the point is proportional to the cube of the velocity. [Use the method of Differential Calculus] [H.S. '88]

$$s=\sqrt{t}. \ \therefore \ \frac{ds}{dt}=\frac{1}{2\sqrt{t}} \text{ and } \frac{d^2s}{dt^2}=-\frac{1}{4}\cdot\frac{1}{t^{3/2}}$$

$$=-\frac{2}{8t^{3/2}}=-\frac{2}{(2t^{1/2})^3}=-\frac{2}{(2\sqrt{t})^3}=-\frac{2}{\left(\frac{ds}{dt}\right)^3}$$

Now, $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ are respectively the velocity and acceleration of the point at time t. As $\frac{d^2s}{dt^2}$ is negative, so the acceleration is retardation and as 2 is a constant, so it is propertional to the cube of the velocity.

Example 15. A particle starts with a velocity u and moves in a straight line, its acceleration being always equal to the displacement. If v be the velocity when its displacement is x, then show that $v^2=u^2+x^2$. [Joint Entrance 1984]

We know accelaration$=v\dfrac{dv}{dx}$.

Again acceleration=displacement [by question]

$\therefore\ v\dfrac{dv}{dx}=x$ or. $vdv=xdx$

or, $\dfrac{v^2}{2}=\dfrac{x^2}{2}+\dfrac{C}{2}$ (Integrating both sides) or, $v^2=x^2+C$.

Now when $x=0$, then $v=u$

$\therefore\ u^2=C. \quad \therefore\ v^2=x^2+u^2.$

Example 16. The distance s cm. moved by a particle in time t seconds is given by the formula $S=2t^3-4t^2+3t$. Find its velocity and acceleration after 3 seconds.

$s=2t^3-4t^2+3t \quad \therefore\ \dfrac{ds}{dt}=6t^2-8t+3. \quad \dfrac{d^2s}{dt^2}=12t-8.$

So its velocity after 3 seconds

$=\left(\dfrac{ds}{dt}\right)_{t=3}=6.3^2-8.3+3=33$ cm/second

and acceleration after 3 seconds

$=\left(\dfrac{d^2s}{dt^2}\right)_{t=3}=12.3-8=28$ cm/second^2

Example 17. If $s=at^2+bt+c$, where t is the time, s is the distance traversed, v is the velocity and $a, b,$ c are constants, prove that $4a(s-c)=v^2-b^2$ [C.U. ; Joint Entrance 1987]

Velocity $v=\dfrac{ds}{dt}=2at+b.$

$$\begin{aligned}\therefore\ v^2-b^2&=(2at+b)^2-b^2=4a^2t^2+4abt+b^2-b^2\\&=4a^2t^2+4abt=4a(at^2+bt)\\&=4a(at^2+bt+c-c)=4a(s-c)\\ \therefore\ 4a(s-c)&=v^2-b^2.\end{aligned}$$

Example 18. A body was moving along a straight line according to the law of motion $x=\frac{1}{2}vt$. Prove that its acceleration, was constant. (x, v and t have their usual meanings).

[Joint Entrance 1981]

$x=\frac{1}{2}vt=\frac{1}{2}\dfrac{dx}{dt}t$ or, $2\dfrac{dt}{t}=\dfrac{dx}{x}$

or, $2\log t=\log x-\log c$ (Integrating both sides) or, $\log(ct^2) = \log x$ or, $x = ct^2$

$$\therefore \frac{dx}{dt} = 2ct, \text{ or, } \frac{d^2x}{dt^2} = 2c.$$

So, acceleration $\frac{d^2x}{dt^2} = 2c$. (constant)

Example 19. Show that the function $f(x) = x^3 - 6x^2 - 36x + 7$ is increasing when $x < -2$ or $x > 6$.

$f(x) = x^3 - 6x^2 - 36 + 7$

$\therefore f'(x) = 3x^2 - 12x - 36 = 3(x^2 - 4x - 12) = 3(x-6)(x+2)$

Now when $x < -2$, $x-6$ and $x+2$ are both negative

So $3(x-6)(x+2)$ is positive

when $x > 6$, $x-6$ and $x+2$ are both positive and so $3(x-6)(x+2)$ is positive

$\therefore$ when $x < -2$ or $x > 6$, then $f'(x)$ is positive i.e., $f(x)$ is increasing.

Example 20. If $x > \frac{1}{2}$, show that $x(4x^2 - 3)$ is steadily increasing.

Let $f(x) = x(4x^2 - 3) = 4x^3 - 3x$. $\therefore f'(x) = 12x^2 - 3 = 3(4x^2 - 1)$

When $x > \frac{1}{2}$, $f'(x) > 3\{4.(\frac{1}{2})^2 - 1\} = 3(1-1) = 0$

So when $x > \frac{1}{2}$, $f'(x)$ is positive i.e. $f(x)$ is steadily increasing.

Example 21. Show that $x^3 - 3x^2 + 3x$ increases as x increases.

[H. S. 1983]

Let $f(x) = x^3 - 3x^2 + 3x$ $\therefore f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$

So, $f'(x) > 0$, when $x \neq 1$. $\therefore$ $f(x)$ is increasing when $x \neq 1$.

$\therefore$ When $x > 1$, $f(x) > f(1)$ and when $x < 1$, $f(x) < f(1)$.

So at $x = 1$, $f(x)$ is increasing.

Example 22. If $f(x) = 4x^3 + 6x^2 - 24x + 1$, show that $f(x)$ decreases in the interval $(-2, 1)$.

[H S., 1984]

$f(x) = 4x^3 + 6x^2 - 24x + 1$

$\therefore f'(x) = 12x^2 + 12x - 24 = 12(x^2 + x - 2) = 12(x+2)(x-1)$.

When $-2 < x < 1$, then $x+2$ is positive and $x-1$ is negative.

So $f'(x) = 12(x-1)(x+2)$ is negative.

$\therefore$ $f(x)$ decreases in the interval $-2 < x < 1$.

Note : Here $(-2, 1)$ denotes the open interval $-2 < x < 1$.

Example 23. If $f(x)=\log_e(1+x)+\frac{1}{1+x}$, $x>0$ show that $f(x)$ increases with x.

$$f(x)=\log_e(1+x)+\frac{1}{1+x}$$

$$\therefore\quad f'(x)=\frac{1}{1+x}-\frac{1}{(1+x)^2}=\frac{1+x-1}{(1+x)^2}=\frac{x}{(1+x)^2}$$

$\because$ $(1+x)^2$ is positive.

So when $x>0$, $f'(x)=\frac{x}{(1+x)^2}$ is positive.

$\therefore$ When $x>0$, $f(x)$ increases with x.

Example 24. Show that if $x>0$, then $x>\log(1+x)>\frac{x}{1+x}$.

Let $f(x)=x-\log(1+x)$

$$\therefore\quad f'(x)=1-\frac{1}{1+x}=\frac{1+x-1}{1+x}=\frac{x}{1+x}.$$

$\because$ $x>0$, so x and $1+x$ are both positive.

So when $x>0$, $f(x)=x-\log(1+x)$ increases with x.

$\therefore$ $f(x)>f(0)$ or $x-\log(1+x)>0-\log 1=0$

or $x>\log(1+x)$ ... ... (i)

Next, let $\phi(x)=\log(1+x)-\frac{x}{1+x}$

$$\therefore\quad \phi'(x)=\frac{1}{1+x}-\frac{(1+x).1-x.1}{(1+x)^2}=\frac{1}{1+x}-\frac{1}{(1+x)^2}$$

$$=\frac{x}{(1+x)^2}>0 \qquad [\because\ x>0]$$

$\therefore$ When $x>0$, $\phi(x)$ increases with x.

$\therefore$ $\phi(x)>\phi(0)$ when $x>0$.

But $\phi(0)=\log(1+0)-\frac{0}{1+0}=\log 1-0=0$

$\therefore$ $\phi(x)>0$ when $x>0$

or, $\log(1+x)-\frac{x}{1+x}>0$ when $x>0$

or, $\log(1+x)>\frac{x}{1+x}$ when $x>0$ ... ... (ii)

Combining the inequalities (i) and (ii) we get

$$x>\log(1+x)>\frac{x}{1+x} \text{ when } x>0.$$

Example 25. If $f(x)=(x-1)\,e^x+1$, show that $f(x)$ is positive for all positive values of x. [Joint Entrance 1985]

$$f(x)=(x-1)\,e^x+1.$$

$$\therefore\; f'(x)=(x-1)\,e^x+e^x=x\,.\,e^x$$

Now e^x is positive for all x and $x>0$ (given)

$$\therefore\; xe^x>0 \quad \text{or,} \quad f'(x)>0.$$

So $f(x)$ increases with x when $x>0$

$$\therefore\; f(x)>f(0) \text{ when } x>0.$$

Now $f(0)=(0-1)\,e^0+1=-1+1=0 \quad \therefore\; f(x)>0$ when $x>0$.

i.e., $f(x)$ is positive for positive values of x.

Example 26. Determine the increment and differential of each of the following functions.

(i) $f(x)=x^2$ when the value of x increases from $x=2$ to $x=2{\cdot}01$.

(ii) $f(x)=x^2+x+1$ when the value of x changes from $x=2$ to $x=1{\cdot}98$.

(iii) $f(x)=\dfrac{3}{x+1}$ when the value of x changes from 1 to $1{\cdot}01$.

(i) $f(2)=2^2=4$. Here increment of $x=\triangle x=2{\cdot}01-2=0{\cdot}01$.

$$f(2{\cdot}01)=(2{\cdot}01)^2=4{\cdot}0401.$$

So increment of the function $=4{\cdot}0401-4=0{\cdot}0401$.

Again $f'(x)=2x \quad \therefore\; f'(2)=2.2=4$.

$\therefore$ differential $=f'(x)\,\Delta x=4\times0{\cdot}01=0{\cdot}04$.

(ii) $f(x)=x^2+x+1 \quad \therefore\; f(2)=2^2+2+1=7.$

$$f(1{\cdot}98)=(1{\cdot}98)^2+1{\cdot}98+1=3{\cdot}9204+1{\cdot}98+1=6{\cdot}9004.$$

$\therefore$ Increment of $f(x)=6{\cdot}9004-7=-0{\cdot}0996$

Also increment in $x=\triangle x=1{\cdot}98-2=-{\cdot}02$.

$$f'(x)=2x+1 \quad \therefore\; f'(2)=2.2+1=5$$

$\therefore$ differential of $f(x)=f'(x)\,\triangle x=5\times(-{\cdot}02)=-0{\cdot}1$.

(iii) $f(x)=\dfrac{3}{x+1} \quad \therefore\; f(1)=\dfrac{3}{1+1}=\dfrac{3}{2}=1{\cdot}5$

$$f(1{\cdot}01)=\frac{3}{1{\cdot}01+1}=\frac{3}{2{\cdot}01}=1{\cdot}4925 \text{ (nearly)}$$

$\therefore$ Increment of $f(x)=1{\cdot}4925-1{\cdot}5=-0{\cdot}0075$

Now $f'(x)=-\dfrac{3}{(x+1)^2}$ $\therefore$ $f'(1)=-\dfrac{3}{(1+1)^2}=-\dfrac{3}{4}=-0{\cdot}75$

Increment of $x=\triangle x=1{\cdot}01-1=0{\cdot}01$

$\therefore$ differential of $f(x)=f'(x).\triangle x=-0{\cdot}75\times0{\cdot}01=-0{\cdot}0075$.

Example 27. (i) If $y=x^2+x$, find the increment $\triangle y$ of y for the increment Δx of x and also the differential dy of y.

$y=x^2+x$. $\therefore$ $y+\Delta y=(x+\Delta x)^2+(x+\triangle x)$, when x increases by $\triangle x$.

$\therefore$ Increment of $y=\triangle y=(x+\triangle x)^2+(x+\Delta x)-(x^2+x)$

$=x^2+2x.\triangle x+(\triangle x)^2+x+\triangle x-x^2-x$

$=2x.\triangle x+(\triangle x)^2+\triangle x=\Delta x(2x+\triangle x+1)$

Again $\dfrac{dy}{dx}=(2x+1)$

$\therefore$ differential of $y=dy=\dfrac{dy}{dx}.\triangle x=(2x+1).\triangle x=(2x+1)dx$.

(ii) If $y=\sin x$, find the increment $\triangle y$ and differential dy for the increment $\triangle x$ of x.

$y=\sin x$. So when x increases by $\triangle x$, y becomes $y+\triangle y$ $=\sin(x+\Delta x)$

So increment of $y=\triangle y=\sin(x+\triangle x)-\sin x$

$=2\cos\left(\dfrac{2x+\triangle x}{2}\right)\sin\dfrac{\triangle x}{2}$.

Also $dy=\dfrac{dy}{dx}.\Delta x=\cos x.\triangle x=\cos x\,dx$.

Example 28. Find the differentials of the following functions.

(i) $y=\sin^2x$ (ii) $y=\log x$ (iii) $y=\log(\log x)$ (iv) $y=\sqrt{2-x}$.

(i) $y=\sin^2x$ $\therefore$ $\dfrac{dy}{dx}=2\sin x\cos x=\sin 2x$.

$\therefore$ $dy=\dfrac{dy}{dx}.dx=\sin 2x\,dx$.

(ii) $y=\log x$ $\therefore$ $\dfrac{dy}{dx}=\dfrac{1}{x}$; $dy=\dfrac{dy}{dx}.dx=\dfrac{1}{x}.dx=\dfrac{dx}{x}$.

(iii) $y=\log(\log x)$ $\therefore$ $\dfrac{dy}{dx}=\dfrac{1}{\log x}.\dfrac{1}{x}$.

$\therefore$ $dy=\dfrac{dy}{dx}.dx=\dfrac{1}{x\log x}\,dx$.

(iv) $y=\sqrt{2-x}$ ∴ $\frac{dy}{dx}=-\frac{1}{2}\frac{1}{\sqrt{2-x}}$.

and $\frac{dy}{dx}=\frac{dy}{dx}.dx=-\frac{1}{2\sqrt{2-x}}\,dx$.

Example 29. (i) If $x=1{\cdot}997$, find the approximate value of x^4+4x^2+1.

(ii) If $\sqrt{5{\cdot}76}=2{\cdot}4$, find the approximate value of $\sqrt{5{\cdot}82}$.

(i) Let $f(x)=x^4+4x^2+1$ ∴ $f'(x)=4x^3+8x$.
Let $x=2$, $x+h=1{\cdot}997$ ∴ $h=1{\cdot}997-2=-{\cdot}003$.
Now $f(x+h)=f(x)+hf'(x)$ ∴ $f(1{\cdot}997)=f(2)+hf'(2)$

$$=(2)^4+(2)^2+1+(-0{\cdot}003)\times\{4.(2)^3+8.2\}$$
$$=33-0{\cdot}003\times48=33-0{\cdot}144=32{\cdot}856.$$

(ii) Let $f(x)=\sqrt{x}$ ∴ $f'(x)=\frac{1}{2\sqrt{x}}$
Let $x=5{\cdot}76$, $x+h=5{\cdot}82$ ∴ $h=5{\cdot}82-5{\cdot}76=0{\cdot}06$
Now $f(x+h)=f(x)+hf'(x)$
∴ $f(5{\cdot}82)=f(5{\cdot}76)+0{\cdot}6\times f'(5{\cdot}76)$

or, $\sqrt{5{\cdot}82}=\sqrt{5{\cdot}76}+{\cdot}06\times\frac{1}{2\sqrt{5{\cdot}76}}=2{\cdot}4+0{\cdot}06\times\frac{1}{2\times2{\cdot}4}$
$$=2{\cdot}4+\tfrac{1}{80}=2{\cdot}4+{\cdot}0125=2{\cdot}4125.$$

Example 30. (i) If $\log_e^{10}=2{\cdot}303$, find the value of $\log_e^{100\cdot2}$
(ii) If $\log_{10}^e={\cdot}4343$, find the value of $\log_{10}^{100\cdot3}$

(i) Let $f(x)=\log_e^x$ ∴ $f'(x)=\frac{1}{x}$
Let $x=100$, $x+h=100{\cdot}2$. ∴ $h=0{\cdot}2$.
Now $f(x+h)=f(x)+hf'(x)$

or, $\log_e^{100\cdot2}=\log_e^{100}+0{\cdot}2\times\frac{1}{100}$
$$=2\log_e^{10}+{\cdot}002=2\times2{\cdot}303+0{\cdot}002$$
$$=4{\cdot}606+{\cdot}002=4{\cdot}608.$$

(ii) Let $f(x)=\log_{10}^x=\log_e^x\times\log_{10}^e$ ∴ $f'(x)=\log_{10}^e\times\frac{1}{x}$
Also let $x=100$, $x+h=100{\cdot}3$ ∴ $h=0{\cdot}3$.
Now, $f(x+h)=f(x)+hf'(x)$
∴ $f(100{\cdot}3)=f(100)+0{\cdot}3\times f'(100)$

or, $\log_{10}{}^{100{\cdot}3} = \log_{10}{}^{100} + 0{\cdot}3 \times \log_{10}{}^{e} \times \dfrac{1}{100}$

$= 2 + 0{\cdot}3 \times 0{\cdot}4343 \times {\cdot}01 = 2 + 0{\cdot}0013029 = 2{\cdot}0013029$

Examdle 31. (i) If $1^\circ = 0{\cdot}0175$ radian, find the value of $\tan 46^\circ$.

(ii) If $\sin 60^\circ = 0{\cdot}86603$, find the value of $\sin 64^\circ$

(i) Let $f(x) = \tan x \quad \therefore \quad f'(x) = \sec^2 x$.

Also let $x = 45^\circ$, $x + h = 46^\circ \quad \therefore \quad h = 1^\circ = 0{\cdot}0175$ radian

Now $f(x+h) = f(x) + hf'(x)$

or, $f(46^\circ) = f(45^\circ) + 0{\cdot}0175 \times f'(45^\circ)$

or, $\tan 46^\circ = \tan 45^\circ + 0{\cdot}0175 \times \sec^2 45^\circ$

$= 1 + 0{\cdot}0175 \times 2 = 1 + 0{\cdot}035 = 1{\cdot}035$

(ii) Let $f(x) = \sin x \quad \therefore \quad f'(x) = \cos x$.

Also let $x = 60^\circ$, $x + h = 64^\circ \quad \therefore \quad h = 4^\circ = 4 \times 0{\cdot}075$

Now $f(x+h) = f(x) + hf'(x)$

or, $f(64^\circ) = f(60^\circ) + 4 \times 0{\cdot}0175 \times f'(60^\circ)$

or, $\sin 64^\circ = \sin 60^\circ + 4 \times 0{\cdot}0175 \times \cos 60^\circ$

$= 0{\cdot}86603 + 4 \times 0{\cdot}0175 \times \frac{1}{2} = 0{\cdot}86603 + 0{\cdot}035 = 0{\cdot}90103$.

Example 32. The radius of a circular area is 7 cm. If there be an error of 0·2 mm in the measurement of the radius, find the error in the computation of the area.

Let A and r denote the area and radius of the circular area in sq. cms. and cms. respectively.

$$\therefore \quad A = \pi r^2 \quad \therefore \quad \frac{dA}{dr} = 2\pi r.$$

Now corresponding to an error $\triangle r$ in r, the corresponding error $\triangle A$ in A is given by $\triangle A = \dfrac{dA}{dr} \cdot \triangle r = 2\pi r \times \triangle r$.

Now when $r = 7$ cm., $\triangle r = 0{\cdot}2$ mm $= {\cdot}02$ cm.

$\therefore \quad \triangle A = 2 \times \frac{22}{7} \times 7 \times 0{\cdot}02 = 0{\cdot}88$.

So, the required error in the area $= 0{\cdot}88$ sq. cm.

Example 33. When $\theta = 60^\circ$, find the approximate change in the value of $\sin\theta$ for a change of $1'$ in the value of θ.

[$1' = 0{\cdot}00029$ radian]

Let $y = \sin\theta \qquad \therefore \quad \dfrac{dy}{d\theta} = \cos\theta$.

Now change Δy in y corresponding to a change $\triangle\theta$ in θ is given by

$$\triangle y=\frac{dy}{d\theta}\cdot\triangle\theta=\cos\theta\,.\,\triangle\theta.$$

Let $\theta=60^\circ$, $\triangle\theta=1'=0{\cdot}00029$.

$\therefore$ Change in $y=\sin\theta$ is

$\triangle y=\cos 60^\circ\times 0{\cdot}00029=\frac{1}{2}\times 0{\cdot}00029={\cdot}00015$ (nearly).

Example 34. The position of the vertices of ΔABC undergo a very small change on the circum-circle of the triangle. Prove that $\dfrac{da}{\cos A}+\dfrac{db}{\cos B}+\dfrac{dc}{\cos C}=0.$

As the vertices of the triangle undergo a very small change on the circum-circle, R, the radius of the circum-circle does not under go any change.

If da be a very small change in a,

then $da=d\,(2R\sin A)=2R\cos A\,dA.$ $\therefore$ $\dfrac{da}{\cos A}=2RdA.$

Similarly $\dfrac{db}{\cos B}=2R\,dB.,$ $\dfrac{dc}{\cos C}=2R\,dC.$

$$\therefore\ \frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=2R\,dA+2R\,dB+2R\,dC$$

$$=2Rd\,(A+B+C)=2Rd\,(\pi)=2R\times 0=0$$

[As, A, B, C are angles of a triangle,

so $A+B+C=\pi=$constant.]

Example 35. Estimate the error made in calculating the area of the triangle ABC in which the sides a and b are measured accurately as 25 cm and 16 cm. While the angle C is measured as 60° but $\frac{1}{2}^\circ$ in error.

[Joint Entrance 1978]

Let the area of $\triangle ABC$ be $\triangle$.

$\therefore$ $\triangle=\frac{1}{2}\,ab\sin C=\frac{1}{2}\,.\,25\,.\,16\sin C=200\sin C.$

$\therefore$ Error in the computation of the area

$$=d\Delta=d\,(200\sin C)=200\cos C\,dC$$

Here $C=60^\circ$ and $\triangle C=\frac{1}{2}^\circ=\frac{1}{2}\times\dfrac{\pi}{180}$

$\therefore$ Error in the area$=200\cos 60^\circ\times\frac{1}{2}\times\frac{22}{7}\times\frac{1}{180}=\frac{55}{63}.$

Hence the required error in the computation of the area is $\frac{55}{63}$ sq. cm.

Example 36. The lengths of the sides AB and AC of the triangle ABC are c and b and the magnitude of the angle C is 45°. There was no error in the measurement of AB and AC but an error of $\frac{1}{2}'$ in the magnitude of C. Determine the relative error in the computation of the area of the triangle.

[$1' = 0\cdot00029$ radian]

Let $\triangle$ be the area of the triangle.

$\therefore\ \triangle = \frac{1}{2}\, ab \sin C.$

So $\log \triangle = \log(\frac{1}{2}\, ab \sin C) = \log(\frac{1}{2}ab) + \log \sin C.$

$$\therefore\ \frac{d\triangle}{\triangle} = \frac{1}{\sin C} \cos C\ \triangle C = \cot C\ \triangle C.$$

$$= \cot 45^\circ \times \tfrac{1}{2} \times 0\cdot00029 = 1 \times \cdot000145 = 0\cdot000145.$$

Note: Generally relative errors are determined by taking logarithms of both sides of a given or known relation and then taking differentials of both sides.

Example 37. Show that the relative error in the computation of the volume of a sphere is three times the relative error in the measurement of its radius. [State Council West Bengal]

Let v and r denote the volume and radius of a sphere in cubic units and units respectively.

$\therefore\ v = \frac{4}{3}\pi r^3$ or, $\log v = \log(\frac{4}{3}\pi r^3)$

$$= \log(\tfrac{4}{3}\pi) + 3 \log r. \quad \therefore\ \frac{dv}{v} = 3\,\frac{dr}{r}.$$

(taking differentials on both sides).

Hence the relative error in the computation of the volume of a sphere is three times the relative error in the measurement of the radius.

Example 38. If there be an error of 1%. in the computation of a number, what will be the error in the determination of its common logarithm ? [$\log_{10}{}^{e} = 0\cdot4343$]

Let the number be x.

So the error in computation of the number is $\triangle x = \frac{x}{100}$.

[as the error is 1%]

Let $y=\log_{10}{}^{x}=\log_{e}{}^{x}\times\log_{10}{}^{e}$.

So if $\triangle y$ be the error in the determination of $\log_{10}{}^{x}$, then

$$\triangle y=\frac{dy}{dx}\triangle x=\frac{1}{x}\log^{e}{}_{10}\times\triangle x=\frac{1}{x}\times{\cdot}4343\times\frac{x}{100}=0{\cdot}004343.$$

Example 39. If the length of a second's pendulum increase by 1%, what will be the decrease in the number of beats in one day.

We know if T and l denote the time-period and length of a pendulum, then $T=2\pi\sqrt{\frac{l}{g}}$.

$$\therefore\ \log T=\log\left(2\pi\sqrt{\frac{l}{g}}\right)=\log\left(\frac{2\pi}{\sqrt{g}}\right)+\tfrac{1}{2}\log l.$$

$$\therefore\ \frac{dT}{T}=\frac{1}{2}\frac{dl}{l}\ \text{ So, }\ \frac{dT}{T}\times100=\frac{1}{2}\ \frac{dl}{l}\times100$$

Here the increase in l is 1% $\therefore\ \frac{dl}{l}\times100=1$

$\therefore\ \frac{dT}{T}\times100=\frac{1}{2}\times1=0{\cdot}5$. So there will be an increase of $\frac{1}{2}$% in T.

Now in a day there are $24\times60\times60=86400$ seconds.

So, actually there should be 86400 beats in a day in a second's pendulum. Hence due to the increase in the length, there will be $86400\times{\cdot}005=432$ less in a day.

Example 40. Electric current C, measured by a galvanometer is given by the relation $C\propto\tan\theta$. Find the percentage error in the current corresponding to an error of 0·7% in the measurement of θ when $\theta=45°$.

$$C\propto\tan\theta\quad\therefore\quad C=k\tan\theta$$

or, $\log C=\log(k\tan\theta)=\log k+\log\tan\theta$

$$\therefore\quad\frac{dC}{C}=\frac{1}{\tan\theta}\sec^2\theta\,d\theta=\frac{1}{\sin\theta\cos\theta}\,d\theta=\frac{2\theta}{\sin2\theta}\times\frac{d\theta}{\theta}$$

or, $\frac{dC}{C}\times100=\frac{2\theta}{\sin2\theta}\times\frac{d\theta}{\theta}\times100$

So, when $\theta=45^\circ=\dfrac{\pi}{4}$, $\dfrac{dc}{C}\times 100$

$$=\frac{2\times\pi}{4\times\sin\frac{\pi}{2}}\times 0{\cdot}7\left[\text{Here } \frac{d\theta}{\theta}\times 100=0{\cdot}7\right]=\frac{2\times 22}{7\times 4\times 1}\times 0{\cdot}7=1{\cdot}1$$

Hence the required error in C is 1·1%.

Example 41. The pressure P and the volume V of a given amount of gas are connected by the relation $PV^{1\cdot 4}=\text{constant}$. Find the percentage error in the calculated volume corresponding to an increase of 0·7% made in reading the pressure.

$PV^{1\cdot 4}=\text{constant}=k$ (say) $\quad\therefore\ \log (PV)^{1\cdot 4}=\log k$

or, $\log P+1{\cdot}4\log V=\log k$

or, $\dfrac{dP}{P}+1{\cdot}4\,\dfrac{dV}{V}=0$ or, $1{\cdot}4\,\dfrac{dV}{V}=-\dfrac{dP}{P}$

or, $\dfrac{dV}{V}=-\dfrac{1}{1{\cdot}4}\,\dfrac{dP}{P}$ $\quad\therefore\quad \dfrac{dV}{V}\times 100=-\dfrac{1}{1{\cdot}4}\,\dfrac{dP}{P}\times 100$

$$=-\frac{1}{1{\cdot}4}\times 0{\cdot}7\left[\text{as percentage increase in the pressure is } 0{\cdot}7\%, \text{ so } \frac{dP}{P}\times 100=0{\cdot}7\right]$$

$$=-\tfrac{1}{2}=-0{\cdot}5$$

Hence there will be a decrease in the pressure by ½%. The negative sign in the value of $\dfrac{dV}{V}\times 100$, shows that there will be a decrease in the volume.

EXERCISE 1

1. A circular plate of metal expands by heat so that when its radius is 7 cm.; the radius increases at the rate of 0·25 cm. per second. Find the rate at which the surface area is increasing at that instant.

2. From a balloon full of air, air is discharged at the rate of 11 cubic cm. per second. When the radius of the balloon is 3½ cm. find the rate at which its radius will be contracted.

3. The bottom of a cistern is a square and the length of each side of the square is 5 ft. If the height of water level in the cistern increases by 6 inches per minute, find the rate at which water is entering the cistern.

4. If the volume of a sphere increases uniformly, show that the rate at which the area of its curved surface changes is inversely proportional to its radius.

5. Prove that the rate of change of the circumference of a circular plate with the radius is a constant and the rate of change in the area of the plate with the radius is in direct variation with the radius.

6. A particle is moving along the parabola $y^2=12x$. Find the co-ordinates of the point of the parabola at which the rate of changes of the abscissa and ordinate of the point are equal.

7. One end of a ladder 13 meter long rests on a horizontal ground and the other end against a vertical wall. The end on the ground slides away from the wall at the rate of 6 metres per second. Find the rate at which the other end will slide downward when the height of the ladder is 12 metres.

8. Water is running into an open inverted conical vessel at the uniform rate of 22 cubic centimetre per minute. The diameter of the base and the height of the cone are both 2 metres. Find the rate at which water level rises at the end of 8 minutes from the time water starts pouring in.

9. The volume of a spherical balloon is increasing at the rate of 10 cubic centimetres per second. Find the rate of change of its surface at the instant when its radius is 16 centimetres.

10. Show that when $\theta=\frac{\pi}{4}$, the rate at which $\tan\theta$ increases is twice the rate at which θ increases.

11. The lengths of two sides of a triangle are 5 cm. and 6 cm. and their included angle is θ. When $\theta=30°$, then the rate of change of θ is $2\sqrt{3}$. Find the rate of change in the area of the triangle at that instant.

12. A man $5\frac{1}{2}$ foot tall is walking towards a lamp post 14ft. high at the rate of 50ft. per second. Find the rate at which the furthest extremity of his shadow is moving ? [Tripura 1983]

13. A man 1·5 m tall walks away from the foot of a lamp

post 3m high, along a straight line and moves at the rate of 0·6m. per second. Find the rate at which his shadow is lengthening.

14. A man is walking at the rate of 5 km. p.h. towards the foot of a building 40m. high. At what rate is he approaching the top when he is 30m. from the foot of the building ?

[State Council W. Bengal 1986]

15. Each side of a square is lengthening at the uniform rate of 1 inch per second. Find the rate at which its area is increasing when the length of each side is 6 inches.

16. Each side of an equilateral triangle is lengthening at the uniform rate of $\sqrt{3}$ cm. per second. Find the rate at which the area of the triangle is increasing when the length of each side is 4 cm.

17. A kite is flying 28 m. high and there are 100 m. of cordout. If the kite moves away from the man in the horizontal direction at the rate of 5 km per hour, then find the rate at which cord is being paidout.

18. A rod 5 metres long, moves with its ends A and B bying on two perpendicular lines OX and OY respectively. If A moves at the of rate 1 metre per second when it is 4 metres from O, find at what rate the end B is moving ?

19. A right circular cone with height equal to the diameter of the base is made by pouring sand on the horizontal ground at the rate of 11 cubic ft. per minute. Find the rate at which the height of the cone is increasing when the radius of the base is 1·75 ft.

20. A stone thrown into still water causes a series of concentric ripples. If the radius of the outer ripple is increasing at the rate of 5ft per second, how fast is the area of the disturbed water increasing when the outer ripple has a radius of 12 feet ?

21. A vessel containing V c.c. of liquid is in the form of a cone of semi-vertical angle 30°, the vertex downwards and axis vertical. At time t_1, a small plug is removed from the bottom of the vessel and liquid flows out at the rate $k\sqrt{y}$ c.c. per minute,

k being a constant and y cm. the depth of the liquid in the vessel at time t minutes. If at time t_1 the vessel contains volume $\frac{1}{2}V$ c.c. of liquid and at $t=t_2$ it is empty, show that $\frac{t_1}{t_2}=1-\frac{1}{2^{5/6}}$

[Joint Entrance 1978]

22. The displacement 's' of a particle at time 't' second is given by $s=\frac{1}{2}t^2+\sqrt{t}$. Find the velocity and acceleration of the particle at time $t=4$ second.

23. The distance of a particle moving along a straight line from a fixed point of the straight line is x at time t. If $x \propto t^2$, show that the velocity of the particle is proportional to time and that its acceleration is constant.

24. If the displacements of particle moving along a straight line is so that $s=at^2+bt+c$. Prove that the velocity of the particle gradually diminishes and the retardation is proportional to the square of the velocity.

25. The displacement of a particle is given by $s=e^{-2t}\sin 5t$. Express the velocity in the form $Ae^{-2t}\cos(5t+\epsilon)$ and find the value of t when the velocity of the particle vanishes.

26. When the distance of a particle moving along a straight line is x from a fixed point of the straight line, then the velocity of the particle is $\mu\sqrt{\frac{c-x}{x}}$. Show that the acceleration of the particle is always towards a fixed point and its magnitude is inversely proportional to the square of the velocity.

27. If $f(x)=10-9x+6x^2-x^3$, examine whether $f(x)$ increases or decreases for values of x for which (i) $x>3$ (ii) $1<x<3$.

[H. S. 1987]

28. Show that $\frac{\sin x}{x}$ is increasing for all values of x in the open interval $0<x<\frac{\pi}{2}$.

29. Find the interval in which the function $x^3-3x^2-24x+30$ is decreasing.

30. In which interval is the function $x^4-4x^3+4x^2+40$ increasing?

31. Show that the function $x^3-9x^2+36x+18$ is increasing for all values of x.

32. Prove that if a, b, c, d are constants, $\dfrac{a\sin x+b\cos x}{c\sin x+d\cos x}$ is either increasing or decreasing for all values of x.

33. Show that $x-\sin x$ is an increasing function for all values of x.

34. Show that if $x>0$, then

$$x-\frac{x^2}{2}<\log(1+x)<x-\frac{x^2}{2(1+x)}$$

35 Show that for all values of x in the interval $0<x<\frac{\pi}{2}$,

(i) $x-\dfrac{x^3}{6}<\sin x<x-\dfrac{x^3}{6}+\dfrac{x^5}{120}$

(ii) $1-\dfrac{x^2}{2}<\cos x<1-\dfrac{x^2}{2}+\dfrac{x^2}{24}$

(iii) $\sin x<x<\tan x$.

36. (i) Show that if $x>0$ then $x>\log(1+x)>x-\frac{1}{2}x^2$.

(ii) Show that $1+x\log_e(x+\sqrt{x^2+1})\geqslant\sqrt{1+x^2}$ if $x\geqslant 0$.

[I. I. T. 1983]

37. Show that the function x^5 steadily increases from $-\infty$ to $+\infty$ but x^6 decreases from ∞ to 0 and then increases.

38. "If $y=f(x)=\dfrac{x+2}{x-1}$ and $x<1$, then y increases with x". Comment on the validity of the statement and correct it if required. [C.f. I. I. T., 1984]

39. Those values of x for which the function $y=2x^2-\log_e(x)$ is monotonic increasing or monotonic decreasing satisfy the inequalities ... ...

Fill up the gap. [C.f. I. I. T. 1983]

40. Determine the increments and differentials of the following functions.

(i) $y=x^3$ when the value of x increases from 2 to 2·1.

(ii) $v=\frac{4}{3}\pi r^3$, when the value of r increases from 10 to 10·01.

(iii) $y=\log_{10}{}^{x}$, when the value of x increases from 4 to 4·01.

[Given $\log_e{}^4=1·3863$, $\log_e{}^{4·01}=1·3888$ and $\log^e{}_{10}=0·4343$]

41. (i) If $y=\dfrac{3}{x}$ and (ii) $y=\cos x$; find the increment Δy of y for the increment Δx of x. Also find the differential dy of y.

42. Determine the differentials of the following functions.

(i) $y=\cos^3 x$ (ii) $y=\log^x{}_{10}$ (ii) $y=e^{2x}$.

43. With the help of the ideas of differentials show that if α be small compared to 1, then $\dfrac{1}{\sqrt{1+\alpha}}=1-\frac{1}{2}\alpha$.

44. With the ideas of differentials show that

(i) $\dfrac{1}{x+dx}=\dfrac{1}{x}-\dfrac{dx}{x^2}$ (ii) $\sqrt{x+dx}=\sqrt{x}+\dfrac{1}{2\sqrt{x}}\,dx$.

45. Find the approximate values of

(i) $\sqrt{83}$ (ii) $\sqrt{6·26}$ (use differentials).

46. Using differentials find the values of

(i) x^4+x^2+1 when $x=3·05$

(ii) x^3+x+1 when $x=2·98$.

47. If $\log^e{}_{10}=0·4343$, find the approximate values of

(i) $\log_e{}^{10·1}$ and $\log_{10}{}^{10·1}$.

48. $\log^x{}_{10}=0·4343\ \log_e{}^x$ and $\log_{10}{}^4=0·6021$, find the approximate value of $\log_{10}{}^{404}$.

49. $1°=0·175$ radian and $1'=0·00029$ radian ; find approximate values of (i) $\operatorname{cosec}^2 46°$ (ii) $\cos 32°$ (iii) $\tan 45°58'$.

50. The radius of a sphere was found by measurement as 20 cm. If the maximum error in this measurement is found to be 0·5 cm., find the maximum error that will occur in the computation of the curved surface of the sphere.

51. Show that the increment in the volume of a sphere is approximately $4\pi r^2$ times the increase in the radius r.

52. When the temperature remains constant, the law of expansion of gas is pv=constant where p and v denote the pressure and corresponding volume of gas at any instant. When the applied pressure is 20 kg/cm², then if the volume decreases from 1000 c.c. to 999 c.c., determine the increase in pressure.

53. At height h the density of the atmosphere is given by $\rho=\rho_0 e^{-\frac{z}{800}}$; where ρ_0 is the density at the sea-level. Find the approximate increase in the height corresponding to a fall of 1% in the density.

54. Find the error in the common logarithm of a number corresponding to an error of 0·5% in the computation of the number. $[\,\log^{e}{}_{10}=0{\cdot}4343\,]$

55. Show that the relative error in the computation of the area of a circle is twice the relative error in the measurement of the radius.

56. Show that the relative error in the computation of the volume of a cube is three times the relative error in the measurement of an edge of the cube.

57. Show that the percentage error in the computation of the n-th power and the n-th root of a number are respectively n-times and $\frac{1}{n}$ times the percentage error in the determination of the number.

58. If there be an error of 1 cm. in the measurement of the radius of a circle of radius 7 cm., find the error and percentage error in the computation of the area of the circle.

59. The time period 'T' of a simple pendulum of length 'l' is given by $T=2\pi\sqrt{\frac{l}{g}}$. Show that the error in the time period is $\frac{1}{2}$% if there be an error of 1% in the measurement of the radius.

60. In the formula $q = kr^3\sqrt{h}$, (k is a constant).

(i) If h remains constant and there be a decrease of 1% in r, find the percentage change in q.

(ii) If r remains constant, find the percentage change in q corresponding to an increase of 2% in h.

61. The pressure p and the volume v of agas are related by the equation $pv^{1.4} = k$ where k is a constant. If there be a decrease of 1% in v, what will be the corresponding percentage change in p ?

62. b and c are the lengths of the sides CA and AB. If they remain unchanged in magnitude but the angles B and C undergo small changes, show that

$$\frac{dB}{\sqrt{b^2 - c^2 \sin^2 B}} = \frac{dC}{\sqrt{c^2 - b^2 \sin^2 C}}.$$

CHAPTER TWO

TANGENT AND NORMAL

§ 2.1. *Equation of tangent to the curve $y=f(x)$ at the point (x_1, y_1).*

We have seen in chapter one (§ 1·4) that if the function $f(x)$ is differentiable at (x_1, y_1) then $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}=f'(x_1)$ is the gradient of the tangent to the curve $y=f(x)$ at the point (x_1, y_1).

So, the equation of the tangent to the curve at the point (x_1, y_1) is $y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$.

§ 2.2. *Equations of tangents of different conics.*

(i) **Equation of the tangent to the circle $x^2+y^2=a^2$ at the point (x_1, y_1).**

The equation of the circle is $x^2+y^2=a^2$. Differentiating both sides of the equation with respect to x we get

$$2x+2y\frac{dy}{dx}=0 \quad \text{or,} \quad \frac{dy}{dx}=-\frac{x}{y}$$

Hence the equation of the tangent to the circle at the point (x_1, y_1) is $y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$

or, $y-y_1=-\frac{x_1}{y_1}(x-x_1)$

or, $yy_1-y_1^2=-xx_1+x_1^2$ or, $xx_1+yy_1=x_1^2+y_1^2$

or, $xx_1+yy_1=a^2$ [$\because$ (x_1, y_1) is a point of the circle]

(ii) **Equation of the tangent to the parabola $y^2=4ax$ at the point (x_1, y_1)**

The equation of the parabola is $y^2=4ax$. Differentiating both sides of the equation with respect to x we get,

$$2y\frac{dy}{dx}=4a \quad \text{or,} \quad \frac{dy}{dx}=\frac{2a}{y}.$$

Now equation of the tangent to the parabola at the point (x_1, y_1) is $y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$

or, $y-y_1=\frac{2a}{y_1}(x-x_1)$ or, $yy_1-y_1^2=2ax-2ax$,

or, $yy_1=2ax+y_1^2-2ax_1$ or, $yy_1=2ax+4ax_1-2ax_1$.

[∴ (x_1, y_1) is a point of the parabola, So $y_1^2=4ax_1$]

or, $yy_1=2ax+2ax_1$ or, $yy_1=2a(x+x_1)$

(iii) **Equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1).**

The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Differentiating both sides of the equation with respect to x we get,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \quad \text{or,} \quad \frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}.$$

∴ The equation of the tangent to the ellipse at the point (x_1, y_1) is $y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$

or, $y-y_1=-\frac{b^2}{a^2}\frac{x_1}{y_1}(x-x_1)$

or, $\frac{yy_1-y_1^2}{b^2}=\frac{-xx_1+x_1^2}{a^2}$ or, $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$

or, $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ [As (x_1, y_1) is a point of the ellipse]

(iv) **Equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1).**

The equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Differentiating both sides of the equation with respect to x we get,

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0 \quad \text{or,} \quad \frac{dy}{dx}=\frac{b^2}{a^2}\frac{x}{y}.$$

Now the equation of the tangent to the hyperbola at the point (x_1, y_1) is

$$y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$$

or, $y-y_1=\frac{b^2}{a^2}\frac{x_1}{y_1}(x-x_1)$ or, $\frac{yy_1-y_1^2}{b^2}=\frac{xx_1-x_1^2}{a^2}$

$$\text{or, } \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2} \text{ or, } \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

[As (x_1, y_1) is a point of the hyperbola]

(v) **Equation of the tangent to the rectangular hyperbola $xy=a^2$ at the point (x_1, y_1).**

Differentiating both sides of the equation $xy=a^2$, of the hyperbola, with respect to x we get

$$y+x\frac{dy}{dx}=0 \quad \text{or, } \frac{dy}{dx}=-\frac{y}{x}.$$

Now, the equation of the tangent to the hyperbola at the point (x_1, y_1) is

$$y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$$

$$\text{or, } y-y_1=-\frac{y_1}{x_1}(x-x_1) \text{ or, } x_1y-x_1y_1=-y_1x+x_1y_1$$

$$\text{or, } x_1y+xy_1=2x_1y_1 \text{ or, } x_1y+xy_1=2a^2$$

[As (x_1y_1) is a point of the rectangular hyperbola.]

(vi) **Equation of the tangent to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ at the point (x_1, y_1).**

The equation of the conic is $ax^2+2hxy+by^2+2gx+2fy+c=0$ Differentiating both sides of the equation with respect to x we get,

$$2ax+2h\left(y+x\frac{dy}{dx}\right)+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$\text{or, } (hx+by+f)\frac{dy}{dx}=-(ax+hy+g) \text{ or, } \frac{dy}{dx}=-\frac{(ax+hy+g)}{hx+by+f}$$

Now the equation of the tangent to the conic at the point (x_1, y_1) is $y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$

$$\text{or, } y-y_1=-\frac{ax_1+hy_1+g}{hx_1+by_1+f}(x-x_1)$$

$$\text{or, } hx_1y+byy_1+fy-hx_1y_1-by_1^2-fy_1$$
$$=-axx_1-hxy_1-gx+ax_1^2+hx_1y_1-gx_1$$

$$\text{or, } axx_1+h(x_1y+xy_1)+byy_1+gx+fy$$
$$=ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1$$

or, $axx_1+h(x_1y+xy_1)+byy_1+g(x+x_1)+f(y+y_1)+c$
$=ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c$
[Adding gx_1+fy_1+c on both sides]

or, $axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c=0$
[As (x_1, y_1) is a point of the conic]

Note 1. In every case above, $\frac{dy}{dx}$ has finite value at every point (x_1, y_1) of the curve.

2. The following artifice can always be followed in writing the equation of the tangent to a curve at any given point of it.

Write x^2 and y^2 as xx and yy in the terms involving x^2 and y^2 and then put x_1 and y_1 for one x and one y respectively in the products. In the terms involving x and y, write x and y as $\frac{x+x}{2}$ and $\frac{y+y}{2}$ respectively. Then put x_1 for one x and y_1 for one y. If there be a term involving xy say hxy, write it as $h\left(\frac{xy}{2}+\frac{xy}{2}\right)$ and then put x_1 for x in one product and y_1 for y in the other product and get $h\left(\frac{x_1y+xy_1}{2}\right)$. For example to write the equation of the tangent to the circle $x^2+y^2+2gx+2fy+c=0$ at (x_1, y_1), first write the equation in the form.

$$xx+yy+g(x+x)+f(y+y)+c=0.$$

Now put x_1 and y_1 for one x and one y respectively in each product and get the equation of the tangent as,

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c.$$

3. You know that the co-ordinates of any point of the circle $x^2+y^2=a^2$ can be taken as $(a \cos \theta, a \sin \theta)$. The equation of the tangent to the circle at this point is $x.a \cos \theta+y.a \sin \theta=a^2$ [Putting $x_1=a \cos \theta$ and $y_1=a \sin \theta$ in the equation of the tangent to the circle at (x_1, y_1)] or, $x \cos \theta+y \sin \theta=a$. The equation of the tangent to the parabola $y^2=4ax$ at the point $(at^2, 2at)$ is

$y \, . \, 2at=2a(x+at^2)$ or, $yt=x+at^2$

[Putting $x_1=at^2$, $y_1=2at$ in the equation of the tangent to the parabola at the point (x_1, y_1)]

Putting $x_1=a\cos\theta$, $y_1=b\sin\theta$ in the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) we get the equation of the tangent to the ellipse at the point $(a\cos\theta,\ b\sin\theta)$ as $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$. Putting $x_1=a\sec\theta$, $y_1=b\tan\theta$ in the equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, at (x_1, y_1) we get the equation of the tangent to the conic at $(a\sec\theta,\ b\tan\theta)$ as $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$.

Putting $x_1=at$ and $y_1=\frac{a}{t}$ in the equation of the tangent to the rectangular hyperbola we get the equation of the tangent to the rectangular hyperbola at the point $\left(at, \frac{a}{t}\right)$ as

$$\frac{x}{t}+yt=2a \quad \text{or,} \quad x+yt^2=2at.$$

§ 2·3. *Condition for the straight line $y=mx+c$ to be a tangent to a conic and the co-ordinates of the point of contact.*

(A) Condition for the straight line $y=mx+c$ to be a tangent to the conic $\frac{x^2}{a}+\frac{y^2}{b}=1$.

Let the straight line $y=mx+c$ ··· ··· ··· ···(i) touch the conic $\frac{x^2}{a}+\frac{y^2}{b}=1$ ··· (ii) at the point (x_1, y_1).

Differentiating both sides of the equation-(ii) with respect to x we get,

$$\frac{2x}{a}+\frac{2y}{b}\frac{dy}{dx}=0 \quad \text{or,} \quad \frac{dy}{dx}=-\frac{b}{a}\frac{x}{y}.$$

Now, the equation of the tangent to the curve at the point (x_1, y_1) is $y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$

$$\text{or,} \quad y-y_1=-\frac{b}{a}\frac{x_1}{y_1}(x-x_1) \quad \text{or,} \quad ayy_1-ay_1^2=-bxx_1+bx_1^2$$

$$\text{or,} \quad bxx_1+ayy_1=bx_1^2+ay_1^2$$

or, $\dfrac{xx_1}{a}+\dfrac{yy_1}{b}=\dfrac{{x_1}^2}{a}+\dfrac{{y_1}^2}{b}$ [Dividing both sides by ab]

or, $\dfrac{xx_1}{a}+\dfrac{yy_1}{b}=1$......(iii) [As (x_1, y_1) is a point on the conic]

So, the equations (i) and (ii) represent the same straight line and their corresponding coefficients are proportional.

$$\therefore\ \frac{\frac{x_1}{a}}{m}=\frac{\frac{y_1}{b}}{-1}=\frac{-1}{c}\quad\text{or,}\quad x_1=\frac{-am}{c},\ y_1=\frac{b}{c}.$$

Now (x_1, y_1), is a point of the conic-(ii), $\therefore\ \dfrac{{x_1}^2}{a}+\dfrac{{y_1}^2}{b}=1.$

or, $\dfrac{a^2m^2}{c^2a}+\dfrac{b^2}{c^2b}=1$ or, $am^2+b=c^2$......(iv)

which is the required condition. The co-ordinates of the point of contact are $\left(-\dfrac{am}{c}, \dfrac{b}{c}\right)$.

Now putting $a=a^2$, $b=a^2$ in the equation—(ii) we get from the condition-(iv), the condition that the straight line $y=mx+c$ will touch the circle $x^2+y^2=a^2$ is $c^2=a^2(1+m^2)$

or, $c\pm=a\sqrt{1+m^2}$.

So, for all real values of x, the two straight lines $y=mx\pm a\sqrt{1+m^2}$ are tangents to the circle $x^2+y^2=a^2$. In these two cases the corresponding points of contact are $\left(\dfrac{-a^2m}{\pm a\sqrt{1+m^2}}, \dfrac{a^2}{\pm a\sqrt{1+m^2}}\right)$ [Putting $c=\pm a\sqrt{1+m^2}$]

i e., $\left(\mp\dfrac{am}{\sqrt{1+m^2}}, \pm\dfrac{a}{\sqrt{1+m^2}}\right)$.

Again putting $a=a^2$ and $b=b^2$ and $a=a^2$ and $b=-b^2$ in the equation (ii) we get from the condition-(iv) above that the straight line $y=mx+c$ will touch the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ if $c^2=a^2m^2+b^2$ and $c^2=a^2m^2-b^2$ respectively.

So, for all real values of m, the straight lines $y=mx\pm\sqrt{a^2m^2+b^2}$ will touch the ellipse at the points

$$\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right) \text{ i.e., } \left(\frac{-a^2m}{\pm\sqrt{a^2m^2+b^2}}, \frac{b^2}{\pm\sqrt{a^2m^2+b^2}}\right)$$

$$\text{or, } \left(\mp\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

Also the straight lines $y=mx\pm\sqrt{a^2m^2-b^2}$ will touch for all real values of m the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The corresponding points of contact are $\left(-\frac{a^2m}{\pm\sqrt{a^2m^2+b^2}}, \frac{-b^2}{\pm\sqrt{a^2m^2+b^2}}\right)$

$$\text{i.e., } \left(\mp\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

(B) **Condition that the straight line y=mx+c will be tangent to the parabola y² =4*a*x.**

Let the straight line $y=mx+c$······(i) touch the parabola $y^2=4ax$······(ii) at the point (x_1, y_1), the equation of the tangent to the parabola at the point (x_1, y_1) is

$$yy_1=2a(x+x_1) \quad \cdots \quad \cdots\text{(iii)}$$

So, the equations (i) and (iii) represent the same straight line and hence the corresponding coefficients are proportional.

$$\therefore \frac{2a}{m}=\frac{y_1}{1}=\frac{2ax_1}{c} \quad \therefore x_1=\frac{c}{m} \text{ and } y_1=\frac{2a}{m}.$$

Now (x_1, y_1) is a point of the parabola $y^2=4ax$.

So, $y_1{}^2=4ax_1$, or, $\frac{4a^2}{m^2}=4a\frac{c}{m}$ or, $c=\frac{a}{m}$ and this is the required condition.

So, $x_1=\frac{c}{m}=\frac{a}{m^2}$ and $y_1=\frac{2a}{m^2}$ [Putting the value of c]

Thus the straight line $y=mx+\frac{a}{m}$, for all real values of m touches the parabola $y^2=4ax$ and the corresponding point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

§ 2.4. *Normal.*

The perpendicular drawn at the point P of a curve on the tangent to the curve at P is called the normal to the curve at P.

In figure 2·1, PT is the tangent to the curve $y=f(x)$ at the point P of the curve. PN is perpendicular to the tangent PT at P. So, PN is the normal to the curve at P.

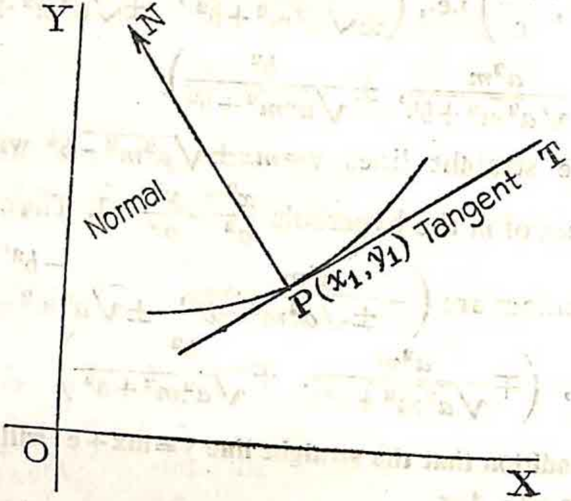


Fig. 2·1

If the co-ordinates of P be (x, y), then the gradient of the tangent to the curve at P is $\frac{dy}{dx}=f'(x)$. So, if 'm' be the gradient of the normal to the curve at P, then $m.\frac{dy}{dx}=-1$

$$\text{or, } m=-\frac{1}{\frac{dy}{dx}}=-\frac{dx}{dy}$$

Generally, to obtain the equation of the normal to a curve at a point $P\ (x_1, y_1)$ we first determine $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$. Then the gradient of the normal to the curve at P is $-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$

So, the equation of the normal at P is

$$y-y_1=-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x-x_1).$$

we now determine the equations of the normals to different conics at a point (x_1, y_1) of the conic.

(A) **Equation of normal to the circle $x^2+y^2+2gx+2fy+c=0$ at a point (x_1, y_1) of it.**

The equation of the circle is $x^2+y^2+2gx+2fy+c=0$……(i)

Differentiating both sides of equation—(i) with respect to x we get, $2x+2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$ or, $(y+f)\frac{dy}{dx}=-(x+g)$

$$\text{or, } \frac{dy}{dx}=-\frac{x+g}{y+f}. \quad \therefore \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}=-\frac{x_1+g}{y_1+f}.$$

Hence equation of normal to the curve at the point (x_1, y_1)

$$\text{is } y-y_1=-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,\, x_1)}}(x-x_1)$$

$$\text{or, } y-y_1=-\frac{1}{-\frac{x_1+g}{y_1+f}}(x-x_1)$$

$$\text{or, } y-y_1=\frac{y_1+f}{x_1+g}(x-x_1). \text{ or, } \frac{y-y_1}{y_1+f}=\frac{x-x_1}{x_1+g} \cdots \cdots \text{(ii)}$$

Corolary : 1. The centre of the circle ⋯ ⋯ (i) is the point $(-g, -f)$. Putting $x=-g, y=-f$ in the equation—(ii) we find that the equation is satisfied. Hence the normal passes through the centre of the circle i.e., the normal is a radius of the circle. So, every tangent to a circle is perpendicular to the radius through the point of contact.

2. In the above dicussion, putting $g=f=0$ and $c=-a^2$, we find the equation of the normal to the circle $x^2+y^2=a^2$ at the point is $\frac{y-y_1}{y_1}=\frac{x-x_1}{x_1}$. [Putting $g=f=0$ in equation (ii)]

$$\text{or, } \frac{y}{y_1}-1=\frac{x}{x_1}-1 \text{ or, } \frac{x}{x_1}=\frac{y}{y_1}.$$

(B) **Equation of normal to the curve $y^2=4ax$ at the point (x_1, y_1).**

The equation of the parabola is $y^2=4ax$ ⋯ ⋯ (i)

Differentiating both sides of the equation with respect to y we get $2y=4a\frac{dx}{dy}$, or, $\frac{dx}{dy}=\frac{y}{2a}$.

In figure 2·1, PT is the tangent to the curve $y=f(x)$ at the point

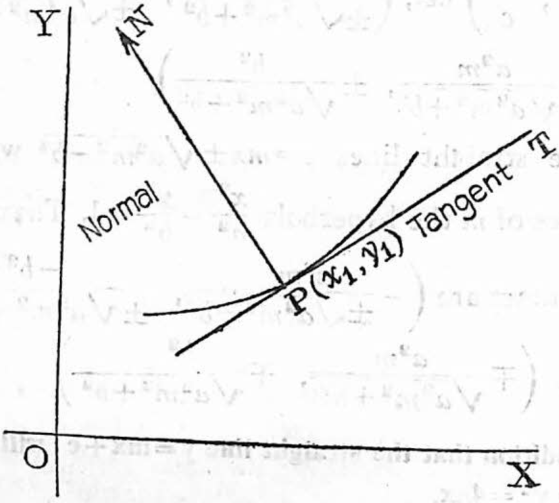


Fig. 2·1

P of the curve. PN is perpendicular to the tangent PT at P. So, PN is the normal to the curve at P.

If the co-ordinates of P be (x, y), then the gradient of the tangent to the curve at P is $\frac{dy}{dx}=f'(x)$. So, if 'm' be the gradient of the normal to the curve at P, then $m.\frac{dy}{dx}=-1$

$$\text{or, } m=-\frac{1}{\frac{dy}{dx}}=-\frac{dx}{dy}$$

Generally, to obtain the equation of the normal to a curve at a point $P\ (x_1, y_1)$ we first determine $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$. Then the gradient of the normal to the curve at P is $-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$

So, the equation of the normal at P is

$$y-y_1=-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x-x_1).$$

we now determine the equations of the normals to different conics at a point (x_1, y_1) of the conic.

(A) **Equation of normal to the circle $x^2+y^2+2gx+2fy+c=0$ at a point (x_1, y_1) of it.**

The equation of the circle is $x^2+y^2+2gx+2fy+c=0$……(i)

Differentiating both sides of equation—(i) with respect to x we get, $2x+2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$ or, $(y+f)\frac{dy}{dx}=-(x+g)$

or, $\frac{dy}{dx}=-\frac{x+g}{y+f}$. $\therefore \left(\frac{dy}{dx}\right)_{(x_1,\ y_1)}=-\frac{x_1+g}{y_1+f}$.

Hence equation of normal to the curve at the point (x_1, y_1) is $y-y_1=-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,\ x_1)}}(x-x_1)$

$$\text{or,}\quad y-y_1=-\frac{1}{-\frac{x_1+g}{y_1+f}}(x-x_1)$$

or, $y-y_1=\frac{y_1+f}{x_1+g}(x-x_1)$. or, $\frac{y-y_1}{y_1+f}=\frac{x-x_1}{x_1+g}$ … …(ii)

Corolary : 1. The centre of the circle …… (i) is the point $(-g, -f)$. Putting $x=-g$, $y=-f$ in the equation—(ii) we find that the equation is satisfied. Hence the normal passes through the centre of the circle i.e., the normal is a radius of the circle. So, every tangent to a circle is perpendicular to the radius through the point of contact.

2. In the above dicussion, putting $g=f=0$ and $c=-a^2$, we find the equation of the normal to the circle $x^2+y^2=a^2$ at the point is $\frac{y-y_1}{y_1}=\frac{x-x_1}{x_1}$. [Putting $g=f=0$ in equation (ii)]

or, $\frac{y}{y_1}-1=\frac{x}{x_1}-1$ or, $\frac{x}{x_1}=\frac{y}{y_1}$.

(B) **Equation of normal to the curve $y^2=4ax$ at the point (x_1, y_1).**

The equation of the parabola is $y^2=4ax$ … … (i)

Differentiating both sides of the equation with respect to y we get $2y=4a\frac{dx}{dy}$, or, $\frac{dx}{dy}=\frac{y}{2a}$.

Hence the equation of the normal to the parabola at the point (x_1, y_1) is $y-y_1=\left[-\frac{dx}{dy}\right]_{(x_1,\, y_1)}(x-x_1)$

or, $y-y_1=-\frac{y_1}{2a}(x-x_1)$ or, $2ay-2ay_1=-y_1x+x_1y_1$

or, $y_1x+2ay=2ay_1+x_1y_1=y_1(2a+x_1)$ ... ... (ii)

Corolary : If the gradient of the normal to the parabola at (x_1, y_1) be m, then $m=-\frac{y_1}{2a}$ or, $y_1=-2am$.

$$\therefore \quad x_1=\frac{y_1^2}{4a}=\frac{4a^2m^2}{4a}=am^2.$$

Putting these values of (x_1, y_1) in equation—(ii)

We get $-2amx+2ay=-4a^2m-2a^2m^3$

or, $y=mx-2am-am^3$ ... ... (iii)

So, for all real values of m, the equation—(iii) is the normal to the parabola $y^2=4ax$ at the point $(am^2, -2am)$ of it.

(C) Equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1).

The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... ... (i)

Differentiating both sides of the equation with respect to x we get, $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0.$

$\therefore \frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$ and so the gradient of the normal to the curve at the point (x_1, y_1) is $-\frac{1}{\left[\frac{dy}{dx}\right]_{(x_1,\, y_1)}}=\frac{a^2y_1}{b^2x_1}$ and the equation of the normal is $y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$

$$\text{or,}\quad \frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}} \quad \cdots \quad \cdots \quad \text{(ii)}$$

Note : As in the case of a parabola we could find $\frac{dx}{dy}$ to determine the equation of the normal.

Corolary : For all values of θ, $x=a\cos\theta$, $y=b\sin\theta$ satisfy the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ i. e., the co-ordinates of a point of the ellipse (i) can be taken as $(a\cos\theta, b\sin\theta)$. Putting $x_1=a\cos\theta$, $y_1=b\sin\theta$ in the equation (ii), we get the equation of the normal to the ellipse at the point $(a\cos\theta, b\sin\theta)$ is

$$\frac{x-a\cos\theta}{\frac{a\cos\theta}{a^2}}=\frac{y-b\sin\theta}{\frac{b\sin\theta}{b^2}}.$$

or, $ax\sec\theta-a^2=by\operatorname{cosec}\theta-b^2$

or, $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$.

(D) Equation of normal to the hyperbola at the point (x_1, y_1).

Differentiating both sides of the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\cdots$(i) with respect to y we get $\frac{2x}{a^2}\frac{dx}{dy}-\frac{2y}{b^2}=0$ or, $\frac{dx}{dy}=\frac{\frac{2y}{b^2}}{\frac{2x}{a^2}}=\frac{a^2}{b^2}\cdot\frac{y}{x}$.

So, the gradient of the normal to the hyperbola at the point $(x_1, y_1)=\left(-\frac{dx}{dy}\right)_{(x_1,\, y)}=-\frac{a^2}{b^2}\frac{y_1}{x_1}$. Hence the equation of the normal at the point (x_1, y_1) is $y-y_1=-\frac{a^2 y_1}{b^2 x_1}(x-x_1)$

$$\text{or,}\quad \frac{x-x_1}{\frac{x_1}{a^2}}+\frac{y-y_1}{\frac{y_1}{b^2}}=0 \quad \cdots \quad \cdots \quad \text{(ii)}$$

Corolary : The co-ordinates of any point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ can be taken as $(a\sec\theta, b\tan\theta)$. So from equation-(ii) above we get the normal to the hyperbola at this point as

$$\frac{x-a\sec\theta}{\frac{a\sec\theta}{a^2}}=\frac{y-b\tan\theta}{\frac{b\tan\theta}{b^2}} \quad \text{or,} \quad \frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2.$$

§ 2·5. *Number of tangents drawn to a conic from an external point.*

(A) In § 2·3 we have seen that for all real values of m the straight line $y=mx+a\sqrt{1+m^2}$ is a tangent to the circle $x^2+y^2=a^2$. If this tangent passes through an external point (h, k) of the circle, then,

$k=mh+a\sqrt{1+m^2}$ or, $k-mh=a\sqrt{1+m^2}$

or, $(k-mh)^2=a^2(1+m^2)$,

or, $(h^2-a^2)m^2-2mkh+k^2-a^2=0$ ... ... (i)

This is a quadratic equation in 'm'; so we shall obtain two values of m from this equation and we shall get for the two values of m two and only two tangents of the circle passing through the external point (h, k). So, from an external point of a circle two and only two tangents of the circle can be drawn.

Note 1. As (h, k) is an external point of the circle, so $h^2+k^2-a^2>0$.

Now the discriminant of equation-(i) is

$4h^2k^2-4(h^2-a^2)(k^2-a^2)=4\{h^2k^2-h^2k^2+h^2a^2+a^2k^2-a^4\}$

$=4\{a^2(h^2+k^2-a^2)\}>0$. as $a^2>0$.

So, the roots of the equation-(i) will be real and unequal.

2. We can take the equation of a circle in the form $x^2+y^2=a^2$ by proper choice of co-ordinate axes.

(B) The straight line $y=mx+\dfrac{a}{m}$ touches for all real values of 'm' the parabola $y^2=4ax$. (The equation of a parabola can always be taken in the form $y^2=4ax$ by proper choice of co-ordinate axes). If this tangent passes through an external point (h, k) then $k=mh+\dfrac{a}{m}$ or, $m^2h-mk+a=0$......(i)

This is a quadratic equation in 'm' and will give two and only two real and unequal values of m [As (h, k) is an external point of the parabola, so $k^2-4ah>0$ and the discriminant of equation-(i) will be positive]. Hence from an external point of a parabola two and only two tangents can be drawn to the parabola.

(C) The equation of an ellipse, by proper choice of co-ordinates can always be taken in the form $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. So, let the given ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The straight line $y=mx+\sqrt{a^2m^2+b^2}$, is for all values of m a tangent to the ellipse. If this tangent passes through an external point (h, k) of the ellipse, then $k=mh+\sqrt{a^2m^2+b^2}$.

or, $k-mh=\sqrt{a^2m^2+b^2}$ or, $(k-mh)^2=a^2m^2+b^2$

or, $m^2(h^2-a^2)-2mkh+k^2-b^2=0$……(i)

This is a quadratic equation in m and the discriminant of this equation is $4\{k^2h^2-(h^2-a^2)(k^2-b^2)]=4(a^2k^2+b^2h^2-a^2b^2)$. As (h, k) is an external point of the ellipse, so $\frac{h^2}{a^2}+\frac{k^2}{b^2}-1>0$ or, $b^2h^2+a^2k^2-a^2b^2>0$. So, the roots of equation-(i) are real and unequal and we shall get corresponding to these two values of m two tangents to the ellipse drawn from the external point (h, k).

(D) In (C) above, putting $-b^2$ for b^2, in the same way as above we can show that two and only two tangents can be drawn to a hyperbola from an external point.

§. 2·6. *Locus of the point of intersection of perpendicular tangents to a conic : director circles of ellipse and hyperbola.*

(i) From § 2·5 (B) we find that the gradients m_1 and m_2 of tangents to the parabola $y^2=4ax$ are roots of the equation $m^2h-mk+a$. If these two tangents are perpendicular, then $m_1m_2=-1$. or, $\frac{a}{h}=-1$ or, $h+a=0$. Hence the locus of the point of intersection of perpendicular tangents of the parabola is $x+a=0$ which is the equation of the directrix of the parabola. As the equation of a parabola can always be taken in the form $y^2=4ax$, so it follows that perpendicular tangents of a parabola intersect on the directrix of the parabola.

(B) **Locus of the point of intersection of perpendicular tangents of a circle.**

The equation of a circle can always be taken in the form $x^2+y^2=a^2$. In § 2·5 (A) we have seen that the gradients m_1 and

m_2 of the tangents to the circle $x^2+y^2=a^2$ drawn from an external point (h, k) are the roots of the equation.

$$(h^2-a^2)\, m^2-2mkh+k^2-a^2=0.$$

If these tangents be perpendicular to each other, then $m_1m_2=-1$. or, $\dfrac{k^2-a^2}{h^2-a^2}=-1$; $k^2-a^2=-h^2+a^2$ or, $h^2+k^2=2a^2$. Hence the locus of the point (h, k) is the circle $x^2+y^2=2a^2$. In other words the locus of the point of intersection of perpendicular tangents of a circle is a concentric circle.

(C) **Locus of the point of intersection of perpendicular tangents to an ellipse.**

The equation of an ellipse, by proper choice of co-ordinate axes, can always be taken in the form $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\cdots$(i). In § 2·5 (C) above, we have seen that the gradients m_1 and m_2 of the tangents drawn to the ellipse from an external point (h, k) are roots of the equation $(h^2-a^2)\, m^2-2mkh+k^2-b^2=0\cdots\cdots$(ii). If these tangents be perpendicular to each other then $m_1m_2=-1$.

or, $\dfrac{k^2-b^2}{h^2-a^2}=-1$ or, $k^2-b^2=-h^2+a^2$ or, $h^2+k^2=a^2+b^2$.

Hence the equation of the locus of the point of intersection (h, k) of the two tangents is $x^2+y^2=a^2+b^2$. In other words tangents, perpendicular to each other, of the ellipse intersect on the curve $x^2+y^2=a^2+b^2$ which is a circle whose centre is the centre of the ellipse. This circle is called the *director circle* of the ellipse.

(D) Putting $-b^2$ for b^2 in the equations-(i) and (iii) above we can prove in the same way as in the case of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, that perpendicular tangents to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ intersect each other in the circle $x^2+y^2=a^2-b^2$. This circle is called the *dierctor circle* of the hyperbola. So perpendicular tangents to a hyperbola intersect on the director circle of the hyperbola.

§ 2·7 *Chord of contact.*

Let the two tangents drawn from an external point P of a conic touch the conic at the two points Q and R. QR is joined.

The chord QR of the conic is called the *chord of contact* of the point P with respect to the conic.

(1) Letus determine the equation of the chord of contract of the external point P of the circle $x^2+y^2=a^2$ with respect to the circle.

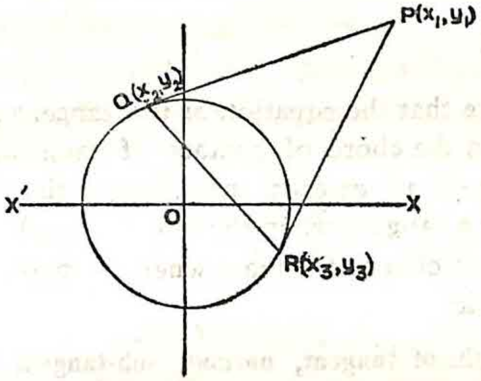


Fig. 2·2

Let the co-ordinates of the point P be (x_1, y_1) and the tangents drawn to the circle touch the circle at the points $Q\,(x_2, y_2)$ and $R\,(x_3, y_3)$.

The equation of the tangent at $Q\,(x_2, y_2)$ to the circle is $xx_2+yy_2=a^2$. As this tangent passes through $P\,(x_1, y_1)$, so

$$x_1x_2+y_1y_2=a^2 \quad \dots \quad \dots \quad \text{(i)}$$

The equation of the tangent at $R\,(x_3, y_3)$ to the circle is $xx_3+yy_3=a^2$. As it passes through the point $P\,(x_1, y_1)$, so

$$x_1x_3+y_1y_3=a^2 \quad \dots \quad \dots \quad \text{(ii)}$$

Equations—(i) and (ii) show that the co-ordinates of the points Q and R both satisfy the equation $xx_1+yy_1=a^2$, i.e., the equation of the chord of contact QR is $xx_1+yy_1=a^2$.

Similarly it can be proved that,

(2) The equation of the chord of contact of tangents drawn to the parabola $y^2=4ax$ from an external point $P\,(x_1, y_1)$ is $yy_1=2a\,(x+x_1)$.

(3) The equation of the chord of contact of the tangents

drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from an external point $P(x_1, y_1)$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

(4) The chord of contact of the external point $P(x_1, y_1)$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to the hyperbola is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

Note : Note that the equation of the tangent to a conic at a point of it and the chord of contact of tangents of the same conic drawn from an external point have the same equation. The equation is a tangent when the point (x_1, y_1) is a point on the conic and is a chord of contact when the point is an external point of the conic.

§ 2·8. Lengths of tangent, normal, sub-tangent and subnormal of a curve.

Let $y=f(x)$ be a given curve and $P(x_1, y_1)$ be a point on it. Let the tangent PT and the normal PN to the curve at P intersect the x-axis at T and N respectively. PT and PN are called the lengths of the tangent and normal respectively.

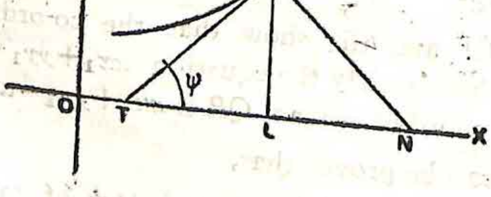


Fig. 2·3

rom P, PL is drawn perpendicular on the x-axis. TL and NL re called the subtangent and sub-normal to the curve at P.

Now, let PT, the tangent at P be inclined at an angle ψ with the positive direction of the x-axis. Then $\tan\psi=\frac{dy}{dx}=y_1$.

Now $PT=PL\operatorname{cosec}\psi=y\sqrt{1+\cot^2\psi}$ [As $PL=y$.]

$$=y\sqrt{1+\frac{1}{\tan^2\psi}}=y\sqrt{1+\frac{1}{y_1{}^2}}=\frac{y}{y_1}\sqrt{1+y_1{}^2}.$$

$PN=PL\operatorname{cosec}\angle PNL=y\operatorname{cosec}(90^\circ-\psi)$

[As $\angle TPN$ is a right angle, so $\angle PNL=90^\circ-\psi$]

$$=y\sec\psi=\sqrt{1+\tan^2\psi}=y\sqrt{1+y_1{}^2}$$

Subtangent $TL=PL\cot\psi=y\frac{1}{\tan\psi}=\frac{y}{y_1}$.

Subnormal $NL=PL\cot(90^\circ-\psi)=PL\tan\psi=yy_1$.

EXAMPLES 2

Example 1. Prove that of the three tangents to the curve $y=(x-1)(x-2)(x-3)$ at the points where it meets the x-axis, two are parallel and the other makes an angle of 135° with the positive direction of the x-axis.

Putting $y=0$ in the equation of the curve we get $x=1, 2, 3$. So, the curve meets the x-axis at the points $(1, 0)$, $(2, 0)$ and $(3, 0)$.

Now $y=(x-1)(x-2)(x-3)=x^3-6x^2+11x-6$.

$$\therefore\quad \frac{dy}{dx}=3x^2-12x+11$$

So, $$\left(\frac{dy}{dx}\right)_{(1,0)}=3.1^2-12.1+11=2.$$

$$\left(\frac{dy}{dx}\right)_{(2,0)}=3.2^2-12.2+11=-1.$$

$$\left(\frac{dy}{dx}\right)_{(3,0)}=3.3^2-12.3+11=2.$$

Now $\left(\frac{dy}{dx}\right)_{(1,0)}$ and $\left(\frac{dy}{dx}\right)_{(3,0)}$ are the gradients of the tangents to the curve at the points $(1, 0)$ and $(3, 0)$ and we find that they are equal (each equal to 2). So these two tangents are parallel. Again $\left(\dfrac{dy}{dx}\right)_{(2,0)}=-1=\tan 135^\circ$ is the gradient of the tangent at the point (2, 0). Hence this tangent makes an angle 135° with the positive direction of the x-axis.

Example 2. Find the equations of the tangents to the following curves at the given points.

(i) The circle $x^2+y^2+4x+6y-87=0$ at (4, 5).

(ii) The parabola $x^2+2x+y=0$ at the point $(-2, 0)$.

(iii) The ellipse $5x^2+3y^2=137$ at the point where $y=2$.

(iv) The curve $x=a\cos^3\theta$, $y=b\sin^3$ at the point 'θ'.

(i) $x^2+y^2+4x+6y-87=0$.

Differentiating both sides of the equation with respect to x we get

$$2x+2y\frac{dy}{dx}+4+6\frac{dy}{dx}=0$$

$$\text{or, } (2y+6)\frac{dy}{dx}=-(2x+4) \text{ or, } \frac{dy}{dx}=-\frac{(2x+4)}{2y+6}$$

$$\text{or, } \frac{dy}{dx}=-\frac{2(x+2)}{2(y+3)} \therefore \left(\frac{dy}{dx}\right)_{(4,5)}=-\frac{4+2}{5+3}=-\tfrac{3}{4}.$$

Hence the equation of the tangent to the curve at the point (4, 5) is

$$y-5=\left(\frac{dy}{dx}\right)_{(4,5)}(x-4) \text{ or, } y-5=-\tfrac{3}{4}(x-4)$$

or, $4y-20=-3x+12$ or, $3x+4y=32$.

(ii) $x^2+2x+y=0$

Differentiating both sides of the equation with respect to x we get

$$2x+2+\frac{dy}{dx}=0 \qquad \therefore \frac{dy}{dx}=-2(x+1)$$

$$\therefore \left(\frac{dy}{dx}\right)_{(-2,0)}=-2(-2+1)=2.$$

Hence the equation of the tangent to the parabola at the point $(-2, 0)$ is

$$y-0=\left(\frac{dy}{dx}\right)_{(-2,0)}(x+2) \text{ or, } y=2(x+2) \text{ or, } y=2x+4.$$

(iii) Putting $y=2$, in the equation of the ellipse we find

$$5x^2+12=137 \quad \text{or,} \quad 5x^2=125 \quad \text{or,} \quad x^2=25 \quad \text{or,} \quad x=\pm 5.$$

So, we are to find the equation of the tangents to the ellipse at the points $(\pm 5, 0)$.

Now differentiating both sides of the equation $5x^2+3y^2=137$ we get

$$10x+6y\frac{dy}{dx}=0 \quad \text{or,} \quad \frac{dy}{dx}=-\frac{5x}{3y}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(5,2)}=-\frac{25}{6} \text{ and } \left(\frac{dy}{dx}\right)_{(-5,2)}=\frac{25}{6}.$$

The equation of the tangent to the ellipse at the point (5, 2) is

$$y-2=\left(\frac{dy}{dx}\right)_{(5,2)}(x-5) \quad \text{or,} \quad y-2=-\frac{25}{6}(x-5)$$

$$\text{or,} \quad 6y-12=-25x+125 \quad \text{or,} \quad 25x+6y=137.$$

The equation of the tangent to the ellipse at the point $(-5, 2)$ is

$$y-2=\left(\frac{dy}{dx}\right)_{(-5,2)}(x+5) \quad \text{or,} \quad y-2=\frac{25}{6}(x+5)$$

$$\text{or,} \quad 6y-12=25x+125 \quad \text{or,} \quad 25x-6y+137=0$$

(iv) The equations of the curve are $x=a\cos^3\theta$, $y=b\sin^3\theta$

Now, $\frac{dx}{d\theta}=3a\cos^2\theta(-\sin\theta)$ and $\frac{dy}{d\theta}=3b\sin^2\theta\cos\theta$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=-\frac{b}{a}\frac{\sin\theta}{\cos\theta}.$$

Hence the equation of the tangent to the curve at the point 'θ' i.e. $(a\cos^3\theta, b\sin^3\theta)$ is

$$y-b\sin^3\theta=-\frac{b\sin\theta}{a\cos\theta}(x-a\cos^3\theta)$$

$$\text{or,} \quad a\cos\theta\, y-ab\sin^3\theta\cos\theta=-b\sin\theta\, x+ab\sin\theta\cos^3\theta$$

$$\begin{aligned}\text{or,} \quad b\sin\theta\, x+a\cos\theta\, y&=ab\sin^3\theta\cos\theta+ab\sin\theta\cos^3\theta\\&=ab\sin\theta\cos\theta(\sin^2\theta+\cos^2\theta)\\&=ab\sin\theta\cos\theta.\end{aligned}$$

Example 3. Show that the straight line $x-y=5$ touch the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and find the co-ordinates of the point of contact.

$x-y=5$ or, $x=y+5$ putting this value of x in the eqution $\frac{x^2}{16}+\frac{y^2}{9}=1$ of the ellipse we get $\frac{(y+5)^2}{16}+\frac{y^2}{9}=1$

or, $25y^2+90y+81=0$ or, $(5y+9)^2=0$ or, $y=-\frac{9}{5}, -\frac{9}{5}$,

$\therefore$ $x=y+5=-\frac{9}{5}+5=\frac{16}{5}$.

So the straight line $x-y=5$ meets the ellipse in two coincident points $(-\frac{9}{5}, \frac{16}{5})$, $(-\frac{9}{5}, \frac{16}{5})$.

Hence the straight line touches the elllpse and the co-ordinates of the point of contact are $(-\frac{9}{5}, \frac{16}{5})$.

Example 4. For which value of k will the straight line $x+3y=k$ touch the circle $x^2+y^2-3x-3y+2=0$.

$x+3y=k$ or, $x=k-3y$.

Putting this value of x in the equation of the circle we get,

$(k-3y)^2+y^2-3(k-3y)-3y+2=0$

or, $10\,y^2+6(1-k)y+(k^2-3k+2)=0$ ... ... (i)

The equation—(i) is a quadratic equation in y and its roots are the ordinates of the points of intersection of the circle and the straight line. So if the straight line touches the circle, then the roots of the equation—(i) will be equal and so its discriminant will be zero.

So, $36(1-k)^2-40(k^2-3k+2)=0$

or, $-4k^2+48k-44=0$ or, $k^2-12k+11=0$.

or, $(k-11)(k-1)=0$ $\therefore$ $k=11$ or, 1.

When $k=11$, then the equation—(i) becomes

$10\,y^2-60\,y^2+90=0$ or, $y^2-6\,y+9=0$ or, $(y-3)^2=0$

$\therefore$ $y=3, 3$. So $x=k-3y=11-11-9=2, 2$.

So, in this case the point of contanct is (2, 3).

When $k=1$, the equation—(i) becomes.

$10\,y^2=0$ or, $y=0$, so $x=k-3y=1$.

So, in this case the point of contact is (1, 0).

Hence if $k=11$ or 1 the straight line $x+3y=k$ will touch the circle and the corresponding points of contact are the points (2, 3) and (1, 0).

Example 5. If the straight line $y=x\sin\alpha+a\sec\alpha$ touches the circle $x^2+y^2=a^2$, show that $\cos^2\alpha=1$.

Let the straight line $y=x\sin\alpha+a\sec\alpha$......(1) touch the circle $x^2+y^2=a^2$......(2) at the point (x_1, y_1) of it.

The equation of the tangent to the circle (2) at the point (x_1, y_1) is $xx_1+yy_1=a^2$……(3).

So, the equations (1) and (3) represent the same straight line. Hence their corresponding coefficients are proportional.

$$\therefore \quad \frac{x_1}{\sin \alpha}=\frac{y_1}{1}=\frac{a^2}{-a \sec \alpha}$$

or, $x_1=-a \sin \alpha \cos \alpha$ and $y_1=-a \cos \alpha$.

Now, (x_1, y_1) is a point of the circle—(2);

so $x_1{}^2+y_1{}^2=a^2$ or, $a^2 \sin^2 \alpha \cos^2 \alpha+a^2 \cos^2 \alpha=a^2$

or, $\sin^2 \alpha \cos^2 \alpha+\cos^2\alpha-1=0$.

or, $\sin^2 \alpha \cos^2 \alpha+\cos^2 \alpha-(\cos^2 \alpha+\sin^2 \alpha)=0$

or, $\sin^2 \alpha(\cos^2 \alpha-1)=0$, or, $-(1-\cos^2 \alpha)(1-\cos^2 \alpha)=0$

or, $(1-\cos^2 \alpha)^2=0$ or, $1-\cos^2 \alpha=0$ $\therefore$ $\cos^2 \alpha=1$.

Example 6. Find the equation of the tangent to the parabola $y^2=8x$, parallel to the straight line $y+3x=0$. [H. S., 1970]

Let the tangent to the parabola $y^2=8x$……(1)

Parallel to the straight line $y+3x=0$ be $y+3x+k=0$……(2) and it touch the parabola at the point (x', y').

Now, the equation of the tangent to the parabola at the point (x', y') is $yy'=4(x+x')$ or, $yy'-4x-4x'=0$……(3)

Hence the straight lines (2) and (3) represent the same straight line.

So, the corresponding coefficients are proportional.

$$\therefore \quad \frac{y'}{1}=\frac{-4}{3}=-\frac{4x'}{k} \quad \therefore \quad y'=-\frac{4}{3} \text{ and } x'=\frac{k}{3}.$$

Now (x', y') is a point of the straight line (2)

$\therefore$ $y'+3x'+k=0$ or, $-\frac{4}{3}+k+k=0$

or, $2k=\frac{4}{3}$ $\therefore$ $k=\frac{2}{3}$ $\therefore$ $x'=\frac{2}{9}$.

Hence the required equation of the tangent is $y+3x+\frac{2}{3}=0$ or, $9x+3y+2=0$, the point of contact being the point $(\frac{2}{9}, -\frac{4}{3})$.

Example 7. Find the equation of the normals to the following curves at the given points.

(i) The parabola $x^2=4y$ at the point (6, 9).

(ii) The hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ at the extremity of the latus rectum situated in the first quadrant.

(iii) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the extremities of the minor-axis.

(iv) The curve $x=a\,(\theta-\sin\theta)$, $y=a\,(1-\cos\theta)$ at the point $\theta=\frac{\pi}{2}$.

(i) $x^2=4y \quad \therefore \quad 2x=4\frac{dy}{dx}$ or, $\frac{dy}{dx}=\frac{2x}{4}$

$$\therefore \quad -\frac{1}{\left(-\frac{dy}{dx}\right)_{(6,\,9)}}=-\frac{1}{3}.$$

Hence the equation of the normal to the parabola at the point (6, 9) is

$y-9=-\frac{1}{3}(x-6)$ or, $3y-27=-x+6$ or, $x+3y=33.$

(ii) If e be the eccentricity of hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, then $4=9\,(e^2-1)$ or, $\frac{4}{9}=e^2-1$ or, $e^2=1+\frac{4}{9}=\frac{13}{9}$ $\therefore$ $e=\frac{\sqrt{13}}{3}$.

So the co-ordinates of the extremity of the latus-rectum situated in the first quadrant are

$$\left(3.\frac{\sqrt{13}}{3},\ \frac{4}{3}\right) \quad \text{i.e.,} \quad (\sqrt{13},\ \tfrac{4}{3})$$

Now differentiating both sides of the equation of the hyperbola with respect to y we get

$$\frac{2x}{9}\frac{dx}{dy}-\frac{2y}{4}=0 \quad \text{or,} \quad \frac{dx}{dy}=\frac{9}{4}\cdot\frac{y}{x}$$

$$\therefore \quad \left(-\frac{dx}{dy}\right)_{(\sqrt{13},\,\frac{4}{3})}=-\frac{9}{4}\cdot\frac{4}{3\sqrt{13}}=-\frac{3}{\sqrt{13}}.$$

Hence the equation of the normal to the hyperbola at the point $(\sqrt{13}, \frac{4}{3})$ is

$$y-\frac{4}{3}=\left(-\frac{dx}{dy}\right)_{(\sqrt{13},\,\frac{4}{3})}(x-\sqrt{13})$$

or, $y-\frac{4}{3}=-\frac{3}{\sqrt{13}}(x-\sqrt{13})$ or, $y+\frac{3x}{\sqrt{13}}=\frac{4}{3}+3=\frac{13}{3}$.

(iii) The extremities of the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$······(i) are the points $(0, \pm b)$.

Now differentiating both sides of the equation—(i) with respect to x we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \text{ or, } \frac{dy}{dx}=-\frac{b^2}{a^2}\,\frac{x}{y}. \quad \therefore \left(\frac{dy}{dx}\right)_{(0,\ \pm b)}=0.$$

Hence the tangents to the ellipse at these points are parallel to the x-axis. So the normals are parallel to the y-axis. $\therefore$ Let the equations of the normals be $x=k$. As the two normals pass through the points $(0, \pm b)$, so, $k=0$. Hence the normals to the ellipse at these points are the straight line $x=0$ i.e., the minor axis.

(iv) $x=a(\theta-\sin\theta)$ $\therefore$ $\dfrac{dx}{d\theta}=a(1-\cos\theta)$

$y=a(1-\cos\theta)$ $\therefore$ $\dfrac{dy}{dx}=a\sin\theta.$

$$\therefore \frac{dx}{dy}=\frac{\dfrac{dx}{d\theta}}{\dfrac{dy}{d\theta}}=\frac{a(1-\cos\theta)}{a\sin\theta}=\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\tan\frac{\theta}{2}.$$

$$\therefore \left(-\frac{dx}{dy}\right)_{\theta=\frac{\pi}{2}}=-\tan\frac{\pi}{4}=-1.$$

Hence the equation of the normal to the curve at the point $\theta=\frac{\pi}{2}$ i.e., the point $\left\{a\left(\frac{\pi}{2}-\sin\frac{\pi}{2}\right), a\left(1-\cos\frac{\pi}{2}\right)\right\}$

i.e., $\{a(\frac{\pi}{2}-1), a\}$ is

$$y-a=-1\left\{x-a\left(\frac{\pi}{2}-1\right)\right\}$$

or, $y-a=-1(x-a\,\frac{\pi}{2}+a)$ or, $y-a=-x+a\,\frac{\pi}{2}-a$

or, $x+y=a\,\frac{\pi}{2}$.

Example 8. Find the equation of the normal to the ellipse $4x^2+9y^2=72$ which is perpendicular to the straight line $2x+3y=6$.

The equation of a straight line perpendicular to the straight line $2x+3y=6$ is $3x-2y=c$ $\cdots$(i)

Let this straight line be normal to the ellipse $4x^2+9y^2=72$ $\cdots\cdots$(ii) at the point (x', y').

Now differentiating both sides of equation-(ii) with respect to y we get

$$8x\,\frac{dx}{dy}+18y=0 \quad \therefore \quad -\frac{dx}{dy}=\frac{9}{4}\,\frac{y}{x}$$

$$\therefore \left(-\frac{dx}{dy}\right)_{(x',\,y')}=\frac{9}{4}\,\frac{y'}{x'}$$

So, the equation of the normal to the ellipse at the point (x', y') is

$$y-y'=\left(-\frac{dx}{dy}\right)_{(x',\,y')}(x-x') \quad \text{or,} \quad y-y'=\frac{9}{4}\,\frac{y'}{x'}\,(x-x')$$

or, $9y'x-4x'y=5x'y'$……(iii)

So, equation-(i) and (iii) respresent the same straight line. Hence the corresponding coefficients are proportional.

$$\therefore \quad \frac{9y'}{3}=\frac{-4x'}{-2}=\frac{5x'y'}{c} \quad \therefore \quad x'=\frac{3c}{5} \text{ and } y'=\frac{2c}{5}$$

Now (x', y') is a point on the ellipse-(ii)

$$\therefore \quad \frac{36c^2}{25}+\frac{36c^2}{25}=72 \quad \text{or,} \quad \frac{72c^2}{25}=72 \quad \text{or,} \quad c^2=25 \quad \therefore \quad c=\pm 5$$

Hence the required equations of the normals are $3x-2y\pm 5=0$.

Example 9. Find the equation of the tangents to the curve $x^2-2xy+2y^2-7x+6y+6=0$. Which are perpendicular to the straight line $6x+5y=4$.

The equation of the given curve is

$x^2-2xy+2y^2-7x+6y+6=0$……(i)

Differentiating both sides of this equation with respect to x we get,

$$2x-2\left(y+x\frac{dy}{dx}\right)+4y\,\frac{dy}{dx}-7+6\,\frac{dy}{dx}=0.$$

or, $\dfrac{dy}{dx}(4y-2x+6)=2y-2x+7$. $\therefore \dfrac{dy}{dx}=\dfrac{2y-2x+7}{4y-2x+6}$.

The gradient of the given straight line

$6x+5y=4$……(ii) is $-\dfrac{6}{5}$.

So, that gradient of the tangent to the curve which is perpendicular to the straight line (ii) is $\dfrac{5}{6}$.

$\therefore \quad \frac{2y-2x+7}{4y-2x+6}=\frac{5}{6}$ or, $12y-12x+42=20y-10x+30.$

or, $-2x=8y-12$ or, $x=6-4y \cdots\cdots$(iii)

Putting $x=6-4y$ in the equation—(i) of the curve we get,

$(6-4y)^2-2(6-4y).y+2y^2-7(6-4y)+6y+6=0$

or, $26y^2-26y=0 \quad \therefore \quad y=0$ or, 1

So from (iii) we get $x=6$ or, 2.

Hence the tangents to the curve at the points (6, 0) or, (2, 1) are perpendicular to the given straight line (ii).

Now the equation of the tangent to curve at the point (6, 0) is

$y-0=\frac{5}{6}(x-6)$ or, $6y=5x-30$ or, $5x-6y=30.$

Also the equation of the tangent to the curve at the point (2, 1) is $y-1=\frac{5}{6}(x-2)$ or, $5x-10=6y-6$ or, $5x-6y=4$

Hence the equations of the tangents are,

$5x-6y=30$ and $5x-6y=4.$

Example 10. Find the equation of the normal to the hyperbola $xy=4$ at the point (2, 2). Also determine the point at which the normal again intersects the hyperpola.

The equation of the hyperbola is $xy=4 \cdots\cdots$(i)

Differentiating both sides the equation with respect to y we get $y\frac{dx}{dy}+x=0$ or, $-\frac{dx}{dy}=\frac{x}{y}$

$$\therefore \quad \left(-\frac{dx}{dy}\right)_{(2,\,2)}=\frac{2}{2}=1.$$

Hence the equation of the normal to the curve at (2, 2) is

$y-2=1.(x-2)$ or, $y=x,$

Now putting $y=x$ in the equation of the hyperbola we get,

$x^2=4$ or, $x=\pm 2. \quad \therefore \quad y=\pm 2.$

So, the normal intersects, the hyperbola at the point (2, 2) and $(-2, -2)$. The point (2, 2) refers to the given foot of the normal and $(-2, -2)$ is the required point where it once again intersects the curve.

Example 11. The normal to the rectangular hyperbola $xy=a^2$

at the point $\left(at, \dfrac{a}{t}\right)$ intersects the curve again at the point $\left(at', \dfrac{a}{t'}\right)$. Prove that $t^3t' = -1$.

The equation of the hyperbola is $xy = a^2 \cdots\cdots$(i).

Differentiating both sides of the equation with respect to y we get $y\dfrac{dx}{dy} + x = 0$ or, $-\dfrac{dx}{dy} = \dfrac{x}{y}$ $\therefore$ $\left(-\dfrac{dx}{dy}\right)_{(at,\ \frac{a}{t})} = t^2$.

Henece the equation of the hyperbola at the point $\left(at, \dfrac{a}{t}\right)$ is

$$y - \frac{a}{t} = t^2(x - at) \quad \text{or,} \quad t^3x - ty = a - at^4 \cdots\cdots \text{(ii)}$$

If this normal intersects the hyperbola again at the point $\left(at', \dfrac{a}{t'}\right)$, then the straight line joining the points $\left(at', \dfrac{a}{t'}\right)$ and $\left(at, \dfrac{a}{t}\right)$ and the normal (ii) will be the same straight line.

The equation of the straight line joining the points $\left(at, \dfrac{a}{t}\right)$ and $\left(at', \dfrac{a}{t'}\right)$ is $\dfrac{x - at}{at - at'} = \dfrac{y - \dfrac{a}{t}}{\dfrac{a}{t} - \dfrac{a}{t'}}$

$$\text{or,} \quad \frac{x - at}{a(t - t')} = \frac{ty - a}{\dfrac{ta(t' - t)}{tt'}} \quad \text{or,} \quad x - at = -t'(ty - a)$$

$$\text{or,} \quad x + tt'y = a(t + t') \quad \cdots \quad \cdots \quad \text{(iii)}.$$

So, equation—(ii) and (iii) represent the same straight line and hence their corresponding coefficients are proportional.

$$\therefore \quad \frac{t^3}{1} = \frac{-t}{tt'} \quad \text{or,} \quad t^3 = -\frac{1}{t'} \quad \text{or,} \quad t^3t' = -1.$$

Example 12. Find the condition that the straight line $x \cos\theta + y \sin\theta = p$ may touch the parabola $y^2 = 4ax$. [H. S. 1984]

Let the straight line $x \cos\theta + y \sin\theta = p \cdots\cdots$(1) touch the parabola $y^2 = 4ax \cdots\cdots$(2) at the point (x', y').

Differentiating both sides of the equation—(ii) with respect to x we get

$$2y\frac{dy}{dx}=4a \quad \text{or,} \quad \frac{dy}{dx}=\frac{2a}{y}. \quad \therefore \left[\frac{dy}{dx}\right]_{(x',\, y')}=\frac{2a}{y'}.$$

Hence the equation of the tangent to the parabola at the point (x', y') is

$$y-y'=\frac{2a}{y'}(x-x') \quad \text{or,} \quad yy'-y'^2=2ax-2ax'.$$

$$\text{or,} \quad yy'-2ax=4ax'-2ax' \qquad [\because \; y'^2=4ax']$$

$$\text{or,} \quad yy'-2ax=2ax' \quad \cdots \quad \cdots \quad \text{(iii)}$$

So equations—(i) and (iii) represent the same straight line. Hence their corresponding coefficients are proportional.

$$\therefore \quad \frac{\cos\theta}{-2a}=\frac{\sin\theta}{y'}=\frac{p}{2ax'} \quad \therefore \quad x'=-\frac{p}{\cos\theta} \text{ and } y'=-2a\tan\theta.$$

Now $y'^2=4ax'$ or, $4a^2\tan^2\theta=-\dfrac{4ap}{\cos\theta}$

or, $a\tan\theta\sin\theta=-p$ and this is the required condition.

Example 13. Find the equation of the tangent to the parabola $y^2=12x$ inclined at an angle 45° with the positive direction of the x-axis. Also find the co-ordinates of the point of contact.

[H. S., 1963]

Let the tangent to the parabola $y^2=12x \cdots\cdots$(i) at the point (x', y') makes an angle 45° with the positive direction of the x-axis. So, the gradient of the tangent is $\tan 45^\circ=1$ and its equation is $y-y'=1(x-x')$ or, $x-y=x'-y'$ ... ... (ii)

Now differentiating both sides of equation—(i)

With respect to x we get $2y\,\dfrac{dy}{dx}=12.$

$$\therefore \quad \frac{dy}{dx}=\frac{6}{y} \quad \text{or,} \quad \left(\frac{dy}{dx}\right)_{(x',\, y')}=\frac{6}{y'}$$

$$\therefore \quad \frac{6}{y'}=1 \quad \text{or,} \quad y'=6.$$

Now (x', y') is a point on the parabola,

$$\therefore \quad y'^2=12x' \quad \text{or,} \quad 36=12x' \quad \therefore \quad x'=3.$$

Hence the equation of the tangent to the parabola $y^2=12x$ at the point (3, 6), inclined at an angle 45° with the positive direction of the x-axis is $y-6=1\,.\,(x-3)$ or, $x-y+3=0$.

Example 14. Find the equations of the two tangents to the ellipse $2x^2+7y^2=14$ drawn from the point (5, 2). Also find the angle between them. [Joint Entrance, 1973]

We know that the straight lines $y=mx\pm\sqrt{a^2m^2+b^2}$ are tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of m.

Now the equation of the ellipse is $2x^2+7y^2=14$

or, $\frac{x^2}{7}+\frac{y^2}{2}$……(i) $\therefore$ $a^2=7$, and $b^2=2$.

$\therefore$ The straight lines $y=mx\pm\sqrt{7m^2+2}$ … … (ii) are tangents to the ellipse—(i).

If these tangents pass through the point (5, 2) then

$2=5m\pm\sqrt{7m^2+2}$ or, $2-5m=\pm\sqrt{7m^2+2}$

or, $4+25m^2-20m=7m^2+2$ (squaring both sides)

or, $18m^2-20m+2=0$ or, $9m^2-10m+1=0$

or, $(m-1)(9m-1)=0$ $\therefore$ $m=1$ or, $\frac{1}{9}$.

Putting $m=1$ in the equation—(ii) we get $y=x\pm3$. But $y=x+3$ is not satisfied by $x=5$ and $y=2$. So $y=x-3$ is the equation of one tangent to the ellipse—(i) through the point (5, 2).

Again putting $m=\frac{1}{9}$ in the equation—(ii)

We get $y=\frac{1}{9}x\pm\sqrt{\frac{7}{81}+2}=\frac{1}{9}x\pm\frac{13}{9}$.

The equation $y=\frac{1}{9}x-\frac{13}{9}$ is not satisfied by $x=5$, $y=2$. Hence $y=\frac{1}{9}x+\frac{13}{9}$ or, $x-9y+13=0$ is the equation of the other tangent to the ellipse through the point (5, 2).

Again the gradients of the tangents are 1 and $\frac{1}{9}$. So the acute angle between the tangents is $\tan^{-1}\frac{1-\frac{1}{9}}{1+\frac{1}{9}}=\tan^{-1}(\frac{4}{5})$.

Example 15. The straight line $5x-3y+9=0$ touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The eccentricity of the hyperbola is $\frac{5}{3}$. Find the distance between the foci of the hyperbola and also the co-ordinates of the point of contact.

The eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ... ... (i) is $\frac{5}{3}$ $\therefore$ $b^2=a^2(e^2-1)=a^2(\frac{25}{9}-1)=\frac{16}{9}a^2$ ... ... (ii) or, $16a^2=9b^2$.

Let the straight line $5x-3y+9=0$ ... (iii) touch the hyperbola—(i) at the point (x', y'). Now the equation of the tangent to the hyperbola at the point (x', y') is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1 \quad \cdots \quad \cdots \quad \text{(iv)}$$

Hence equations (iii) and (iv) represent the same straight line and so their corresponding coefficients are proportional.

$$\therefore \ \frac{x'}{5a^2}=\frac{-y'}{-3b^2}=\frac{1}{-9} \quad \text{So,} \ x'=-\tfrac{5}{9}a^2 \ \text{and} \ y'=-\frac{b^2}{3}$$

Now (x', y') is a point of the hyperbola ... ... (v)

$$\therefore \ \frac{25\,a^4}{81\,a^2}-\frac{b^4}{9b^2}=1 \ \text{ or, } \ \frac{25}{81}a^2-\frac{1}{9}b^2=1$$

or, $\frac{25}{81}a^2-\frac{1}{9}\cdot\frac{16}{9}a^2=1$ [From (ii)]

or, $\frac{a^2}{9}=1$ $\therefore$ $a^2=9$ $\therefore$ $b^2=\frac{16}{9}a^2=16.$

So $x'=-\frac{5}{9}a^2=-\frac{5}{9}\cdot 9=-5$ and $y'=-\frac{b^2}{3}=-\frac{16}{3}$.

$\therefore$ The co-ordinates of the point of contact are $(-5, -\frac{16}{3})$. Also the distance between the foci is $2\,ae=2\cdot 3\cdot\frac{5}{3}=10$ units.

Example 16. Find the equation of the common tangent of the circle $x^2+y^2=2a^2$ and the parabola $y^2=8ax$.

We know that the straight line $y=mx+\frac{2a}{m}$ ... (i) is a tangent of the parabola $y^2=8ax$ ... ... (ii)

Now putting $mx+\frac{2a}{m}$ for y in the equation $x^2+y^2=2a^2$ ... (iii) of the circle we get

$$x^2+\left(mx+\frac{2a}{m}\right)^2=2a^2 \quad \text{or,} \quad x^2(1+m^2)+4ax+\frac{4a^2}{m^2}=2a^2$$

or, $m^2(1+m^2)x^2+4am^2x+4a^2=2a^2m^2$

or, $m^2(1+m^2)x^2+4am^2x+2a^2(2-m^2)=0$ ... ... (iv)

This equation (iv) is a quadratic equation in x and its roots are the x co-ordinates of the straight line (i) and the circle (iii). So, if the straight line (i) also touches the circle (iii), then the roots of equation (iv) will be equal and the discriminant of the equation will be zero.

i.e., $16a^2m^4-8m^2(1+m^2)a^2(2-m^2)=0$

or, $8a^2m^2(2m^2-2-2m^2+m^2+m^4)=0$

or, $8a^2m^2(m^4+m^2-2)=0$

or, $8a^2m^2(m^2+2)(m^2-1)=0$

$\therefore\ m^2-1=0$ as $m^2+2\neq0$ and $m\neq0$ [since a straight line parallel to the x-axis cannot touch the parabola—(ii)]

or, $m=\pm1$.

So, the equation—(i) becomes $y=\pm x\pm2a$ and it is the equation of the common tangent to both the parabola—(ii) and the circle—(iii).

Example 17. Find the equation of the common tangent to the parabolas $y^2=4ax$ and $x^2=4by$. [H. S.]

The straight line $y=mx+\dfrac{a}{m}\cdots$(i) is a tangent to the parabola $y^2=4ax\cdots$(ii) for all values of m ($\neq0$). Putting $mx+\dfrac{a}{m}$ for y in the equation of the parabola $x^2=4by\cdots$(iii)

We get $x^2=4b\left(mx+\dfrac{a}{m}\right)$

or, $mx^2-4bm^2x-4ab=0$ ... ... (iv)

The roots of the quadratic equation (iv) in x give the x co-ordinates of the points of intersection of the straight line (i) and the parabola (iii). So if the straight line (i) also touches the parabola (iii), then the roots of the equation (iv) will be equal and the discriminant of the equation will be zero.

So, $16b^2m^4+16abm=0$ or, $16bm(bm^3+a)=0$.

$\therefore bm^3+a=0$ (as $m\neq0$) $\therefore m^3=-\dfrac{a}{b}$ or, $m=-\sqrt[3]{\dfrac{a}{b}}$.

Hence the required equation of the common tangent of the two parabolas is $y=-\sqrt[3]{\frac{a}{b}}\,x-\frac{a}{\sqrt[3]{\frac{a}{b}}}$ or, $y=-\sqrt[3]{\frac{a}{b}}\,x-\sqrt[3]{a^2b}.$

or, $\sqrt[3]{a}\,x+\sqrt[3]{b}\,y+\sqrt[3]{a^2b^2}=0.$

Example 18. The normal to the parabola $y^2=5x$ makes an angle of 45° with the x-axis. Find the equation of the normal and the co-ordinates of its foot. [H. S. 1980]

Let the normal to the parabola $y^2=5x$ ⋯ (i) at the point $(\frac{5}{4}t^2, \frac{5}{2}t)$ makes an angle 45° with the positive direction of the x-axis [Here $4a=5 \quad \therefore \; a=\frac{5}{4}$].

Hence the gradient of the normal is $\tan 45°=1$.

Differentiating both sides of equation (i) with respect to y we get, $2y=5\frac{dx}{dy}$. $\quad \therefore \; \frac{dx}{dy}=\frac{2y}{5} \quad \therefore \; \left[-\frac{dx}{dy}\right]_{(\frac{5}{4}t^2,\ \frac{5}{2}t)}$

$$=-\frac{2.\frac{5}{2}t}{5}=-t \quad \therefore \; -t=1 \quad \text{or,} \quad t=-1.$$

Here the foot of the normal is the point

$$\{\tfrac{5}{4}.(-1)^2,\ \tfrac{5}{2}(-1)\}=(\tfrac{5}{4}, -\tfrac{5}{2})$$

And the equation of the normal is $y+\frac{5}{2}=1\,(x-\frac{5}{4})$
or, $4x-4y=15$.

Example 19. If the normal to the parabola $y^2=4ax$ drawn at the point $(am_1^2, 2am_1)$ meets the parabola again at the point $(am_2^2, 2am_2)$, then prove that $m_1^2+m_1m_2+2=0$.

Differentiating both sides of the equation $y^2=4ax$ ⋯ ⋯ (i) with respect to y we get, $2y=4a\,\frac{dx}{dy}$.

or, $\frac{dx}{dy}=\frac{y}{2a} \quad \therefore \; \left[-\frac{dx}{dy}\right]_{(am_1^2,\ 2am_1)}=-\frac{2am_1}{2a}=-m_1$

So, the equation of the normal to the parabola at the point $(am_1^2, 2am_1)$ is $y-2am_1=-m_1\,(x-am_1^2)$.

or, $m_1x+y=2am_1+am_1^3$ ⋯ ⋯ (ii)

If this normal meets the parabola again at the point $(am_2^2, 2am_2)$, then $x=am_2^2$, $y=2am_2$ will satisfy the equation (ii).

So, $am_1m_2^2+2am_2=2am_1+am_1^3$.

or, $m_1^3+2m_1-m_1m_2^2-2m_2=0$

or, $m_1(m_1^2-m_2^2)+2(m_1-m_2)=0$

or, $(m_1-m_2)(m_1^2+m_1m_2+2)=0$

or, $m_1^2+m_1m_2+2=0$ $\quad$ [assuming $m_1 \neq m_2$]

Example 20. Prove that for all values of n, the tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at the point (a, b) is $\frac{x}{a}+\frac{y}{b}=2$.

Differentiating both sides of the equation, $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=1$ (i)

With respect to x we get $\frac{n\,x^{n-1}}{a^n}+\frac{n\,y^{n-1}}{b^n}\frac{dy}{dx}=0$.

$$\therefore\ \frac{dy}{dx}=-\frac{b^n}{a^n}\frac{x^{n-1}}{y^{n-1}} \qquad \therefore\ \left[\frac{dy}{dx}\right]_{(a,\,b)}=-\frac{b^n}{a^n}\frac{a^{n-1}}{b^{n-1}}=-\frac{b}{a}.$$

Hence the equation of the tangent to the curve (i) at the point (a, b) is $y-b=-\frac{b}{a}(x-a)$ or, $ay-ab=-bx+ab$.

or, $bx+ay=2ab$ or, $\frac{x}{a}+\frac{y}{b}=2$ and this equation is independent of n. Hence for all values of n, the tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at the point (a, b) is $\frac{x}{a}+\frac{y}{b}=2$.

Example 21. Prove that if the straight line $lx+my=1$ be normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\frac{a^2}{l^2}-\frac{b^2}{m^2}=(a^2+b^2)^2$.

Let the given straight line $lx+my=1$ ... (1) be normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ... (ii) at the point $(a\sec\theta,\ b\tan\theta)$ of it.

Now differentiating both sides of equation (ii) with respect to y we get $\frac{2x}{a^2}\frac{dx}{dy}-\frac{2y}{b^2}=0$.

$$\text{or, } \frac{dx}{dy}=\frac{a^2}{b^2}\frac{y}{x}. \qquad \therefore\ \left[-\frac{dx}{dy}\right]_{(a\sec\theta,\ b\tan\theta)}$$

$$=-\frac{a^2}{b^2}\frac{b\tan\theta}{a\sec\theta}=-\frac{a}{b}\sin\theta.$$

So, the equation of the normal to the hyperbola (ii) at the point $(a \sec \theta, b \tan \theta)$ is $y - b \tan \theta = -\frac{a}{b} \sin \theta \, (x - a \sec \theta)$

or, $by - b^2 \tan \theta = -a \sin \theta x + a^2 \tan \theta$

or, $a \sin \theta x + by = (a^2 + b^2) \tan \theta$ ⋯ ⋯ (iii)

So, the equations (i) and (iii) represent the same straight line and so the corresponding coefficients are proportional.

$$\therefore \quad \frac{a \sin \theta}{l} = \frac{b}{m} = \frac{(a^2+b^2) \tan \theta}{1}$$

$$\therefore \quad \frac{a}{l} = (a^2+b^2) \sec \theta \text{ and } \frac{b}{m} = (a^2+b^2) \tan \theta.$$

$$\therefore \quad \frac{a^2}{l^2} - \frac{b^2}{m^2} = (a^2+b^2)^2 \sec^2 \theta - (a^2+b^2)^2 \tan^2 \theta$$

$$= (a^2+b^2)^2 (\sec^2 \theta - \tan^2 \theta) = (a^2+b^2)^2.1$$

$$\text{or,} \quad \frac{a^2}{l^2} - \frac{b^2}{m^2} = (a^2+b^2)^2.$$

Example 22. Prove that if the straight line $lx+my=1$ be normal to the parabola $y^2=4ax$, then $al^3+2alm^2=m^2$

Let the straight line $lx+my=1$ ⋯ (i) be normal to the parabola $y^2=4ax$ ⋯ (2) at the point $(at^2, 2at)$. Differentiating both sides of equation (ii) with respect to y we get

$$2y = 4a \frac{dx}{dy} \qquad \therefore \quad \frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}.$$

$\therefore$ The gradient of the normal to the parabola at the point $(at^2, 2at)$ is $\left[-\frac{dx}{dy}\right]_{(at^2, 2at)} = -\frac{2at}{2a} = -t.$

So, the equation of the normal to the parabola at the point is $y - 2at = -t(x - at^2)$.

or, $tx + y = 2at + at^3$ ⋯ ⋯ (iii)

So, the equations (i) and (iii) are equations of the same straight line and so their corresponding coefficients are proportional.

$$\therefore \quad \frac{l}{t} = \frac{m}{1} = \frac{1}{2at+at^3} \quad \therefore \quad t = \frac{l}{m} \text{ and } \frac{1}{m} = 2at + at^3$$

$$= 2a\frac{l}{m} + a\frac{l^3}{m^3} \quad \text{or,} \quad al^3 + 2alm^2 = m^2$$

Example 23. Prove that if the straight line $x \cos \alpha + y \sin \alpha = p$ touch the curve $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$, then $(a \cos \alpha)^{\frac{m}{m-1}}+(b \text{ in } \alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$. [C. U.]

Let the straight line $x \cos \alpha + y \sin \alpha = p$......(1) touch the curve $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$......(2) at the point (x_1, y_1) of the curve.

Differentiating both sides of the equation-(2) of the curve with respect to x we get $\frac{m}{a^m} x^{m-1} + \frac{m}{b^m} y^{m-1} \frac{dy}{dx} = 0$.

$\therefore \quad \frac{dy}{dx} = -\frac{b^m}{a^m} \frac{x^{m-1}}{y^{m-1}}$. $\therefore \quad \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = -\frac{b^m}{a^m} \frac{{x_1}^{m-1}}{{y_1}^{m-1}}$.

Hence the equation of the tangent to the curve at the point (x_1, y_1) is, $y - y_1 = -\frac{b^m}{a^m} \frac{{x_1}^{m-1}}{{y_1}^{m-1}} (x - x_1)$

or, $b^m {x_1}^{m-1} x + a^m {y_1}^{m-1} y = a^m {y_1}^m + b^m {x_1}^m$

or, $\frac{{x_1}^{m-1} x}{a^m} + \frac{{y_1}^{m-1} y}{b^m} = \frac{{x_1}^m}{a^m} + \frac{{y_1}^m}{b^m}$ [dividing both sides by $a^m b^m$]

or, $\frac{{x_1}^{m-1} x}{a^m} + \frac{{y_1}^{m-1} y}{b^m} = 1$......(3) [As (x_1, y_1) is a point of the curve (2)]

So, equations—(1) and (3) respresent the same straight line and hence their corresponding coefficient are proportional.

$$\therefore \quad \frac{\frac{{x_1}^{m-1}}{a^m}}{\cos \alpha} = \frac{\frac{{y_1}^{m-1}}{b^m}}{\sin \alpha} = \frac{1}{p}$$

$\therefore \quad \left(\frac{x_1}{a}\right)^{m-1} = \frac{a \cos \alpha}{p}$ and $\left(\frac{y_1}{b}\right)^{m-1} = \frac{b \sin \alpha}{p}$.

or, $\frac{x_1}{a} = \left(\frac{a \cos \alpha}{p}\right)^{\frac{1}{m-1}}$, $\frac{y_1}{b} = \left(\frac{b \sin \alpha}{p}\right)^{\frac{1}{m-1}}$

Now, (x_1, y_1) is a point of the curve...(2)

So $\frac{{x_1}^m}{a^m} + \frac{{y_1}^m}{b^m} = 1$ or, $\left\{\left(\frac{a \cos \alpha}{p}\right)^{\frac{1}{m-1}}\right\}^m + \left\{\left(\frac{b \sin \alpha}{p}\right)^{\frac{1}{m-1}}\right\}^m = 1$

or, $(a \cos \alpha)^{\frac{m}{m-1}} + (b \sin \alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$.

Example 24. Prove that if the straight line $x \cos \alpha + y \sin \alpha = p$ touch the curve $x^m y^n = a^{m+n}$, then

$p^{m+n}m^m n^n=(m+n)^{m+n}a^{m+n}\sin^m\alpha\cos^n\alpha$.

Let the straight line $x\cos\alpha+y\sin\alpha=p$···(i) touch the curve $x^m y^n=a^{m+n}$······(ii) at the point (x_1, y_1) of it.

Taking logarithm of both sides of equation-(ii) we get,

$$m\log x+n\log y=\log(a^{m+n}).$$

Now differentiating both sides with respect to x we get,

$$\frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=0 \quad \therefore \quad \frac{dy}{dx}=-\frac{my}{nx}.$$

$$\therefore \left[\frac{dy}{dx}\right]_{(x_1, y_1)}=-\frac{m}{n}\frac{y_1}{x_1}$$

Hence the equation of the tangent to the curve-(ii) at the point (x_1, y_1) is $y-y_1=-\frac{m}{n}\frac{y_1}{x_1}(x-x_1)$

$$\text{or, } my_1x+nx_1y=(m+n)\,x_1y_1 \quad \cdots \quad \cdots \text{(iii)}$$

Hence equation-(i) and (ii) represent the same straight line and so their corresponding coefficients are proportional.

$$\text{or, } \frac{my_1}{\cos\alpha}=\frac{nx_1}{\sin\alpha}=\frac{(m+n)x_1y_1}{p}$$

$$\therefore \quad x_1=\frac{mp}{(m+n)\cos\alpha} \text{ and } y_1=\frac{np}{\sin\alpha\,(m+n)}$$

Now (x_1, y_1) is a point of the curve $x^m y^n=a^{m+n}$

So, $x_1^m y_1^n=a^{m+n}$

$$\therefore \quad \frac{m^m p^m}{(m+n)^m\cos^m\alpha}\cdot\frac{n^n p^n}{(m+n)^n\sin^n\alpha}=a^{m+n}$$

$$\text{or, } p^{m+n}m^m n^n=a^{m+n}(m+n)^{m+n}\cos^m\alpha\sin^n\alpha.$$

Example 25. Prove that if the straight line $\frac{ax}{3}+\frac{by}{4}=c$ is normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $5c=a^2e^2$.

Let the straight line $\frac{ax}{3}+\frac{by}{4}=c$······(i) be normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$······(ii) at the point (x_1, y_1). Now, the equation of the normal to the ellipse...(ii) at the point (x_1, y_1) is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}} \text{ or, } \frac{a^2}{x_1}x-\frac{b^2}{y_1}y=a^2-b^2$$

or, $\frac{a^2}{x_1}x-\frac{b^2}{y_1}y=a^2e^2$......(iii)

$[\because a^2-b^2=a^2-a^2(1-e^2)=a^2e^2]$

So, equations-(i) and (iii) represent the same straight line and hence their corresponding coefficients are proportional.

$$\therefore \frac{\frac{a}{3}}{\frac{a^2}{x_1}}=\frac{\frac{b}{4}}{\frac{b^2}{-y_1}}=\frac{c}{a^2e^2} \text{ or, } \frac{x_1}{a}=\frac{3c}{a^2e^2}, \frac{y_1}{b}=\frac{-4c}{a^2e^2}$$

Now (x_1, y_1) is a point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\therefore \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$ or, $\frac{9c^2}{a^4e^4}+\frac{16c^2}{a^4e^4}=1$

or, $25c^2=a^4e^4$ $\therefore$ $5c=a^2e^2$.

Example 26. Prove that the sum of the intercepts on the co-ordinate axes of tangents to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ is constant.

The equation of the curve is $\sqrt{x}+\sqrt{y}=\sqrt{a}$.........(i). Differentiating both sides of equation-(i) with respect to x we get,

$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$. or, $\frac{dy}{dx}=-\sqrt{\frac{y}{x}}$.

So, gradient of the tangent to the curve-(i) at any point (x_1, y_1) of it is $\left[\frac{dy}{dx}\right]_{(x_1, y_1)}=-\sqrt{\frac{y_1}{x_1}}$.

Hence equation of the tangent to the curve at the point (x_1, y_1) is $y-y_1=-\sqrt{\frac{y_1}{x_1}}(x-x_1)$ or, $\sqrt{x_1}y+\sqrt{y_1}x=\sqrt{x_1}y_1+\sqrt{y_1}x_1=\sqrt{x_1y_1}(\sqrt{x_1}+\sqrt{y_1})$ or, $\sqrt{x_1}y+\sqrt{y_1}x=\sqrt{x_1y_1}\sqrt{a}$ [as (x_1, y_1) is a point of the curve so $\sqrt{x_1}+\sqrt{y_1}=\sqrt{a}$]

or, $\frac{x}{\sqrt{x_1}\sqrt{a}}+\frac{y}{\sqrt{y_1}\sqrt{a}}=1$

Hence the intercepts of the tangent on the axes of co-ordinates are $\sqrt{x_1}\sqrt{a}$ and $\sqrt{y_1}\sqrt{a}$ and their sum is $\sqrt{x_1}\sqrt{a}+\sqrt{y_1}\sqrt{a}=(\sqrt{x_1}+\sqrt{y_1})\sqrt{a}=\sqrt{a}\sqrt{a}=a$ which is a constant.

Example 27. If the lengths of the intercepts on the co-ordinate axes of the tangent to the curve $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$ at any point of it be x_1 and y_1, then show that $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$ [C.U.]

The equation of the curve is $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$.

Differentiating both sides of the equation with respect to x we get, $\frac{2}{3}\ \frac{1}{a^{2/3}}\ x^{-1/3}+\frac{2}{3}\cdot\frac{1}{b^{2/3}}\ y^{-1/3}\ \frac{dy}{dx}=0$

$$\therefore\quad \frac{dy}{dx}=-\frac{b^{2/3}}{a^{2/3}}\ \frac{y^{1/3}}{x^{1/3}}.$$

Hence the equation of the tangent to the curve at any point (α,β) of it is

$$y-\beta=\left[\frac{dy}{dx}\right]_{(\alpha,\ \beta)}(x-\alpha)\quad\text{or.}\quad y-\beta=-\frac{b^{2/3}}{a^{2/3}}\ \frac{\beta^{1/3}}{\alpha^{1/3}}(x-\alpha)$$

$$\text{or,}\quad b^{2/3}\beta^{1/3}x+a^{2/3}\alpha^{1/3}y=\alpha^{1/3}\beta^{1/3}a^{2/3}b^{2/3}\left(\frac{\alpha^{2/3}}{a^{2/3}}+\frac{\beta^{2/3}}{b^{2/3}}\right)$$

$$\text{or.}\quad b^{2/3}\beta^{1/3}x+a^{2/3}\alpha^{1/3}y=\alpha^{1/3}\beta^{1/3}a^{2/3}b^{2/3}$$

$$\left[\text{as } (\alpha,\beta) \text{ is a point of the curve-(i) so } \frac{\alpha^{2/3}}{a^{2/3}}+\frac{\beta^{2/3}}{b^{2/3}}=1\right]$$

$$\text{or,}\quad \frac{x}{a^{2/3}\alpha^{1/3}}+\frac{y}{b^{2/3}\beta^{1/3}}=1.\qquad\left[\text{Dividing both sides by } \alpha^{1/3}\beta^{1/2}a^{1/3}b^{1/3}\right]$$

So, the lengths of the intercepts of the tangent on the axes of co-ordinates are $a^{2/3}\alpha^{1/3}$ and $b^{2/3}\beta^{1/3}$.

$$\therefore\quad x_1=a^{2/3}\alpha^{1/3},\quad y_1=b^{2/3}\beta^{1/3}.$$

$$\therefore\quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=\frac{a^{4/3}\alpha^{2/3}}{a^2}+\frac{b^{4/3}\beta^{2/3}}{b^2}=\frac{\alpha^{2/3}}{a^{2/3}}+\frac{\beta^{2/3}}{b^{2/3}}=1.$$

Example 28. Find the equations of tangents to the curve $y=\cos(x+y)$ $[-2\pi\leqslant x\leqslant 2\pi]$ which are parallel to the straight line $x+2y=0$. [I.I.T. 1984]

$$y=\cos(x+y)\quad\therefore\quad \frac{dy}{dx}=-\sin(x+y)\left(1+\frac{dy}{dx}\right)$$

$$\text{or,}\quad \frac{dy}{dx}\{1+\sin(x+y)\}=-\sin(x+y)$$

$$\text{or,}\quad \frac{dy}{dx}=-\frac{\sin(x+y)}{1+\sin(x+y)}.$$

We are to find equations of tangents parallel to the straight line $x+2y=0$ and its gradient is $-\frac{1}{2}$. So the gradient of each tangent parallel to the straight line $x+2y=0$ is also $-\frac{1}{2}$.

So if (x, y) be the point of contact then

$$\frac{-\sin(x+y)}{1+\sin(x+y)}=-\tfrac{1}{2}.$$

or, $2\sin(x+y)=1+\sin(x+y)$ or, $\sin(x+y)=1$.

$\therefore$ $\sin(x+y)=1$ and $\cos(x+y)=y$.

$\therefore$ $1+y^2=\sin^2(x+y)+\cos^2(x+y)=1$.

or, $y^2=0 \therefore y=0$. Again if $y=0$, then $\sin x=1$ and $\cos x=0$

[Purting $y=0$ in $\sin(x+y)=1$]

$$\therefore \quad x=\frac{\pi}{2}, -\frac{3\pi}{2} \quad [\because \; -2\pi \leqslant x \leqslant 2\pi]$$

Hence the co-ordinates of the points of contact are $\left(\frac{\pi}{2}, 0\right)$ and $\left(-\frac{3\pi}{2}, 0\right)$ and the equations of the tangents at these two points are $y=-\frac{1}{2}\left(x-\frac{\pi}{2}\right)$ and $y=-\frac{1}{2}\left(x+\frac{3\pi}{2}\right)$.

Example 29. Find the point of intersection of the tangents to the ellipse $3x^2+2y^2=35$ at its points of intersection with the straight line $x+2y=7$.

Let the point of intersection be (x_1, y_1). Then the equation of the chord of the ellipse joining the points of contact of the tangents drawn from (x_1, y_1) is $x+2y=7$. In otherwords, the equation of the chord of contact of the point (x_1, y_1) with respect to the ellipse $3x^2+2y^2=35$ is $x+2y=7$......(i). Again the equation of this chord of contact is $3xx_1+2yy_1=35$......(ii).

Hence the two equations-(i) and (ii) represent the same straight line. So, their corresponding coefficients are proportional.

$$\therefore \quad \frac{3x_1}{1}=\frac{2y_1}{2}=\frac{35}{7} \quad \text{or,} \quad x_1=\frac{5}{3},\ y_1=5.$$

Hence the co-ordinates of the point of contact are $(\frac{5}{3}, 5)$.

Example 30. Prove that the portion of a tangent to a parabola at any point on it, intercepted between the point of contact and the directrix subtends a right angle at the focus.

Let S be the focus of the parabola $y^2=4ax$. and $P(x_1, y_1)$ be any point on it. Let the tangent to the parabola at the point P cut the directrix at R. To prove that $\angle PSR$ is a right angle.

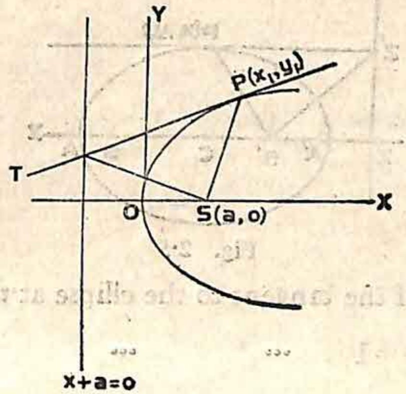


Fig. 2·4

Here the co-ordinates of S are $(a, 0)$. Now the equation of the directrix is $x=-a$······(i) and that of the tangent to the parabola at P is $yy_1=2a(x+x_1)$······(ii).

Putting $x=-a$ in equation (ii) we get $yy_1=2a(x_1-a)$ or, $y=\frac{2a}{y_1}(x_1-a)$. Hence the co-ordinates of the point of intersection R of the tangent and the directrix are $\{-a, \frac{2a}{y_1}(x_1-a)\}$.

Now the gradient of PS is $m_1=\frac{y_1}{x_1-a}$ and that of RS is

$$m_2=\frac{\frac{2a}{y_1}(x_1-a)-0}{-a-a}=-\left(\frac{x_1-a}{y_1}\right)$$

$$\therefore\ m_1m_2=\frac{y_1}{x_1-a}\left(-\frac{x_1-a}{y_1}\right)=-1.$$

Hence PS and PR are perpendicular to each other and $\angle PSR$ is a right angle.

Example 31. Prove that the portion of the tangent at any point of an ellipse intercepted between the point of contact and a directrix subtends a right angle at the corresponding focus.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and e be the eccentricity. So the co-ordinates of the focus S' are $(-ae, 0)$

and the equation of the corresponding direction is $x=-\frac{a}{e}$ ... (i)

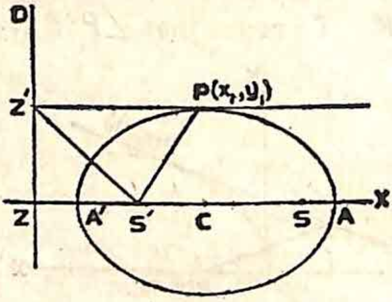


Fig. 2·5

The equation of the tangent to the ellipse at the point (x_1, y_1) of it is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ... ... (ii)

Let this tangent (ii) intersect the directrix (i) at the point Z'.

To prove that $\angle PS'Z'$ is a right angle. Putting $x=-\frac{a}{e}$ in the equation (ii) we get

$$-\frac{x_1}{ae}+\frac{yy_1}{b^2}=1 \quad \text{or,} \quad y=\frac{b^2}{y_1}\left(1+\frac{x_1}{ae}\right)=\frac{b^2}{y_1}\left(\frac{x_1+ae}{ae}\right)$$

So, the co-ordinates of Z' are $\left\{-\frac{a}{e}, \frac{b^2}{y_1}\left(\frac{x_1+ae}{ae}\right)\right\}$.

Now the gradient of the straight line $Z'S'$ is

$$m_1=\frac{\frac{b^2}{y_1}\left(\frac{x_1+ae}{ae}\right)-0}{-\frac{a}{e}+ae}=\frac{b^2\,(x_1+ae)\,.\,e}{y_1\,ae\,a(e^2-1)}$$

$$=\frac{b^2\,(x_1+ae)}{y_1a^2\,(e^2-1)}=-\frac{b^2\,(x_1+ae)}{y_1a^2\,(1-e^2)}.$$

$$=-\left(\frac{x_1+ae}{y_1}\right) \quad [\text{as } b^2=a^2\,(1-e^2)\,]$$

Again the gradient of the straight line PS' is $m_2=\frac{y_1}{x_1+ae}$

$$\therefore \quad m_1m_2=-\frac{(x_1+ae)}{y_1}\cdot\frac{y_1}{x_1+ae}=-1.$$

Hence the straight lines $Z'S'$ and PS' are perpendicular to each other i.e., $\angle PS'Z'$ is a right angle.

Example 32. Prove that the normal at any point of an ellipse bisects the angle between the focal distances of the point.

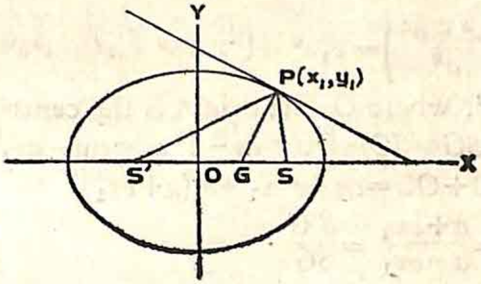


Fig. 2·6

Let the equation of a given ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its eccentricity be e. The co-ordinates of the two foci S and S' are $(ae, 0)$ and $(-ae, 0)$. Let $P\ (x_1, y_1)$ be any point of the ellipse.

Now $PS=\sqrt{(x_1-ae)^2+y_1^2}=\sqrt{(x_1-ae)^2+\frac{b^2}{a^2}(a^2-x_1^2)}$

[∵ (x_1, y_1) is a point of the ellipse, so

$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$ ∴ $y_1^2=\frac{b^2}{a^2}(a^2-x_1^2)$]

$$=\sqrt{(x_1-ae)^2+\frac{a^2(1-e^2)}{a^2}(a^2-x_1^2)} \quad [\because\ b^2=a^2(1-e^2)]$$

$$=\sqrt{(x_1-ae)^2+(1-e^2)(a^2-x_1^2)}$$

$$=\sqrt{x_1^2-2aex_1+a^2e^2+a^2-a^2e^2-x_1^2+x_1^2e^2}$$

$$=\sqrt{a^2-2aex_1+x_1^2e^2}=\sqrt{(a-ex_1)^2}=a-ex_1.$$

Similarly $PS'=a+ex_1$

The equation of the normal to the ellipse at the point $P\ (x_1, y_1)$ is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}} \quad \dots \quad \dots \quad \text{(i)}$$

Let the normal intersects the major axis of the ellipse whose equation is $y=0$ at G.

Putting $y=0$ in equation (i) we get

$$\frac{x-x_1}{\frac{x_1}{a^2}}=-b^2 \quad \therefore \quad x-x_1=-\frac{b^2}{a^2}x_1 \quad \text{or,} \quad x=x_1\left(1-\frac{b^2}{a^2}\right)$$

$$\text{or,} \quad x=x_1\left(\frac{a^2-b^2}{a^2}\right)=x_1e^2 \quad [\because \; b^2=a^2-a^2e^2]$$

$\therefore$ $OG=x_1e^2$, where O the origin is the centre of the ellipse.
From Figure $SG=SO-OG=ae-e^2x_1=e(a-ex_1)$
and $S'G=S'O+OG=ae+e^2x_1=e\,(a+ex_1)$

$$\therefore \quad \frac{PS'}{PS}=\frac{a+ex_1}{a-ex_1}=\frac{S'G}{SG}.$$

So, PG is the bisector of $\angle S'PS$.

Example 33. A is a point on the parabola $y^2=4ax$. The normal at A cuts the parabola again at B. If AB subtends a right angle at the vertex of the parabola, find the slope of AB.

[I. I. T. 1982]

Let the co-ordinates of the point A of the parabola $y^2=4ax$ be $(at_1^2, 2at_1)$ and the normal to the parabola drawn at A again meets the curve at the point $B\ (at_2^2, 2at_2)$.

The equation of the normal to the parabola at $A\ (at_1^2, 2at_1)$ is

$$y+t_1x=2at_1+at_1^3 \quad \cdots \quad \cdots \quad \text{(i)}$$

As $B\ (at_2^2, 2at_2)$ is a point on this line,

$$\text{so,} \quad 2at_2+at_1t_2^2=2at_1+at_1^3$$

$$\text{or,} \quad 2a\,(t_2-t_1)+at_1(t_2^2-t_1^2)=0.$$

or, $2+t_1(t_2+t_1)=0$ [as A and B are two different points so $t_1\neq t_2$ i.e., $t_2-t_1\neq 0$. Also $a\neq 0$]

$$\text{or,} \quad 2+t_1^2=-t_1t_2 \quad \therefore \quad t_2=-\left(\frac{2+t_1^2}{t_1}\right)$$

Again the vertex of the parabola is the origin O. So, the gradients of OA and OB are $\frac{2at_1}{at_1^2}=\frac{2}{t_1}$ and $\frac{2at_2}{at_2^2}=\frac{2}{t_2}$. According to the given condition, $\angle AOB$ is a right angle.

$$\therefore \quad \frac{2}{t_1}\cdot\frac{2}{t_2}=-1 \quad \text{or,} \quad t_1t_2=-4 \quad \text{or,} \quad t_2=-\frac{4}{t_1}$$

$$\therefore \quad -\frac{4}{t_1}=t_2=-\left(\frac{2+t_1^2}{t_1}\right)$$

$$\text{or,} \quad 4=2+t_1^2 \quad \therefore \quad t_1^2=2 \quad \text{or,} \quad t_1=\pm\sqrt{2}$$

Hence the gradient of AB is $t_1=\pm\sqrt{2}$.

Example 34. A tangent to a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intercepts length of unity upon each of the co-ordinate axes. Show that the point (a, b) lies on the rectangular hyperbola $x^2-y^2=1$.

[Joint Entrance 1980]

The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at any point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ or, $\frac{x}{\frac{a^2}{x_1}}-\frac{y}{\frac{b^2}{y_1}}=1$. So, the intercepts of the tangent on the axes of co-ordinates are $\frac{a^2}{x_1}$ and $\frac{b^2}{y_1}$.

So by question $\frac{a^2}{x_1}=\frac{b^2}{y_1}=1 \quad \therefore \quad x_1=a^2, \; y_1=b^2$.

Now, (x_1, y_1) is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

So, $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$ or, $\frac{a^4}{a^2}-\frac{b^4}{b^2}=1$ or, $a^2-b^2=1$. i.e., the co-ordinates of the point $(a. b)$ satisfies the equation $x^2-y^2=1$ or in otherwords the locus of the point (a, b) is the rectangular hyperbola $x^2-y^2=1$.

Example 35. Show that the product of the perpendiculars drawn from the foci on any tangent to an ellipse is constant.

Let us take the ellipse as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let (x_1, y_1) be any point of the ellipse. The equation of the tangent to this ellipse at this point is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \cdots \quad \cdots \quad$ (i).

The lengths of the perpendiculars from the foci $(ae, 0)$ and $(-ae, 0)$ of the ellipse on these tangents are

$$p_1=\frac{\frac{aex_1}{a^2}-1}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}}=\frac{\frac{ex_1}{a}-1}{\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^4}}} \text{ and}$$

$$p_2=\frac{\frac{-aex_1}{a^2}-1}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}}=\frac{\frac{-ex_1}{a}-1}{\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^4}}}$$

Hence the point (h, k) of intersection of the tangents lie on the directrix $x=\frac{a}{e}$ i.e., the tangents intersect on the directrix corresponding to the focus $(ae, 0)$.

Example 37. Find the points on the curve $ax^2+2hxy+by^2=1$ at which the tangents are perpendiculars to the x-axis. [C. U.]

The equation of the curve is $ax^2+2hxy+by^2=1$. Differentiating both sides of the equation with respect to x we get, $2ax+2h\left(y+x\frac{dy}{dx}\right)+2by\frac{dy}{dx}=0$. or, $\frac{dy}{dx}=-\frac{ax+hy}{hx+by}$.

If the tangent to the curve at the point (x, y) be perpendicular to the x-axis, then its gradient $\frac{dy}{dx}=-\frac{ax+hy}{hx+by}$ will be undefined.

So $hx+by$ will be 0. So, the point (x, y) lies on the straight line $hx+by=0$ i.e., it is the point of intersection of the curve with the straight line $hx+by=0$.

Example 38. Prove that the points other than the origin, at which the tangents to the curve $x^3+y^3=3axy$ are parallel to the y-axis all lie on the parabola $y^2=ax$.

The equation of the curve is $x^3+y^3=3axy$. Differentiating both sides of the equation with respect to x

we get $3x^2+3y^2\frac{dy}{dx}=3a\left(y+x\frac{dy}{dx}\right)$ or, $\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$

If the tangent to the curve at the point (x, y) be parallel to the y-axis, then the gradient $\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$ of the tangent at the point is undefined. So, $y^2-ax=0$ or, $y^2=ax$. Hence the point (x, y) lies on the porabola $y^2=ax$.

Example 39. Tangents are drawn from the origin to the curve $y=\sin x$. Show that the points of contact lie on the curve $x^2y^2=x^2-y^2$.

$y=\sin x \quad \therefore \quad \frac{dy}{dx}=\cos x.$

So the equation of the tangent to the curve at a point (α, β) of it is $y-\beta=\left[\frac{dy}{dx}\right]_{(\alpha, \beta)}(x-\alpha)$ or, $y-\beta=\cos\alpha\,(x-\alpha)$.

$$\therefore\ p_1 . p_2 = \frac{\frac{ex_1}{a}-1}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}} \cdot \frac{\frac{-ex_1}{a}-1}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}} = \frac{(-1)^2-\frac{e^2x_1^2}{a^2}}{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}$$

$$= \frac{1-\frac{e^2x_1^2}{a^2}}{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}} = \frac{1-e^2\left(1-\frac{y_1^2}{b^2}\right)}{\frac{1}{a^2}\left(1-\frac{y_1^2}{b^2}\right)+\frac{y_1^2}{b^4}}$$

$\Big[\because$ (x_1, y_1) is a point of the ellipse, so $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

or, $\frac{x_1^2}{a^2}=1-\frac{y_1^2}{b^2}\Big]$

$$= \frac{\frac{b^2-b^2e^2+e^2y_1^2}{b^2}}{\frac{b^2(b^2-y_1^2)+y_1^2a^2}{a^2b^4}} = a^2b^2\frac{b^2(1-e^2)+e^2y_1^2}{b^4-b^2y_1^2+a^2y_1^2}$$

$$= b^2\ \frac{a^2(1-e^2)b^2+a^2e^2y_1^2}{b^4+y_1^2\ (a^2-b^2)} = b^2\ \frac{b^2b^2+a^2e^2y_1^2}{b^4+y_1^2a^2e^2}$$

$[\because\ b^2=a^2(1-e^2)$, so $a^2-b^2=a^2e^2\ ] = b^2\ \frac{b^4+a^2e^2y_1^2}{b^4+a^2e^2y_1^2}=b^2$.

which is a constant.

Example 36. Prove that the tangents drawn to an ellipse at the extremities of a focal chord of an ellipse intersect on the corresponding directrix.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then $(ae, 0)$ are the co-ordinates of a focus of the ellipse and $x=\frac{a}{e}$ is the equation of the directrix corresponding to this focus. Let the tangents drawn at the extremities of a chord of the ellipse passing through this focus intersect at the point (h, k). Hence the chord is the chord of contact of the point (h, k) with respect to the ellipse. Therefore the equation of this focal chord is $\frac{xh}{a^2}+\frac{yk}{b^2}=1$. Again this chord passes through the focus $(ae, 0)$. So $\frac{hae}{a^2}+\frac{0\ .\ k}{b^2}=1$ or $h=\frac{a}{e}$.

If this tangent is drawn from the origin, that is if it passes through the origin, then $-\beta=-\alpha\cos\alpha$.

[Putting $x=0, y=0$ in the equation of the tangent]

$\therefore\ \cos\alpha=\dfrac{\beta}{\alpha}$. Also $\sin\alpha=\beta$ as (α, β) is a point on the curve $y=\sin x$.

$$\therefore\ \cos^2\alpha+\sin^2\alpha=\frac{\beta^2}{\alpha^2}+\beta^2, \text{ or, } 1=\frac{\beta^2+\alpha^2\beta^2}{\alpha^2}$$

$$\text{or, }\ \alpha^2=\beta^2+\alpha^2\beta^2 \quad \text{or, }\ \alpha^2-\beta^2=\alpha^2\beta^2.$$

Hence the equation of the locus of the point of contact (α, β) is $x^2-y^2=x^2y^2$.

So, the points of contact lie on the curve $x^2-y^2=x^2y^2$.

Example 40. Prove that the points of contact of tangents to the curve $y^2=4a\left\{x+a\sin\left(\dfrac{x}{a}\right)\right\}$ which are parallel to the x-axis all lie on a parobola.

The equation of the curve is $y^2=4a\left\{x+a\sin\left(\dfrac{x}{a}\right)\right\}$ ......(i)

Differentiating both sides with respect to x we get

$$2y\frac{dy}{dy}=4a\left\{1+a.\frac{1}{a}\cos\left(\frac{x}{a}\right)\right\}=4a\left\{1+\cos\left(\frac{x}{a}\right)\right\}$$

So, if the tangent to the curve (i) which is parallel to the x-axis will have its gradient

$$\frac{dy}{dx}=\frac{4a}{2y}\left\{1+\cos\left(\frac{x}{a}\right)\right\}=0$$

$$\text{or, }\ 1+\cos\left(\frac{x}{a}\right)=0 \quad \text{or, }\ \cos\left(\frac{x}{a}\right)=-1.$$

So the value of $\sin\left(\dfrac{x}{a}\right)$ at the point is

$$\sqrt{1-\cos^2\left(\frac{x}{a}\right)}=\sqrt{1-1}=0.$$

Again the point, (x, y) being a point on the curve (i),

$$y^2=4a\left\{x+\frac{1}{a}\sin\left(\frac{x}{a}\right)\right\} \text{ or, } y^2=4ax \quad \left[\text{as } \sin\left(\frac{x}{a}\right)=0\right]$$

Hence the points of contact all lie on the parabola $y^2=4ax$.

Example 41. Show that the curves $xy=12$ and $yx^2=36$ intersect at the point (3, 4) and find the angle of intersection of the curve.

Each of the equations $xy=12 \cdots\cdots$(i) and $yx^2=36 \cdots\cdots$(ii) are satisfied by $x=3$, $y=4$.

Hence the curves represented by the equations (i) and (ii) intersect at the point (3, 4).

Differentiating both sides of equation (i) with respect to x we get, $y+x\dfrac{dy}{dx}=0$ or, $\dfrac{dy}{dx}=-\dfrac{y}{x}$.

Hence the gradient of the tangent to the curve (i) at the point (3, 4) is $m_1=\left[\dfrac{dy}{dx}\right]_{(3,\ 4)}=-\dfrac{4}{3}$.

Again differentiating both sides of equation (ii) with respect to x we get $2xy+x^2\dfrac{dy}{dx}=0$. $\therefore$ $\dfrac{dy}{dx}=-\dfrac{2y}{x}$.

Hence the gradient of the tangent to the curve (ii) at the point (3, 4) is $m_2=-\dfrac{2.4}{3}=-\dfrac{8}{3}$.

Hence the angle between these tangents is $\tan^{-1}\dfrac{m_1-m_2}{1+m_1m_2}$

$$=\tan^{-1}\frac{-\frac{4}{3}-(-\frac{8}{3})}{1+(-\frac{4}{3})(-\frac{8}{3})}=\tan^{-1}\frac{12}{41}.$$

Hence the angle θ of intersection of the curve is given by $\tan\theta=\frac{12}{41}$.

Example 42. Find the angle of intersection of the parabola $y^2=2x$ and the circle $x^2+y^2=8$.

Putting $2x$ in place of y^2 from equation $y^2=2x \cdots\cdots$(i) in the equation $x^2+y^2=8 \cdots\cdots$(ii), we find $x^2+2x=8$ or, $x^2+2x-8=0 \cdots\cdots$(iii). or, $(x+4)(x-2)=0$ $\therefore$ $x=-4, 2$

So, $y^2=2\times-4=-8$ or, $y^2=2\times2=4$.

But $y^2\neq-8$. $\therefore$ $y^2=4$ or, $y=\pm2$.

So, the curves intersect at the points (2, 2) and (2, -2)

Differentiating both sides of the equation $y^2=2x$ with respect to x we get $2y\dfrac{dy}{dx}=2$ or, $\dfrac{dy}{dx}=\dfrac{1}{y}$.

So the gradients of the tangents to the curve $y^2=2x$ at the point (2, 2) and (2, −2) are

$$m=\left[\frac{dy}{dx}\right]_{(2,\ 2)}=\tfrac{1}{2} \text{ and } m_1=\left[\frac{dy}{dx}\right]_{(2,\ -2)}=-\tfrac{1}{2}.$$

Again, differentiating both sides of the equation $x^2+y^2=8$ with respect to x we get

$$2x+2y\frac{dy}{dx}=0 \quad \text{or,} \quad \frac{dy}{dx}=-\frac{x}{y}$$

Hence the gradients of the tangents to the curve $x^2+y^2=8$ at the point (2, 2) and (−2, 2) are

$$m'=\left(\frac{dy}{dx}\right)_{(2,\ 2)}=-1 \text{ and } m_2=\left(\frac{dy}{dx}\right)_{(2,\ -2)}=1.$$

Hence the angle between the curves at the point of intersection (2, 2) is

$$\tan^{-1}\frac{m-m'}{1+mm'}=\tan^{-1}\frac{\frac{1}{2}-(-1)}{1+\frac{1}{2}.(-1)}=\tan^{-1}\frac{\frac{3}{2}}{\frac{1}{2}}=\tan^{-1}(3)$$

Again the angle between the curves at the point of intersection (2, −2) is

$$\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}=\tan^{-1}\frac{-\frac{1}{2}-1}{1+\left(-\frac{1}{2}\right).1}=\tan^{-1}\frac{-\frac{3}{2}}{\frac{1}{2}}$$

$$=\tan^{-1}(-3)$$

Example 43. Prove that if the curves $ax^2+by^2=1$ and $a_1x^2+b_1y^2=1$ intersect orthogonally then

$$\frac{1}{a}-\frac{1}{a_1}=\frac{1}{b}-\frac{1}{b_1}.$$

[To intersect orthogonally means intersect at right angles]

Let the curves intersect at the point (α, β). Differentiating both sides of the equation $ax^2+by^2=1$ with respect to x we get

$$2ax+2by\frac{dy}{dx}=0 \quad \text{or,} \quad \frac{dy}{dx}=-\frac{ax}{by}.$$

So the gradient of the tangent to the curve at the point (α, β) is $m_1=\left(\dfrac{dy}{dx}\right)_{(\alpha,\ \beta)}=-\dfrac{a\alpha}{b\beta}$

Similarly the gradient of the tangent to the curve $a_1x^2+b_1y^2=1$ at the point (α, β) is $m_2=-\dfrac{a_1\alpha}{b_1\beta}$.

Now if the curves cut orthogonally, then the tangents at the point of intersection (α, β) are perpendicular to each other.

$$\therefore\ m_1m_2=-1 \text{ or, } \left(-\frac{a\alpha}{b\beta}\right)\left(-\frac{a_1\alpha}{b_1\beta}\right)=-1$$

$$\text{or, } \frac{aa_1\alpha^2}{bb_1\beta^2}=-1 \text{ or, } \frac{\alpha^2}{\beta^2}=-\frac{bb_1}{aa_1} \quad \cdots\cdots (1)$$

Again (α, β) is a point on both the curves,

$$\therefore\ a\alpha^2+b\beta^2=1 \cdots \cdots (2) \text{ and } a_1\alpha^2+b_1\beta^2=1 \cdots \cdots (3)$$

Subtracting equation (3) from equation (2)

we get $(a-a_1)\alpha^2+(b-b_1)\beta^2=0$

$$\text{or, } \frac{\alpha^2}{\beta^2}=-\frac{b-b_1}{a-a_1} \quad \therefore -\frac{bb_1}{aa_1}=-\frac{b-b_1}{a-a_1} \text{ [From (1)].}$$

$$\text{or, } \frac{a-a_1}{aa_1}=\frac{b-b_1}{bb_1} \text{ or, } \frac{1}{a_1}-\frac{1}{a}=\frac{1}{b_1}-\frac{1}{b}$$

$$\text{or, } \frac{1}{a}-\frac{1}{a_1}=\frac{1}{b}-\frac{1}{b_1}.$$

Example 44. If the chord of contact of the tangents drawn from a point P to an ellipse passes through the point Q, then the chord of contact of the point Q with respect to the ellipse will pass through P.

Let the co-ordinates of the points P and Q be (x_1, y_1) and (x_2, y_2) and the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Then the equation of the chord of contact of the tangents to the ellipse drawn from the point P is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$. As this chord of contact passes through the point Q,

$$\text{So, } \frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}=1. \quad \cdots \quad \cdots (1)$$

Again the chord of contact of the tangents to the ellipse drawn from Q is the straight line $\dfrac{xx_2}{a^2}+\dfrac{yy_2}{b^2}=1$, $\quad \cdots \quad \cdots (2)$

Equation (1) shows that this chord of contact (2) passes through the point P.

Example 45.. The points of contact of the tangents to the circle $x^2+y^2=a^2$ drawn from a point $P(\alpha, \beta)$ are Q and R. Prove that the area of the triangle PQR is $\dfrac{a(\alpha^2+\beta^2-a^2)^{3/2}}{\alpha^2+\beta^2}$.

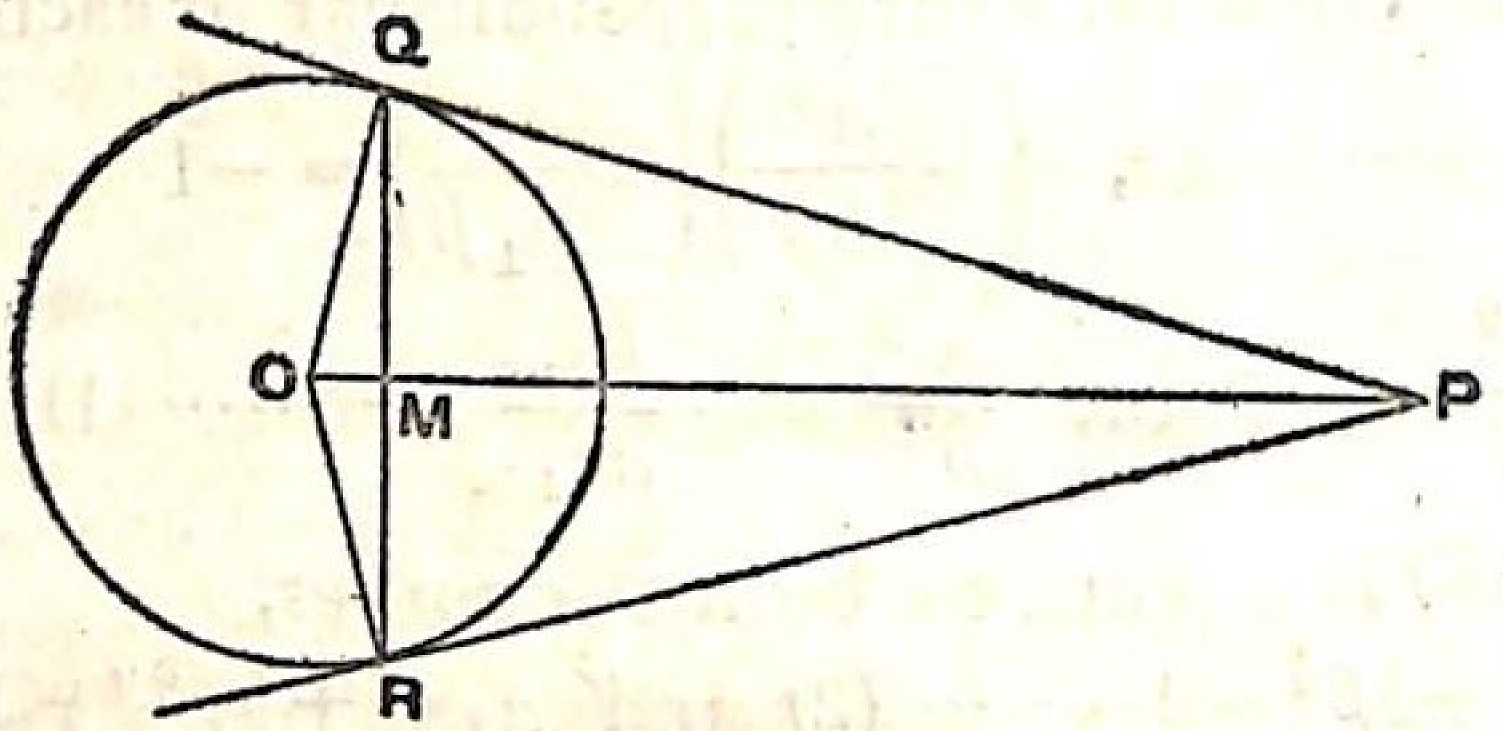


Fig. 2·7

Clearly, QR is the chord of contact of the point $P(\alpha, \beta)$ with respect to the circle $x^2+y^2=a^2$.

Hence the equation of the straight line QR is $x\alpha+y\beta=a^2$.

So the length of the perpendicular PM drawn from P on QR is $\dfrac{{}^2+\beta^2-a^2}{\sqrt{\alpha^2+\beta^2}}$

Again PQ is the length of the tangent from P to the circle. So length $PQ=(\alpha^2+\beta^2-a^2)^{\frac{1}{2}}$.

$$\therefore \quad MQ^2=PQ^2-PM^2=(\alpha^2+\beta^2-a^2)-\frac{(\alpha^2+\beta^2-a^2)^2}{\alpha^2+\beta^2}.$$

Again $QM=\frac{1}{2}QR$. $\therefore$ $QR=2QM$

$\therefore$ Area of $\Delta PQR=\frac{1}{2}QR.PM=\frac{1}{2}.2QM.PM.$

$$=\left\{(\alpha^2+\beta^2-a^2)-\frac{(\alpha^2+\beta^2-a^2)^2}{\alpha^2+\beta^2}\right\}^{1/2}\frac{(\alpha^2+\beta^2-a^2)}{\sqrt{\alpha^2+\beta^2}}$$

$$=\left\{(\alpha^2+\beta^2-a^2)\left(1-\frac{\alpha^2+\beta^2-a^2}{\alpha^2+\beta^2}\right)\right\}^{1/2}\frac{(\alpha^2+\beta^2-a^2)}{\sqrt{\alpha^2+\beta^2}}$$

$$=(\alpha^2+\beta^2-a^2)^{1/2}.\frac{a}{\sqrt{(\alpha^2+\beta^2)}}.\frac{\alpha^2+\beta^2-a^2}{\sqrt{(\alpha^2+\beta^2)}}$$

$$=\frac{(\alpha^2+\beta^2-a^2)^{3/2}.a}{(\alpha^2+\beta^2)} \text{ square units.}$$

Example 46. Show that at any point of a parabola the subnormal is of constant length and the subtangent is proportional to the abscissa of the point of contact.

The equation of any parabola can be taken as $y^2=4ax$.

From $y^2=4ax$ we get $2y\dfrac{dy}{dx}=4a$ or, $\dfrac{dy}{dx}=\dfrac{2a}{y}$.

Hence the length of the subnormal at any point (x, y) is $y\dfrac{dy}{dx}=y\,.\,\dfrac{2a}{y}=2a$ which is constant.

Again the length of the subtangent of the parabola at any point (x, y) is $\dfrac{y}{\frac{dy}{dx}}=\dfrac{y}{\frac{2a}{y}}=\dfrac{y^2}{2a}=\dfrac{4ax}{2a}=2ax$

which is proportional to x (as $2a$ is a constant), i.e. the abscissaa of the point.

Example 47. Prove that at any point of any curve

$$\frac{\text{length of subnormal}}{\text{length of subtangent}}=\left(\frac{\text{length of normal}}{\text{length of tangent}}\right)^2.$$

Let the curve be $y=f(x)$ and (x, y) be any point of the curve. Lengths of tangent, normal, sub-tangent and sub-normal of the curve at this point are $\dfrac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}$, $y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$; $\dfrac{y}{\frac{dy}{dx}}$ $y.\dfrac{dy}{dx}$ respectively.

$$\therefore\ \frac{\text{length of sub-normal}}{\text{length of sub-tangent}}=\frac{y\frac{dy}{dx}}{\frac{y}{\frac{dy}{dx}}}=\left(\frac{dy}{dx}\right)^2$$

$$\text{Also } \frac{\text{length of normal}}{\text{length of tangent}}=\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}}=\frac{dy}{dx}$$

$$\therefore\ \frac{\text{length of sub-normal}}{\text{length of sub-tangent}}=\left(\frac{\text{length of normal}}{\text{length of tangent}}\right)^2$$

Example 48. Prove that the length of the tangent to the curve $x=a(\cos t+\log\tan\frac{1}{2}t)$, $y=a\sin t$ intercepted between the point of contact and the x-axis is constant.

$$x=a(\cos t+\log\tan\tfrac{1}{2}t)\ \therefore\ \frac{dx}{dt}=a\left(-\sin t+\frac{1}{\tan\frac{1}{2}t}\cdot\frac{1}{2}\sec^2\tfrac{1}{2}t\right)$$

$$=a\left(-\sin t+\frac{1}{\dfrac{\sin\frac{t}{2}t}{\cos\frac{t}{2}t}}\cdot\frac{1}{2\cos^2 t}\right)=a\left(-\sin t+\frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$=a\left(-\sin t+\frac{1}{\sin t}\right)=a\left(\frac{1-\sin^2 t}{\sin t}\right)=a\ \frac{\cos^2 t}{\sin t}$$

Also $\dfrac{dy}{dt}=\dfrac{d}{dt}(a\sin t)=a\cos t.$

$$\therefore\ \frac{dy}{dx}=\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}=\frac{a\cos t}{\dfrac{a\cos^2 t}{\sin t}}=\tan t.$$

Now the length of the tangent to the curve at (x, y) intercepted between the point of contact and the x-axis is

$$\frac{y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{\dfrac{dy}{dx}}=\frac{y\sqrt{1+\tan^2 t}}{\tan t}=\frac{a\sin t\,.\sec t}{\dfrac{\sin t}{\cos t}}=a$$

which is constant.

EXERCISE 2

1. Write the equations of the tangents to

(i) The circle $x^2+y^2=5$ at the point $(1, 2)$;

(ii) The circle $x^2+y^2+4x+6y-87=0$ at the point $(4, 5)$;

(iii) The parabola $x^2=-12y$ at the point $(6, -3)$;

(iv) The parabola $y^2=6x$ at the point whose ordinate is 12 ;

(v) To the parabola $y^2=4a(x-a)$ at the two extremities of the latus-rectum ;

(vi) The ellipse $9x^2+16y^2=144$ at the two extremities of a latus rectum ;

(vii) The ellipse $4x^2+9y^2=72$ at the point $(3, 2)$;

(viii) The hyperbola $3x^2-4y^2=8$ at the point $(2, 1)$;

(ix) The curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$ at the point (x_1, y_1);

(x) The curve $x=a(t+\sin t)$, $y=a(1-\cos t)$ at the point 't';

(xi) At the point (x_1, y_1) of the curve $ax^2+2hxy+by^2=1$.

(xii) At the point (x_1, y_1) of the curve $y=a \log \sin x$.

2. Determine the equations of the normals to the following curves at the specified points:

(i) The parabola $y^2+12x=0$ at the point $(-3, 6)$;

(ii) The parabola $y^2=12x$ at the two extremities of the latus-rectum.

(iii) The ellipse $7x^2+8y^2=36$ at the points whose abscissa is 2.

(iv) The ellipse $3x^2+4y^2=12$ at the extremity of the latus-rectum situated in the first quadrant.

(v) The circle $x^2+y^2+4x+6y=87$ at the point $(6, 3)$.

(vi) The curve $x=a\cos^3 t$, $y=a\sin^3 t$ at the point 't'.

(vii) The curve $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$ at the point (x_1, y_1).

3. Determine the equations of the tangent and normal to the curve $y(x-2)(x-3)-x+7=0$ at the point where it meets the x axis.

4. Prove that the tangent to a circle is perpendicular to the radius through the point of contact.

5. Find the equations of the tangents to the circle $x^2+y^2=20$ which are parallel to the straight line $x+2y+5=0$.

6. Find the equations of those tangents to the circle $x^2+y^2-3x+10y-15=0$ which are perpendicular to the straight line $12x+5y=1$.

7. Prove that the straight line $4x-2y+3=0$ touches the parabola $y^2=12x$ and find the point of contact.

8. If the straight line $y=3x+5$ touches the parabola $y^2=8ax$, find the co-ordinates of the point of contact and also of the focus.

9. The straight line $y=mx+c$ touches the parabola $y^2=12x$ and is parallel to the straight line $5y+3x+25=0$. Find the values of m and c.

10. Find the equation of the tangent to the parabola $y^2=4x$ which is parallel to the straight line $x+2y=3$.

11. Show that the straight line $y=x+2$ touches the circle $x^2+y^2=2$ and find the point of contact.

12. For which value of k, the straight line $y=kx+13$ touches the circle $x^2+y^2=144$.

13. Find those tangents to the circle $x^2+y^2=9$ which are parallel to the straight line $3x+4y=0$.

14. Find the equations of the tangents to the circle $x^2+y^2=25$ which are perpendicular to the straight line $4x-3y=12$.

15. Find those tangents to the circle $x^2+y^2=25$,

(i) which are parallel to the straight line $3x+4y=0$.

(ii) which passes through the point $(13, 0)$.

16. Find those tangents to the circle $x^2+y^2-6x+4y=12$ which are parallel to the straight line $4x+3y+5=0$.

17. Find the equations of those tangents to the circle $x^2+y^2-6x+4y=7$ which are perpendicular to the straight line $2x-y+3=0$.

18. Prove that the straight line $x+y=2+\sqrt{2}$ touches the circle $x^2+y^2-2x-2y+1=0$ and also find the point of contact.

19. For which value of c, the straight line $y=mx+c$ touches the circle $x^2+y^2=4y$ for all values of m.

20. Find the equations of the tangents to the circle $x^2+y^2=a^2$ which are (i) parallel to the straight line $y=mx+c$. (ii) perpendicular to the straight line $y=mx+c$. (iii) makes a triangle of area a^2 with the axes of co-ordinates.

21. The extremities of a diameter of a circle are the points $(1, 2)$ and $(3, 4)$. Find the equation of the circle and the equations of the tangents to this circle which are parallel to this diameter.

22. Prove that the tangents to the circle $x^2+y^2=169$ drawn at the points $(5, 12)$ and $(12, -5)$ are perpendicular to each other.

23. Show that if the tangent to the circle $x^2+y^2=a^2$ drawn at the point (x_1, y_1) passes through the centre of the circle $x^2+y^2=2a(x+y)$, then $x_1=a, y_1=0$ or, $x_1=0, y_1=a$.

24. If the tangents to the circle $x^2+y^2+2gx+2fy+c=0$ at the points (x_1, y_1) and (x_2, y_2) be perpendicular, show that $x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+g^2+f^2=0$.

25. Prove that

(i) the straight line $y=2x+1$ touches the parabola $y^2=8x$.

(ii) the straight line $8y=16x+3$ touches the parabola $y^2=3x$.

(iii) the straight line $3y=x+3$ touches the parabola $3y^2=4x$.

Find the point of contact in each case.

26. Find the equation of the tangent to the parabola $y^2=7x$ which is parallel to the straight line $4y-x-5=0$. Find the point of contact.

27. Find the equation of the tangent to the parabola $y^2=8x$, perpendicular to the straight line $2x-3y=6$. Find the co-ordinates of the point of contact.

28. If the straight line $y=3x+1$ touches the parabola $y^2=4ax$, find the length of the latus rectum.

29. Prove that the straight line $x+my+am^2=0$ is a tangent of the parabola $y^2=4ax$. Find the co-ordinates of the point of contact.

30. Show that the straight line $y=x+5$ touches the ellipse $9x^2+16y^2=144$ and determine the co-ordinates of the point of contact.

31. Prove that the straight line $y=x+\sqrt{\frac{7}{12}}$ touches the ellipse $3x^2+4y^2=1$. Also find the co-ordinates of the point of contact.

32. Prove that the straight line $y=x+\sqrt{\frac{5}{6}}$ is a tangent to the ellipse $2x^2+3y^2=1$.

33. Find the values of m for which the straight line $3y=mx+7$ touches the ellipse $2x^2+3y^2=14$. In those cases find the points of contact.

34. Find the equations of the tangents to the ellipse $4x^2+3y^2=5$ which are parallel to the straight line $y=3x+4$.

35. Find equations of the tangents to the ellipse $x^2+9y^2=2$ which are perpendiculur to the straight line $3x+y=2$. Also find the co-ordinates of the points of contact.

36. Find the equation of the tangent to the ellipse $9x^2+16y^2$

$=144$ which intercept two equal and positive segments from the axes of co-ordinates. [H. S. '61]

37. (i) Find the equations of the tangents of the hyperbola $4x^2-9y^2=1$, which are parallel to the straight line $4y=5x+3$.

(ii) Find the equations of the tangents of the hyperbola $x^2-5y^2=40$, that are perpendicular to the straight line $2x-y+3=0$.

38. Show that the straight line $y=2x+3$ touches the hyperbola $7x^2-4y^2=28$ and also find the co-ordinates of the point of contact.

39. Show that the straight line $3y=4x+11$ touches the ellipse $2x^2+3y^2=11$ and find the co-ordinates of the point of contact.

40. Find the equations of the two tangents of the circle $x^2+y^2=3$ which make angles of 60° with the x-axis.

41. Prove that the straight line $2x+4y=9$ is a normal to the parabola $y^2=8x$. Find the foot of the normal.

42. Find the equation of the normal to the parabola $y^2=4x$ which is parallel to the straight line $y=2x$.

43. Find the normal to the parabola $y^2=3x$ which is parallel to the straight line $y=2x+1$. Find also the foot of the normal.

44. Show that the straight line $3x+4y-10=0$ is a normal to the hyperbola $2x^2-3y^2=5$. Also find the foot of the normal.

45. Find the equation of the normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$. Show that if this normal intersects the parabola again at the point $(at_1^2, 2at_1)$, then $t_1=-t-\dfrac{2}{t}$.

46. Find the point of the parabola $y^2=4ax$ at which the normal to the parabola is inclined to the axis of the parabola at an angle 30°.

47. A tangent to the parabola $y^2=12x$ is inclined to the x-axis at an angle 60°. Find the equation of the tangent and also the co-ordinates of the point of contact.

48. Find the co-ordinates of the point of the parabola $y^2=8x$ at which the normal is inclined at an angle of 60° with the axis.

49. Find the equations of the two tangents to the parabola

$y^2=8x$ which pass through the point $(-2, \frac{16}{3})$. Find the angle between the tangents.

50. Prove that the tangent to the parabola $y^2=4ax$ at an extremity of a focal chord of the parabola is parallel to the normal at the other extremity of the chord.

51. A tangent to the parabola $y^2=8x$ makes an angle 45° with the straight line $y=3x+5$. Find its equation and also the co-ordinates of the point of contact.

52. Find the co-ordinates of the point of the parabola $y^2=4ax$ at which the tangent is inclined at an angle 30° with the axis.

53. Show that the straight line $4a(y-b)=x$ touches the parabola $ay^2=bx$.

54. The tangents to the parabola $y^2=4ax$ at the points (x_1, y_1) and (x_2, y_2) are perpendicular to each other. Show that

(i) $x_1x_2=a^2$ (iii) $y_1y_2=-4a^2$ (iii) $4x_1x_2+y_1y_2=0$

55. If the tangents to the parabola $y^2=4ax$ at the points (α_1, β_1) and (α_2, β_2) intersect at the point (x_1, y_1), then show that $x_1=\frac{\beta_1\beta_2}{4a}$.

56. P and P' are two points of the parabola $y^2=4ax$ so that the tangent at P to the parabola is parallel to the normal at P'. Show that the chord PP' passes through the focus of the parabola.

57. Prove that a circle drawn on a focal chord of a parabola as diameter touches the directrix of the parabola.

58. If a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts off intercepts of lengths h and k from the axes of co-ordinates, show that $\frac{a^2}{h^2}+\frac{b^2}{k^2}=1$.

59. Show that the tangent drawn at any point of the curve $x=a(t+\sin t \cos t)$, $y=a(1+\sin t)^2$ makes an angle $\frac{\pi}{4}+\frac{t}{2}$ with the x-axis.

60. Show that the length of the portion of the tangent to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ at any point of it, intercepted between the axes of co-ordinates is constant.

61. Show that the equation of the tangent to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, inclined at an angle ϕ with the x-axis is

$$y\cos\phi-x\sin\phi=a\cos 2\phi.$$

62. Prove that the tangent and normal to the curve

$$x=ae^{\theta}(\sin\theta-\cos\theta),\quad y=ae^{\theta}(\sin\theta+\cos\theta)$$

drawn at any point of the curve are equidistant from the origin.

63. Prove that the distance from the origin of the normal drawn at any point of the curve.

$x=a\cos\theta+a\theta\sin\theta$, $y=a\sin\theta-a\theta\cos\theta$ is constant.

64. Prove that the equation of the tangent to the curve $y^2=x^3$ at the point $(4m^2, 8m^3)$ is $y=3mx-4m^3$ and it meets the curve again at the point $(m^2, -m^3)$. Also show that if it is also normal to the curve, then $9m^2=2$.

65. Find the condition that the straight line $lx+my+n=0$ will be (i) a tangent (ii) a normal to the circle $x^2+y^2+2gx+2fy+c=0$.

66. Find the value of p if the straight line $x\cos\alpha+y\sin\alpha=p$ touches the circle $x^2+y^2-2ax\cos\alpha-2ay\sin\alpha=0$.

67. Show that the straight line $y=mx$ will be a tangent to the circle $x^2+y^2+2gx+2fy+c=0$ if $(g+mf)^2=c(1+m^2)$.

68. Show that the straight line $y=mx+c$ will be a tangent to the circle $(x-a)^2+(y-b)^2=r^2$ if

$$m^2(a^2-r^2)+2ma(c-b)+(c-b)^2=r^2.$$

69. Find the that condition the straight line $lx+my+n=0$ will touch the parabola $y^2=4ax$.

70. The eccentricity of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{3}{5}$ and the straight line $5y=3x+25$ touches the curve. Find the values of a and b.

71. Find the condition that the straight line $lx+my+n=0$ will touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

72. Find the condition that the straight line $lx+mx=n$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

73. The straight line $lx+my=1$ touches the circle $x^2+y^2=a^2$. Show that the locus of the point (l, m) is a circle. Find the equation of the circle.

74. Find the condition that the straight line $lx+my=n$ is a normal to the parabola $y^2=4ax$.

75. Prove that the straight line $y=mx+c$ will be a tangent to the parabola $y^2=4a(x+a)$ if $c=am+\frac{a}{m}$.

76. Prove that the condition that the straight line $x\cos\alpha+y\sin\alpha=p$ will touch the parabola $y^2=4ax$ if $p=-a\sin\alpha\tan\alpha$.

77. Prove that the straight line $lx+my+n=0$ will touch the parabola $y^2=4a(x-b)$ if $am^2=bl^2+nl$.

78. Find the conditions, so that the following straight lines will touch (i) the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and (ii) the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

(a) $y=mx+c$ (b) $lx+my+n=0$.

79. The tangents to the hyperbola at the points (x_1, y_1) and (x_2, y_2) of it intersect each other perpendicularly. Show that

$$\frac{x_1x_2}{y_1y_2}-\frac{a^4}{b^4}=0.$$

80. Prove that the distance of the point of intersection of the normal to a parabola and the axis of a parabola and the foot of the ordinate of the point on the axis is always the same.

81. Find the equation of the common tangent to the parabolas $y^2=32x$ and $x^2=108y$.

82. Two equal parabolas have the same vertex and their axes intersect perpendicularly. Prove that their common tangent intersects each parabola at the same extremity of their latus-rectum.

83. Find the equations of the common tangents to the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

84. Find the angle between the following pairs of curves.

(i) $x^2-y^2=a^2$ and $x^2+y^2=\sqrt{2}a^2$

(ii) $y^2=4ax$ and $x^2=4by$

(at the point of intersection other than the origin).

85. If the curves $yx^2=a$ and $xy=b$ pass through the point (3, 4) find the angle between the curves.

86. Show that the curves $x^2=ay$ and $y^2=ax$ intersect on the curve $x^3+y^3=3axy$. Find the angles between the two curves.

87. Show that the curves $\dfrac{x^2}{a^2+k_1}+\dfrac{y^2}{b^2+k_1}=1$ and $\dfrac{x^2}{a^2+k_2}+\dfrac{y^2}{b^2+k_2}$ intersect each other orthogonally.

88. Show that the curves $x^2+4y^2=8$ and $x^2-2y^2=4$ intersect each other at right angles at four points.

89. Show that if the two ellipses $\dfrac{x^2}{a_1^{\,2}}+\dfrac{y^2}{b_1^{\,2}}=1$ and $\dfrac{x^2}{a_2^{\,2}}+\dfrac{y^2}{b_2^{\,2}}=1$ intersect each other orthogonally then $a_1^{\,2}-a_2^{\,2}=b_1^{\,2}-b_2^{\,2}$.

90. A tangent is drawn to the circle $x^2+y^2=b^2$ from a point on the circle $x^2+y^2=a^2$. The chord of contact of this tangent touches the circle $x^2+y^2=c^2$. Show that a, b, c are in geometric progression.

91. Prove that the normal to the parabola $y^2=4ax$ at the point whose abscissa and ordinate are equal subtend a right angle at the focus.

92. Prove that the area of the triangle formed by joining any three points of a parabola is double of the area of the triangle formed by the tangents to the curve drawn at the point.

93. Prove that the normal drawn to the ellipse $3x^2+4y^2=48$ at the point (2, 3) bisects the angle between the focal distances of the point.

94. Prove that the normal drawn to the hyperbola $4x^2-3y^2=1$ at the point (1, 1) bisects the angle between the focal distances of the point.

95. Find the distance of the origin from the point of intersection of the x-axis with the tangent to the curve $x^2-y^2=9$ at the point (5, 4).

96. SY and $S'Y'$ are perpendiculars from the foci S and S' of an ellipse on the tangent to the curve at any point of it. Prove that $SY.S'Y'=b^2$.

97. P is a point of a parabola and S is the focus of the curve. Prove that the straight line drawn parallel to the axis of the parabola from P in the direction in which the curve opens bisects the angle between the tangent to the curve at the point and the ray $\overrightarrow{PS}$.

98. Prove that the tangents to a parabola drawn at the extremities of any focal chord intersect on the directrix of the curve at right angles.

99. Prove that the chord of the parabola $y^2=ax$ normal at the point $\left(\frac{1}{2}a, \frac{1}{\sqrt{2}}a\right)$ subtends a right angle at the vertex.

100. Find the point of intersection of the two tangents to the ellipse $3x^2+7y^2=35$ at its point of intersection with the straight line $x+2y=7$.

101. Prove that the tangents drawn at the ends of a chord meet on the diameter bisecting the chord.

102. If the chord of contact of an external point P of a hyperbola passes through another external point Q, then show that the chord of contact of Q passes through P.

103. Find the equation of the tangents drawn from the origin to the circle $x^2+y^2+20(x+y)+20=0$.

104. Prove that the length of the chord of contact of the tangents drawn from an external point (α, β) to the parabola $y^2=4ax$ is $\frac{1}{a}\sqrt{\beta^2+4a^2}\sqrt{\beta^2-4a\alpha}$.

105. Find the equations of the tangents drawn from the point $(-15, -7)$ to the ellipse $9x^2+25y^2=225$. Also find the points of contact.

106. Find the co-ordinates of the points of contact of the tangents drawn from the point (0, 5) to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$.

107. The angle between the pair of tangents drawn from an external point to the circle $x^2+y^2=a^2$ is 60°. Show that the locus of the point is the circle $x^2+y^2=4a^2$.

108. The angle between the tangents drawn to the circle $x^2+y^2=a^2$ from an external point P is 120°. Show that the locus of the point P is the circle $x^2+y^2=\frac{4a^2}{3}$.

109. Find the locus of the point from which the tangents drawn to the circle $x^2+y^2=a^2$ are at right angles.

110. Find the locus of the point of intersection of tangents drawn to the parabola $y^2=4ax$ and inclined at an angle α with each other.

111. The normal drawn to the parabola $y^2=4ax$ at the point $P(at^2, 2at)$ intersects the axis of the parabola at the point G. If the line GP be produced to Q so that $PQ=GP$, then show that the locus of the point Q is the parabola $y^2=16a(x+2a)$.

112. Prove that the equation of locus of the middle points of normal chords of the parabola $y^2=4ax$ is $\frac{y^2}{2a}+\frac{4a^3}{y^2}=x-2a$.

113. Find the locus of the foot of the perpendicular drawn from a focus of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangents to the ellipse.

114. Prove that the tangents drawn at the extremities of a latus-rectum of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ intersect on the major axis of the ellipse.

115. Find the locus of the point of intersection of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ inclined at an angle θ with each other.

116. Find the locus of the foot of the perpendicular drawn from the centre of an ellipse on tangents to the ellipse.

117. The normal drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point P of it intersect the x-axis at the point G. Show that the locus of the middle point of PG is $\left(\frac{2ax}{2a^2-b^2}\right)^2+\left(\frac{2y}{b}\right)^2=1$.

118. $P(x, y)$ is any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The line joining P and the centre of the ellipse makes an angle θ with the normal to the ellipse at P. Show that $\tan\theta=\frac{(a^2-b^2)xy}{a^2b^2}$.

119. The normal drawn to an ellipse at an extremity of a latus-rectum passes through an extremity of the minor axis of the ellipse. Show that the eccentricity e of the ellipse satisfies the equation $e^4+e^2-1=0$.

120. Find the locus of the foot of the perpendicular to the tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ from a focus of the hyperbola.

121. The normal drawn at a point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intersect the axes of x and y at the points P and Q. Perpendiculars drawn at P and Q on the x-axis and y-axis respectively intersect at the point R. Show that the equation of the locus of R is $a^2x^2-b^2y^2=(a^2+b^2)^2$.

122. The lengths of the tangents drawn to the three circles $x^2+y^2-16x+60=0$, $x^2+y^2-12x+27=0$ and $x^2+y^2-16x-12y+84=0$ from an external point are equal in length. Find the co-ordinates of the point and also the length of each tangent.

123. The length of the tangent drawn to the circle $x^2+y^2+2x=0$ from a point P is three times the length of the tangent drawn to the circle $x^2+y^2=4$ from the same point P. Show that the locus of the point P is the circle $4x^2+4y^2-x-18=0$.

124. Find the lengths of the tangent, normal sub-tangent and sub-normal at the point 't' of the curve $x=a(t+\sin t)$, $y=a(1-\cos t)$.

106. Find the co-ordinates of the points of contact of the tangents drawn from the point (0, 5) to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$.

107. The angle between the pair of tangents drawn from an external point to the circle $x^2+y^2=a^2$ is 60°. Show that the locus of the point is the circle $x^2+y^2=4a^2$.

108. The angle between the tangents drawn to the circle $x^2+y^2=a^2$ from an external point P is 120°. Show that the locus of the point P is the circle $x^2+y^2=\frac{4a^2}{3}$.

109. Find the locus of the point from which the tangents drawn to the circle $x^2+y^2=a^2$ are at right angles.

110. Find the locus of the point of intersection of tangents drawn to the parabola $y^2=4ax$ and inclined at an angle α with each other.

111. The normal drawn to the parabola $y^2=4ax$ at the point $P(at^2, 2at)$ intersects the axis of the parabola at the point G. If the line GP be produced to Q so that $PQ=GP$, then show that the locus of the point Q is the parabola $y^2=16a(x+2a)$.

112. Prove that the equation of locus of the middle points of normal chords of the parabola $y^2=4ax$ is $\frac{y^2}{2a}+\frac{4a^3}{y^2}=x-2a$.

113. Find the locus of the foot of the perpendicular drawn from a focus of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangents to the ellipse.

114. Prove that the tangents drawn at the extremities of a latus-rectum of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ intersect on the major axis of the ellipse.

115. Find the locus of the point of intersection of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ inclined at an angle θ with each other.

116. Find the locus of the foot of the perpendicular drawn from the centre of an ellipse on tangents to the ellipse.

117. The normal drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point P of it intersect the x-axis at the point G. Show that the locus of the middle point of PG is $\left(\frac{2ax}{2a^2-b^2}\right)^2+\left(\frac{2y}{b}\right)^2=1$.

118. $P(x, y)$ is any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The line joining P and the centre of the ellipse makes an angle θ with the normal to the ellipse at P. Show that $\tan\theta=\frac{(a^2-b^2)xy}{a^2b^2}$.

119. The normal drawn to an ellipse at an extremity of a latus-rectum passes through an extremity of the minor axis of the ellipse. Show that the eccentricity e of the ellipse satisfies the equation $e^4+e^2-1=0$.

120. Find the locus of the foot of the perpendicular to the tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ from a focus of the hyperbola.

121. The normal drawn at a point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intersect the axes of x and y at the points P and Q. Perpendiculars drawn at P and Q on the x-axis and y-axis respectively intersect at the point R. Show that the equation of the locus of R is $a^2x^2-b^2y^2=(a^2+b^2)^2$.

122. The lengths of the tangents drawn to the three circles $x^2+y^2-16x+60=0$, $x^2+y^2-12x+27=0$ and $x^2+y^2-16x-12y+84=0$ from an external point are equal in length. Find the co-ordinates of the point and also the length of each tangent.

123. The length of the tangent drawn to the circle $x^2+y^2+2x=0$ from a point P is three times the length of the tangent drawn to the circle $x^2+y^2=4$ from the same point P. Show that the locus of the point P is the circle $4x^2+4y^2-x-18=0$.

124. Find the lengths of the tangent, normal sub-tangent and sub-normal at the point 't' of the curve $x=a(t+\sin t)$, $y=a(1-\cos t)$.

125. Prove that the length of the subtangent to the curve $y=be^{x/a}$ at any point of it is constant and the length of the sub-normal at a point of the curve is proportional to the square of the ordinate of the point.

126. Find the lengths of the subtangent and subnormal of the curve $y=\frac{1}{2}a\,(e^{x/a}+e^{-x/a})$ at any point (x, y) of the curve.

127. Show that the length of the subtangent at any point of the curve $x^m y^n=a^{m+n}$ is proportional to the abscissa of the point.

128. Show that at any point of the curve $x^{m+n}=a^{m-n}\,y^{2n}$ the mth power of the length of the subtangent is proportional to the n-th power of the length of the sub-normal.

129. Prove that the length of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point of the ellipe is inversely proportional to the length of the perpendicular dropped from the origin on the tangent to the curve at the point.

CHAPTER THREE

MAXIMA AND MINIMA

§ 3·1. *Maximum and minimum values of a function.*

The concept of maxima and minima of a function is very important in Mathematics. You have already learnt determination of greatest and least values of certain classes of Algebraic and Trigonometric functions by algebraic and trigonometric manipulations. Greatest and least values of a function are its maximum and minimum values. But maximum and minimum values of a function are not always its greatest and least values. There are differences between greatest and maximum values and least and minimum values. We shall discuss the difference lateron ; prior to that let us understand what are meant by maximum or minimum values of a function.

Maximnm value : When the value of x is a, then the corresponding value of a function $f(x)$ is maximum if there exists an interval $a-\delta<x<a+\delta$ ($\delta>0$, however small it may be) including the point $x=a$ such that $f(x)$ is continuous in the interval and $f(a)$ is the greatest value of $f(x)$ in the interval. In this case $f(a)$ is a maximum value of $f(x)$ and if $a-\delta<c<a+\delta$, then $f(a)>f(c)$.

Minimum value : When the value of $x=a$, then the corresponding value $f(a)$ of a function $f(x)$ is a minimum if there exists an interval $a-\delta<x<a+\delta$ ($\delta>0$, may be as small as necessary) including a, so that $f(x)$ is continuous in the interval and $f(a)$ is the least value of $f(x)$ in the interval. So if $a-\delta<c<a+\delta$ be any other point of the interval then $f(a)<f(c)$.

§ 3·2. *Geometrical Discussions.*

The graph of a function $f(x)$ is the curve shown in fig. 3·1 and the curve is continuous in the interval $a-\delta<x<a+\delta$ ($\delta>0$). For values $a_1, a_2, a_3, a, a_4, a_5, a_6$ of x in the interval, the corresponding points of the curve are $P_1\{a_1, f(a_1)\}$, $P_2\{a_2, f(a_2)\}$, $P_3\{a_3, f(a_3)\}$, $P\{a, f(a)\}$, $P_4\{a_4, f(a_4)\}$, $P_5\{a_5, f(a_5)\}$, $P_6\{a_6, f(a_6)\}$. P_1A_1, P_2A_2, P_3A_3,

$PA, P_4A_4, P_5A_5, P_6A_6$ are the respective ordinates (y) of these points. From the figure it is seen that AP is the greatest amongst these ordinates. In the interval $a-\delta<x<a+\delta$, there are infinite number of points and from the figure is it is seen that $f(a)$ is the greatest amongst the infinite number of ordinates drawn at these points. So, in the interval $a-\delta<x<a+\delta$, $AP=f(a)$ is the

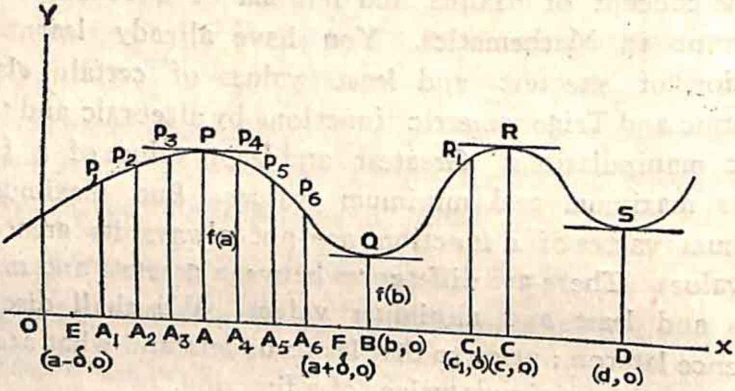


Fig 3·1

greatest value of $f(x)$. So $f(a)$ is a maximum value of $f(x)$. It is seen in the figure that AP is not the greatest amongst the ordinates of the points of the curve. Notice that the ordinate C_1R_1 of the point $R_1\{c_1, f(c_1)\}$ of the curve is greater than AP. So $f(a)$ is not the greatest value of $f(x)$. Similarly the ordinates $f(b)$, $f(c)$, $f(d)$ of the points Q, R, S corresponding to $x=b$, $x=c$ and $x=d$ are minimum, maximum, minimum values of $f(x)$. Note that in the figure the tangent to the curve at each of the points P, Q, R, S is parallel to the x-axis.

So $\frac{dy}{dx}=0$ at each of those points. So a necessary condition for the existence of a maximum or a minimum of the function $y=f(x)$ at a point x is $\frac{dy}{dx}=0$. In the next section we shall give the mathematical enunciation and proof of this proposition.

The above geometrical discussion may be summarised as follows :

1. The maximum and greatest value or the minimum or least value of a function are not the same. Maximum and minimum

values are locally greatest or least values. A function may possess more than one maximum or minimum value. The greatest value of a function is its largest maximum value and the least value of a function is the least amongt its minimum values. Again a maximum value of a function may be less than its minimum value.

2. If a function $f(x)$ possesses a maximum or a minimum at $x=a$, then the tangent to the curve $y=f(x)$ at the point corresponding to $x=a$ will be parallel to the x-axis and $\frac{dy}{dx}$ is 0 at $x=a$.

§ 3.3 *Theorem.*

If a function $y=f(x)$ be differentiable at a point $x=a$, then a necessary condition for the existence of a maximum or a minimum value of the function at $x=a$ is $\frac{dy}{dx}=f'(x)=0$.

Proof : First let $f(x)$ be maximum at $x=a$. So, there exists an interval $a-\delta<x<a+\delta$ $(\delta>0)$ with centre a, small at pleasure such that $f(a)>f(x)$ for all $x \neq a$ in the interval.

So if $h>0$ and $a-\delta<a-h<a+\delta$, then $f(a)>f(a-h)$.

$$\therefore \quad f(a-h)-f(a)<0.$$

$$\text{So,} \quad \frac{f(a-h)-f(a)}{-h}>0 \quad \cdots \quad \cdots \quad \text{(i)}$$

If now $a-h$ $(\neq a)$, be close to a at pleasure, the relation (i) will be satisfied i.e.,

$$\underset{h\to 0}{Lt} \frac{f(a-h)-f(a)}{-h} \geqslant 0 \quad \text{i.e., } f'(a-)\geqslant 0.$$

[$a-h$ is close to a at pleasure but not equal to a means $h\to 0$]

Again if $h>0$ and $a-\delta<a+h<a+\delta$, then $f(a)>f(a+h)$

$$\therefore \quad f(a+h)-f(a)<0.$$

$$\text{or,} \quad \frac{f(a+h)-f(a)}{h}<0 \quad \cdots \quad \cdots \quad \text{(ii),}$$

If now $a+h$ $(\neq a)$, be close to a at pleasure then the relation (ii) will be satisfied ; i.e.,

$$\underset{h\to 0}{Lt} \frac{f(a+h)-f(a)}{h} \leqslant 0. \quad \text{i.e., } f'(a+)\leqslant 0.$$

But when $x=a$, $f(x)$ is differentiable.

So, $f'(a-)=f'(a+)=f'(a)$

But $f'(a-)\geqslant 0$ and $f'(a+)\leqslant 0$ $\quad\therefore\quad f'(a)=0.$

Let now $f(x)$ be minimum at $x=a$ so, there exists an interval $a-\delta<x<a+\delta$ (with centre a), $\delta>0$, so that $f(a)<f(x)$ for every x in the interval.

So, if $h>0$ and $a-\delta<a-h<a+\delta$. then $f(a-h)-f(a)>0$

$$\text{or,}\quad \frac{f(a-h)-f(a)}{-h}<0 \quad \cdots \quad \cdots \quad \text{(iii)}$$

If now $a-h$ $(\neq a)$ be close to a at pleasure, then the relation (iii) will be satisfied,

$$\therefore \quad \underset{h\to 0}{Lt}\ \frac{f(a-h)-f(a)}{-h}\leqslant 0 \quad \text{i.e. } f'(a-)\leqslant 0$$

Again if $a-\delta<a+h<a+\delta$, then $f(a)<f(a+h)$

$$\text{or,}\quad f(a+h)-f(a)>0$$

$$\text{or,}\quad \frac{f(a+h)-f(a)}{h}>0 \quad \cdots \quad \cdots \quad \text{(iv)}$$

The relation (iv) is satisfied if h be close to a at pleasure ; i.e.,

$$\underset{h\to 0}{Lt}\ \frac{f(a+h)-f(a)}{h}\geqslant 0 \quad \text{or,}\quad f'(a+)\geqslant 0.$$

Now $f(x)$ is differentiable at $x=a$.

$$\therefore \quad f'(a-)=f'(a+)=f'(a)$$

But $f'(a-)\leqslant 0$, and $f'(a+)\geqslant 0$. $\quad\therefore\quad f'(a)=0.$

Hence if $f(x)$ be maximum or minimum at $x=a$, then $f'(a)=0$.

§ 3·4. *Geometrical discussions.*

We have seen in the last section that a necessary condition for the existence of a maximum or a minimum of $y=f(x)$ at $x=a$ is $f'(a)=0$, when $f'(a)$ exists. But even if $f'(a)$ does not exist, then also $f(x)$ may be maximum or minimum at $x=a$. We are now discussing, in this section, the circumstances under which $f(x)$ may possess a maximum or a minimum at $x=a$.

The curve in fig. 3·2 represents the graph of the function $y=f(x)$. P and Q are points on the graph corresponding to the values a and b respectively of x. From figure it is evident that $f(a)$ and $f(b)$ are respectively maximum and minimum values of $f(x)$. E and F are points of the graph very close to P situated respectively on the left and right of P. The tangents to the curve at E and F make acute angle ψ_1 and obtuse angle ψ_2 with the positive direction of the x-axis. For all positions of E and F close to P at pleasure (of course in its left and right), these angles will be acute and obtuse respectively. So for values of x very close to a but $<a$, $\tan\psi_1 = f'(a-) > 0$ and $\tan\psi_2 = f'(a+) < 0$ for values very close to a but $>a$.

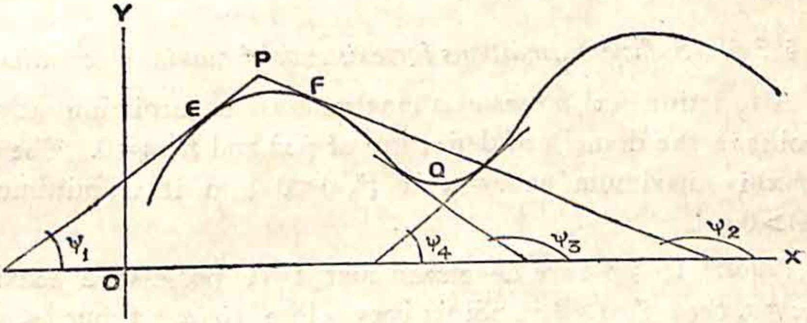


Fig. 3·2

So $\tan\psi_1 = f'(a-) > 0$ and $\tan\psi_2 = f'(a+) < 0$. i.e., as x just assumes values $>a$ from values $<a$, $\tan\psi = \dfrac{dy}{dx}$ immediately changes its sign from positive to negative. Now if the points E and F are taken close to P at pleasure, $\tan\psi_1$ and $\tan\psi_2$ will become $f'(a-)$ and $f'(a+)$ respectively. So if $f(x)$ is maximum at $x=a$, then $f'(a-)$ is positive and $f'(a+)$ is negative.

Similarly it can be proved that if $f(x)$ possesses a minimum at $x=b$, then $f'(b-) < 0$ and $f'(b+) > 0$. In this case if the point of the curve corresponding to $x=b$ be Q, then tangents to the curve at points very close to Q situated on the left and right of Q make obtuse and acute angles respectively with the positive direction of the x-axis.

So, if $f(x)$ possesses a maximum at $x=a$, then the value of $f'(x)$ becomes negative from positive values and if $f(x)$ possesses a minimum value then the value of $f'(x)$ becomes positive from left to right through $x=a$.

Note : In this case existence of $f'(x)$ is not necessary. [See Ex.—16].

§ 3·5. *Sufficient conditions for existence of maximum or minimum.*

A function $f(x)$ possesses a maximum or a minimum at $x=a$, a point in the domain of definition of $f(x)$ and $f'(a)=0$. The value of $f(x)$ is maximum at $x=a$ if $f''(a)<0$ and it is minimum if $f''(a)>0$.

Proof. In § 3·4 we heve seen that if $f(x)$ possesses a maximum at $x=a$, then $f'(a)>0$ at points very close to $x=a$ but $<a$ and $f'(a)<0$ at points very close to $x=a$ but $>a$. So $f'(x)$ is a decreasing function at $x=a$. $\therefore \frac{d}{dx}\{f'(x)\}<0$ or, $f''(x)>0$. We have also seen in the same section that if $f(x)$ possesses a minimum at $x=a$, then $f'(x)<0$ at points very close to a but $<a$ and $f'(x)>0$ at points very close to a but $>a$. So $f'(x)$ is an increasing function at $x=a$, so $f''(a)>0$.

Note 1. The proof of this theorem is not included in the syllabus.

2. Here we have assumed $f''(a)\neq 0$

3. The case when $f''(a)=0$ is not included in the syllabus. We state below for interested students, the general conditions of existence of maximum or minimum values of a function.

General condition : $x=a$ is a point in the domain of definition of a function $f(x)$ and $f'(a)=f''(a)=\cdots=f^{n-1}(a)=0$ but $f^{n}(a)\neq 0$.

Now if n be even, then $f(x)$ will have a maximum value at $x=a$ if $f^{n}(a)<0$ and a minimum value at $x=a$ if $f^{n}(a)>0$.

If n is odd $f(x)$ does not possess any maximum or minimum at $x=a$.

Note : Maximum or minimum values are also called **entreme values** or stationary values.

§3.6. *Some special artifices for determination of maximum or minimum.*

(i) $f(x)$ increases or decreases according as $\frac{1}{f(x)}$ decreases or increases. So if it is more convenient to ditermine maximum of minimum values of $\frac{1}{f(x)}$ then for determination of maximum or minimum values of $f(x)$, at first determine minimum or maximum values of $\frac{1}{f(x)}$.

(ii) If n be a positive integer, then $\{f(x)\}^n$ will possess maximum or minimum values if $f(x)$ possesses maximum or minimum values respectively and conversely. Hence according to convenience determine the maximum or minimum values of $\{f(x)\}^n$ or $f(x)$ at first.

(iii) If y be a function of x,

then $\frac{dy}{dx}$ and $\frac{d}{dx}\{\log y\} = \frac{1}{y}\frac{dy}{dx}$

have the same sign ; for if $\log y$ is defined then $y > 0$. You may use this proposition according to convenience.

EXAMPLES 3

Example 1. Find the maximum and minimum values of

$f(x) = 2x^3 - 21x^2 + 36x - 20$.

$f(x) = 2x^3 - 21x^2 + 36x - 20$.

$\therefore\ f'(x) = 6x^2 - 42x + 36 = 6(x^2 - 7x + 6) = 6(x-1)(x-6)$.

For maximum or minimum values of $f(x)$, $f'(x) = 0$

$\therefore\ x = 1$ or, $x = 6$. $f''(x) = 12x - 42. = 6(2x - 7)$

$f''(1) = 6(2.1 - 7) = -30 < 0$.

So $f(x)$ is maximum at $x = 1$ and the maximum value is $f(1)$ $= 2.1^3 - 21.1^2 + 36.1 - 20 = -3$.

$f''(6) = 6(2.6 - 7) = 30 > 0$.

So $f(x)$ is minimum when $x = 6$ and the minimum value is

$f(6) = 2.6^3 - 21.6^2 + 36.6 - 20 = -128$.

Example 2. Show that the maximum value of $x + \frac{1}{x}$ is less than its minimum value.

Let $y=x+\frac{1}{x}$. $\therefore \frac{dy}{dx}=1-\frac{1}{x^2}$

For maximum or minimum $\frac{dy}{dx}=0$ or, $1-\frac{1}{x^2}=1$

or, $x^2-1=0$ or, $x=\pm1$. $\frac{d^2y}{dx^2}=\frac{2}{x^3}$.

when $x=1$, $\frac{d^2y}{dx^2}=2>0$ and so y is minimum when $x=1$ and the minimum value is $1+\frac{1}{1}=2$.

when $x=-1$, $\frac{d^2y}{dx^2}=-2<0$ and so y is maximum when $x=-1$, and the maximum value is $-1-\frac{1}{1}=-2$.

Hence the maximum value of $x+\frac{1}{x}$ is less than its minimum value.

Example 3. Find the maximum and minimum value of (i) $\sin x$ and (ii) $a \sin x+b \cos x$.

(i) Let $y=\sin x$. $\therefore \frac{dy}{dx}=\cos x$.

For maximum and minimum, $\frac{dy}{dx}=0$ or, $\cos x=0$

$\therefore$ $x=(2n+1)\frac{\pi}{2}$; $[n=0, \pm1, \pm2, \ldots\ldots]$

Again $\frac{d^2y}{dx^2}=-\sin x$.

when n is even, let $n=2p$. $[p=0, \pm1, \pm2 \ldots\ldots]$

then $\frac{d^2y}{dx^2}=-\sin\{(4p+1)\frac{\pi}{2}\}=-\sin\frac{\pi}{2}=-1<0$.

So when $x=(4p+1)\frac{\pi}{2}$, $y=\sin x$ is maximum and the maximum value is $\sin(4p+1)\frac{\pi}{2}=1$.

when n is odd, let $n=2q+1$. $[q=0, \pm1, \pm2, \ldots\ldots]$

$\frac{d^2y}{dx^2}=-\sin\{(4q+3)\frac{\pi}{2}\}=-\sin\frac{3\pi}{2}=1>0$.

So, when $n=2q+1$, $\sin x$ is minimum and the minimum value is $\sin\{(4q+3)\frac{\pi}{2}\}=-1$.

(ii) Let $y=a \sin x+b \cos x$ $\therefore \frac{dy}{dx}=a \cos x-b \sin x$.

For maximum and minimum, $\frac{dy}{dx}=0$.

or, $a\cos x-b\sin x=0$ or $\frac{\cos x}{b}=\frac{\sin x}{a}=\pm\frac{1}{\sqrt{a^2+b^2}}$

$$\therefore \quad \cos x=\frac{b}{\sqrt{a^2+b^2}},\ \sin x=\frac{a}{\sqrt{a^2+b^2}} \qquad \dots \quad \dots \text{(i)}$$

$$\text{or,}\quad \cos x=-\frac{b}{\sqrt{a^2+b^2}},\ \sin x=-\frac{a}{\sqrt{a^2+b^2}} \qquad \dots \quad \dots \text{(ii)}$$

$$\frac{d^2y}{dx^2}=-a\sin x-b\cos x.$$

In case (i), $\frac{d^2y}{dx^2}=-\frac{a^2}{\sqrt{a^2+b^2}}-\frac{b^2}{\sqrt{a^2+b^2}}$

$$=-\frac{a^2+b^2}{\sqrt{a^2+b^2}}=-\sqrt{a^2+b^2}<0.$$

So, in this case, y is maximum and the maximum value is

$$a.\frac{a}{\sqrt{a^2+b^2}}+b.\frac{b}{\sqrt{a^2+b^2}}=\frac{a^2+b^2}{\sqrt{a^2+b^2}}=\sqrt{a^2+b^2}$$

Incase (ii), $\frac{d^2y}{dx^2}=\frac{a^2}{\sqrt{a^2+b^2}}+\frac{b^2}{\sqrt{a^2+b^2}}$

$$=\frac{a^2+b^2}{\sqrt{a^2+b^2}}=\sqrt{a^2+b^2}>0.$$

So, in this case y is minimum and the minimum value is

$$-\frac{a^2}{\sqrt{a^2+b^2}}-\frac{b^2}{\sqrt{a^2+b^2}}=-\frac{a^2+b^2}{\sqrt{a^2+b^2}}=-\sqrt{a^2+b^2}.$$

Example 4. Find the least value of $9\tan^2\theta+4\cot^2\theta$.
[Joint Entrance, 1984]

Let $y=9\tan^2\theta+4\cot^2\theta$.

$$\therefore \quad \frac{dy}{d\theta}=18\tan\theta\sec^2\theta-8\cot\theta\operatorname{cosec}^2\theta.$$

For maximum or minimum $\frac{dy}{d\theta}=0$

$\therefore$ $18\tan\theta\sec^2\theta-8\cot\theta\operatorname{cosec}^2\theta=0$

or, $9\tan\theta\sec^2\theta=4\cot\theta\operatorname{cosec}^2\theta$

$$\text{or, } \frac{\tan\theta \cdot \dfrac{1}{\cos^2\theta}}{\dfrac{1}{\tan\theta}\cdot\dfrac{1}{\sin^2\theta}}=\tfrac{4}{9} \quad \text{or, } \tan^4\theta=\tfrac{4}{9} \quad \therefore \quad \tan^2\theta=\tfrac{2}{3}.$$

[As $\tan^2\theta$ cannot be negative]

$$\text{Now } \frac{d^2y}{d\theta^2}=18\sec^4\theta+36\sec^2\theta\tan^2\theta$$
$$-8\{-\operatorname{cosec}^4\theta-2\operatorname{cosec}^2\theta\cot^2\theta\}$$
$$=18\sec^4\theta+36\tan^2\theta\sec^2\theta+8\operatorname{cosec}^4\theta$$
$$+16\operatorname{cosec}^2\theta\cot^2\theta>0$$

for all real values of θ and so when $\tan^2\theta=\frac{2}{3}$.

$\therefore$ y is minimum when $\tan^2\theta=\frac{2}{3}$ i.e, $\cot^2\theta=\frac{3}{2}$.

Hence the minimum (here least) value of y is $9.\frac{2}{3}+4.\frac{3}{2}=12$.

Example 5. Show that when $x=\frac{\pi}{3}$, the value of $\sin x\,(1+\cos x)$ is maximum.

Let $y=\sin x\,(1+\cos x)=\sin x+\sin x\cos x=\sin x+\frac{1}{2}\sin 2x$.

$$\therefore \quad \frac{dy}{dx}=\cos x+\tfrac{1}{2}\,.\,2\cos 2x=\cos x+\cos 2x.$$

$$\text{When } x=\frac{\pi}{3},\ \frac{dy}{dx}=\cos\frac{\pi}{3}+\cos\frac{2\pi}{3}=\tfrac{1}{2}-\tfrac{1}{2}=0.$$

So when $x=\frac{\pi}{3}$, y may be maximum or minimum.

$$\frac{d^2y}{dx^2}=-\sin x-2\sin 2x.$$

$$\text{When } x=\frac{\pi}{3},\ \frac{d^2y}{dx^2}=-\sin\frac{\pi}{3}-2\sin\frac{2\pi}{3}=-\frac{\sqrt{3}}{2}-2\,.\,\frac{\sqrt{3}}{2}<0.$$

So, when $x=\frac{\pi}{3}$, y is maximum.

Example 6. Show that when $\theta=\frac{\pi}{2}$, $\dfrac{2\theta-\sin 2\theta}{\theta^2}$ $(\theta>0)$ is maximum.

$$\text{Let } f(\theta)=\frac{2\theta-\sin 2\theta}{\theta^2}=\frac{2}{\theta}-\frac{\sin 2\theta}{\theta^2}$$

$$\therefore\ f'(\theta)=-\frac{2}{\theta^2}-\frac{\theta^2\,.\,2\cos 2\theta-2\theta\sin 2\theta}{\theta^4}$$

$$=-\frac{2}{\theta^2}-2\frac{\cos 2\theta}{\theta^2}+2\frac{\sin 2\theta}{\theta^3}$$

$$\therefore\ f'(\tfrac{\pi}{2})=\frac{-2\cdot\dfrac{\pi^2}{4}-2\cdot\dfrac{\pi^2}{4}\cdot\cos\pi+2\cdot\dfrac{\pi}{2}\sin\pi}{\dfrac{\pi^4}{16}}=\frac{-\dfrac{\pi^2}{2}+\dfrac{\pi^2}{2}}{\dfrac{\pi^4}{16}}=0.$$

So when $\theta=\frac{\pi}{2}$, $f(\theta)$ may be maximum or minimum.

$$\text{Now } f''(\theta)=\frac{4}{\theta^3}-2\left\{\frac{\theta^2(-2\sin 2\theta)-\cos 2\theta\cdot 2\theta}{\theta^4}\right\}$$

$$+2\left\{\frac{\theta^3\cdot 2\cos 2\theta-3\theta^2\sin 2\theta}{\theta^6}\right\}$$

$$=\frac{4}{\theta^3}+\frac{4\sin 2\theta}{\theta^2}+\frac{4\cos 2\theta}{\theta^3}+\frac{4\cos 2\theta}{\theta^3}-\frac{6\sin 2\theta}{\theta^4}$$

$$\therefore\ f''(\tfrac{\pi}{2})=\frac{32}{\pi^3}+\frac{4\sin\pi}{\pi^2}+\frac{32\cos\pi}{\pi^3}+\frac{32\cos\pi}{\pi^3}-\frac{6\sin\pi.16}{\pi^4}$$

$$=\frac{32}{\pi^3}+0-\frac{32}{\pi^3}-\frac{32}{\pi^3}-0=-\frac{32}{\pi^3}<0.$$

So when $\theta=\dfrac{\pi}{2}$, then $f(\theta)=\dfrac{2\theta-\sin 2\theta}{\theta^2}$ is maximum.

Example 7. Show that the difference of the maximum and minimum values of $\left(\alpha-\dfrac{1}{\alpha}-x\right)(4-3x^2)$ is $\frac{4}{9}\left(\alpha+\dfrac{1}{\alpha}\right)^3$.

[C. U. B. Sc. 1985]

$$\text{Let } y=\left(\alpha-\frac{1}{\alpha}-x\right)(4-3x^2)$$

$$=4\left(\alpha-\frac{1}{\alpha}\right)-4x-3\left(\alpha-\frac{1}{\alpha}\right)x^2+3x^3.$$

$$\therefore\ \frac{dy}{dx}=-4-6\left(\alpha-\frac{1}{\alpha}\right)x+9x^2$$

For maximum and minimum values of y, $\dfrac{dy}{dx}=0$

$$\therefore\ -4-6\left(\alpha-\frac{1}{\alpha}\right)x+9x^2=0$$

$$\text{or,}\quad 9x^2-6\alpha x+\frac{6}{\alpha}x-4=0.$$

$$\text{or,}\quad 3x(3x-2\alpha)+\frac{2}{\alpha}(3x-2\alpha)=0$$

or, $(3x-2\alpha)\left(3x+\frac{2}{\alpha}\right)$ $\therefore$ $x=\frac{2\alpha}{3}$ or, $-\frac{2}{3\alpha}$.

Now $\frac{d^2y}{dx^2}=18x-6\left(\alpha-\frac{1}{\alpha}\right)$.

$\therefore$ when $x=\frac{2\alpha}{3}$, $\frac{d^2y}{dx^2}=12\alpha-6\alpha+\frac{6}{\alpha}=6\left(\alpha+\frac{1}{\alpha}\right)$.

when $x=-\frac{2\alpha}{3}$, $\frac{d^2y}{dx^2}=18\left(-\frac{2}{3\alpha}\right)-6\left(\alpha-\frac{1}{\alpha}\right)$

$$=-6\left(\alpha+\frac{1}{\alpha}\right).$$

Now when $\alpha>0$, then $6\left(\alpha+\frac{1}{\alpha}\right)>0$ and $-6\left(\alpha+\frac{1}{\alpha}\right)<0$.

So in this case the maximum value of $f(x)$ will be when $x=-\frac{2}{3\alpha}$ and the minimum value of $f(x)$ will be when $x=\frac{2\alpha}{3}$.

So, when $\alpha>0$. The maximum value of $f(x)$

$$=4\left(\alpha-\frac{1}{\alpha}\right)+4.\frac{2}{3\alpha}-3\left(\alpha-\frac{1}{\alpha}\right)\left(-\frac{2}{3\alpha}\right)^2+3\left(-\frac{2}{3\alpha}\right)^3$$

$$=4\left(\alpha-\frac{1}{\alpha}\right)+\frac{8}{3\alpha}-\frac{4}{3}\left(\alpha-\frac{1}{\alpha}\right)\frac{1}{\alpha^2}-\frac{8}{9\alpha^3}.$$

And the minimum value is

$$4\left(\alpha-\frac{1}{\alpha}\right)-\frac{8\alpha}{3}-3\left(\alpha-\frac{1}{\alpha}\right)\frac{4\alpha^2}{9}+\frac{8}{9}\alpha^3$$

$$=4\left[\alpha-\frac{1}{\alpha}\right]-\frac{8\alpha}{3}-\frac{4}{3}\left[\alpha-\frac{1}{\alpha}\right]\alpha^2+\frac{8}{9}\alpha^3.$$

So the difference of the maximum and minimum values is

$$=\frac{8}{3}\left[\frac{1}{\alpha}+\alpha\right]-\frac{4}{3}\left[\frac{1}{\alpha}-\alpha\right]\left(\alpha^2-\frac{1}{\alpha^2}\right)-\frac{8}{9}\left(\alpha^3+\frac{1}{\alpha^3}\right)$$

$$=\frac{8}{3}\left(\alpha+\frac{1}{\alpha}\right)+\frac{4}{3}\left(\frac{1}{\alpha}-\alpha\right)^2\left(\alpha+\frac{1}{\alpha}\right)$$

$$-\frac{8}{9}\left(\alpha+\frac{1}{\alpha}\right)\left(\alpha^2-1+\frac{1}{\alpha^2}\right)$$

$$=\frac{4}{3}\left(\alpha+\frac{1}{\alpha}\right)\left\{2+\left(\alpha-\frac{1}{\alpha}\right)^2-\frac{2}{3}\left(\alpha^2-1+\frac{1}{\alpha^2}\right)\right\}$$

$$=\frac{4}{3}\left(\alpha+\frac{1}{\alpha}\right)\left\{2+\alpha^2+\frac{1}{\alpha^2}-2-\frac{2}{3}\alpha^2+\frac{2}{3}-\frac{2}{3\alpha^2}\right\}$$

$$=\frac{4}{3}\left(\alpha+\frac{1}{\alpha}\right)\left\{\frac{1}{3}\alpha^2+\frac{2}{3}+\frac{1}{3\alpha^2}\right\}=\frac{4}{9}\left(\alpha+\frac{1}{\alpha}\right)\left\{\alpha^2+2+\frac{1}{\alpha^2}\right\}$$

$$=\frac{4}{9}\left(\frac{1}{\alpha}+\alpha\right)\left(\frac{1}{\alpha}+\alpha\right)^2=\frac{4}{9}\left(\frac{1}{\alpha}+\alpha\right)^3.$$

When $\alpha<0$, the maximum and minimum values of y are the minimum and maximum values respectively of y when $\alpha>0$. and so in this case the difference of the maximum and minimum values is the same as in the case $\alpha>0$ and is $\frac{4}{9}\left(\alpha+\frac{1}{\alpha}\right)^3$.

Example 8. $P(x)=a_0+a_1x^2+a_2x^4+\cdots+a_nx^{2n}$ is a pohynomial with real coefficients and $0<a_0<a_1<a_2<\cdots\cdots<a_n$. Determine the maximum or minimum values of $P(x)$ if any.

$$P(x)=a_0+a_1x^2+a_2x^4+\cdots\cdots+a_nx^{2n}$$

$$\therefore\quad P'(x)=2a_1x+4a_2x^3+\cdots\cdots+2n\,a_n\,x^{2n-1}$$

$$=2x(a_1+2a_2x^2+\cdots\cdots+na_u\,x^{2n-2})$$

For maximum and minimum values of $P(x)$, $P'(x)=0$. or, $x=0$

[as for all values of x, $(a_1+2a_2x^2+\cdots\cdots+na_nn^{2n-2})>0$, because $0<a_0<a_1<a_2\cdots\cdots<a_n$ and each of x^2, $x^4,\cdots\cdots$, x^{2n-2} is positive or 0.]

$$\text{Now } P''(x)=2(a_1+2a_2x^2+\cdots\cdots+na_n\,x^{2n-2})$$

$$+2x\,\{4a_2x+8a_3x^3+\cdots\cdots+(2n-2)\,nx^{2n-3}\}$$

$$\therefore\quad P''(0)=2a_1>0.$$

So, if $x=0$, $P(x)$ is minimum and the minimum value is $P(0)=a_0$.

Example 9. Show that $\sin^3 x\cos x$ is minimum when $x=\frac{\pi}{3}$.

Let $f(x)=\sin^3 x\cos x$. $f'(x)=3\sin^2x\cos^2x-\sin^4 x$.

$$f'\left(\tfrac{\pi}{3}\right)=3\,.\left(\frac{\sqrt{3}}{2}\right)^2\left(\tfrac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}\right)^4=\frac{9}{16}-\frac{9}{16}=0.$$

So, when $x=\frac{\pi}{3}$, $f(x)$ may possess a maximum or a minimum.

Now $f'(x)=3\sin^2 x\cos^2x-\sin^4 x=\frac{3}{4}\sin^2 2x-\sin^4x$.

$$\therefore\quad f''(x)=\tfrac{3}{4}\,2\,.\sin 2x\;2\cos 2x-4\sin^3x\cos x$$

$$=3\sin 2x\cos 2x-4\sin^3 x\cos x$$

$$f''\left(\frac{\pi}{3}\right)=3\sin\frac{2\pi}{3}\cos\frac{2\pi}{3}-4\sin^3\frac{\pi}{3}\cos\frac{\pi}{3}$$

$$=3\cdot\frac{\sqrt{3}}{2}\left(-\tfrac{1}{2}\right)-4\frac{3\sqrt{3}}{8}\cdot\tfrac{1}{2}<0.$$

So $f(x)$ possesses a maximum at $x=\frac{\pi}{3}$.

Example 10. Find the minimum value of $a^2e^{2x}+b^2e^{-2x}$ ($a>0$, $b>0$).

Let $y=a^2e^{2x}+b^2e^{-2x}$ $\therefore$ $\frac{dy}{dx}=2a^2e^{2x}-2b^2e^{-2x}$.

For maximum and minimum values of y,

$\frac{dy}{dx}=0.$ or, $2a^2e^{2x}-2b^2e^{-2x}=0.$

or, $(e^{2x})^2=\frac{b^2}{a^2}$ $\therefore$ $e^{2x}=\frac{b}{a}$ [as e^{2x} cannot be negative]

Now $\frac{d^2y}{dx^2}=4a^2e^{2x}+4b^2e^{-2x}$.

When $e^{2x}=\frac{b}{a}$, $\frac{d^2y}{dx^2}=4a^2\cdot\frac{b}{a}+4b^2\cdot\frac{a}{b}=8ab>0$

So when $e^{2x}=\frac{b}{a}$, then y is minimum and the minimum value is $a^2\cdot\frac{b}{a}+b^2\cdot\frac{a}{b}=2ab$.

Example 11. Find the maximum or minimum values of $x^{1/x}$ if any.

[C. U. 1986]

Let $y=x^{1/x}$ $\therefore$ $\log y=\log x^{1/x}=\frac{1}{x}\log x$.

Differentiating both sides with respect to x we get·

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}\cdot\frac{1}{x}-\frac{1}{x^2}\cdot\log x=\frac{1}{x^2}(1-\log x)\ \cdots\ \cdots\ \text{(i)}$$

For maximum and minimum values of y, $\frac{dy}{dx}=0$.

or, $(1-\log x)=0$ or, $\log x=1$ or, $x=e$.

Again differentiating both sides of equation (i) we get,

$$\frac{1}{y}\frac{d^2y}{dx^2}-\frac{1}{y^2}\frac{dy}{dx}=-\frac{2}{x^3}(1-\log x)+\frac{1}{x^2}\left(-\frac{1}{x}\right)$$

When $x=e$, $\frac{1}{e}\frac{d^2y}{dx^2}-\frac{1}{e^2}\cdot 0=-\frac{2}{e^3}(1-\log e)-\frac{1}{e^3}=-\frac{1}{e^3}<0.$

$$\left[\text{As when } x=e,\ \frac{dy}{dx}=0\right]$$

or, $\frac{d^2y}{dx^2}=-\frac{1}{e^4}$ [as $\log e=1$] <0

So when $x=e$, y is maximum and the maximum value is $e^{1/e}$.

Example 12. Show that the following functions have no maximum or minimum value,

(i) $\frac{\sin(x+a)}{\sin(x+b)}$ $(a\neq b)$ (ii) $x^3-3x^2+24x+30$.

(i) Let $y=\frac{\sin(x+a)}{\sin(x+b)}$

$$\therefore\ \frac{dy}{dx}=\frac{\sin(x+b)\cos(x+a)-\sin(x+a)\cos(x+b)}{\sin^2(x+b)}$$

$$=\frac{\sin(x+b-x-a)}{\sin^2(x+b)}=\frac{\sin(b-a)}{\sin^2(x+b)}$$

$\therefore$ As $b\neq a$, for every value of x, $\sin(b-a)\neq 0$

$\therefore$ So, for every value of x, $\frac{dy}{dx}\neq 0$.

So for any value of x, $y=\frac{\sin(x+a)}{\sin(x+b)}$ does not possess any maximum or minimum.

(ii) $f(x)=x^3-3x^2+24x+30$

or, $f'(x)=3x^2-6x+24=3(x^2-2x+8)=3\{(x-1)^2+7\}$

As x, is real, $(x-1)^2$ is always non-negative and so $3\{(x-1)^2+7\}$ is always positive and so not equal to 0. So, $f(x)$ cannot possess any maximum or minimum for any real value of x.

Example 13. Show that $x^5-5x^4+5x^3-1$ has a maximum at $x=1$ and a minimum at $x=3$; but possesses neither at $x=0$.

Let $f(x)=x^5-5x^4+5x^3-1$.

$\therefore\ f'(x)=5x^4-20x^3+15x^2=5x^2(x^2-4x+3)=5x^2(x-1)(x-3)$.

For maximum or minimum values of $f(x)$, $f'(x)=0$.

i.e., $x=0,\ 1,\ 3$. Now $f''(x)=20x^3-60x^2+30x$

$f''(1)=20-60+30=-10<0$.

So when $x=1$, $f(x)$ is maximum.
Again $f''(3)=20.3^3-60.3^2+30.3=90>0$
So, $f(x)$ is minimum when $x=3$.
$f''(0)=0$. In this case let us find $f'''(0)$.
Now $f'''(x)=60x^2-120x+30$. $\therefore$ $f'''(0)=30\neq0$.

Thus the derivative of the third order of $f(x)$ is the first derivative of $f(x)$ which is not 0 and since it is an odd order derivative, so $f(x)$ does not possess any maximum or minimum at $x=0$.

Example 14. Find the greatest and least values of x^3-18x^2+96x in the interval (0, 9).

Let $f(x)=x^3-18x^2+96x$.
$\therefore$ $f'(x)=3x^2-36x+96=3(x^2-12x+32)=3(x-4)(x-8)$.
For maximum or minimum values of $f(x)$, $f'(x)=0$
i. e., $x=4$ or, 8. $f''(x)=6x-36$ $\therefore$ $f''(4)=24-36=-12<0$.
So $f(x)$ is maximum when $x=4$ and the maximum value is $f(4)=4^3-18.4^2+96.4=64-288+384=160$.
Again $f''(8)=48-36=12>0$.
So $f(x)$ is minimum when $x=8$ and the minimum value is $f(8)=8^3-18.8^2+96.8=512-1152+768=128$
Also $f(0)=0$ and $f(9)=9^3-18.9^2+96.9=135$.
So the greatest and least values of $f(x)$ are 160 and 0.

Note : The greatest value of a function in an interval is the greatest among the maximum values and the values at the extremities of the interval. Similarly, the minimum value of the function is determined.

Example 15. Find the maximum and minimum values of $f(x)=(x-1)(x-2)^2$.

Alternative method : $f(x)=(x-1)(x-2)^2$
$f'(x)=(x-2)^2+2(x-1)(x-2)=(x-2)(3x-4)$.
For maximum and minimum values of $f(x)$, $f'(x)=0$. i.e., $f(x)$ may possess maximum or minimum values at $x=2$ or $\frac{4}{3}$.
Now if $x<\frac{4}{3}$, $f'(x)=(x-2)(3x-4)>0$
if $x=\frac{4}{3}$, $f'(x)=0$. if $\frac{4}{3}<x<2$, $f'(x)<0$.
Thus when x assumes values, from values less than $\frac{4}{3}$ to values

greater than $\frac{4}{3}$, through the value $\frac{4}{3}$, then $f(x)$ becomes positive from negative. So $f(x)$ is maximum at $x=\frac{4}{3}$ and the maximum value is $(\frac{4}{3}-1)(\frac{4}{3}-2)^2=\frac{4}{27}$.

Again $f'(2)=0$ and when $x>2$, $f'(2)>0$. So, as $f'(x)$ assumes values from values less than 2 to values greater than 2, through the value 2, $f'(x)$ assumes positive values from negatives values. Thus $f(x)$ is minimum when $x=2$ and the minimum value is $(2-1)(2-2)^2=0$.

Example 16. Can $f(x)$ assume a maximum or a minimum value if $f'(c)$ does not exist?

Taking $f(x)=|x|$ and $c=0$, show the validity of your answer.

[C. U. B. Sc. 1987]

"A function $f(x)$ may possess a maximum or minimum at $x=c$ even if $f'(c)$ does not exist". We show below the validity of the statement taking $f(x)=|x|$ and $c=0$.

Now if $f(x)=|x|$, then the left hand derivative $f'(0-)$ of $|x|$ at $x=0$

$$=\underset{h\to 0-}{Lt}\frac{f(0+h)-f(0)}{h}=\underset{h\to 0-}{Lt}\frac{f(h)-f(0)}{h}$$

$$=\underset{h\to 0-}{Lt}\frac{-h-0}{h}=\underset{h\to 0-}{Lt}\frac{-h}{h}=\underset{h\to 0-}{Lt}(-1)=-1$$

[$\because$ When $h<0$, $|h|=-h$]

The right hand derivative $f'(0+)$ of $|x|$ at $x=0$

$$=\underset{h\to 0+}{Lt}\frac{f(0+h)-f(0)}{h}=\underset{h\to 0+}{Lt}\frac{|h|-|0|}{h}$$

$$=\underset{h\to 0+}{Lt}\frac{h-0}{h}=\underset{h\to 0+}{Lt}(1)=1.$$

So, $f'(0+)\neq f'(0-)$.

So, $f'(0)$ does not exist. But if $h>0$ be small at pleasure, $f'(0-)<0$ and $f'(0+)>0$.

So $f(0)=0$ is the minimum value of $f(x)$

Example 17. x and y are two real variables and $x>0$, $xy=1$. Find the minimum value of $x+y$. [I. I. T. 1981]

For maximum or minimum, $f'(x)=0$.

So, $2\{(x-a_1)+(x-a_2)+\cdots+(x-a_n)\}=0$.

or $nx=a_1+a_2+\cdots+a_n$ or, $x=\dfrac{a_1+a_2+\cdots+a_n}{n}$

which is the arithmetic mean of $a_1, a_2, \cdots, a_n$.

Also $f''(x)=2\{1+1+\cdots+1\}=2n>0$ [$\because n>0$] for all x and so when $x=\dfrac{a_1+a_2+\cdots+a_n}{n}$

$\therefore$ $f(x)$ is minimum when x is the arithmetic mean of $x_1, x_2, x_3, \cdots\cdots, x_n$.

Example 21. Show that of all rectangles inscribed in a circle, the area of the square is maximum.

O is the centre and r is the radius of a given circle. $ABCD$ is a rectangle inscribed in this circle. Let OF and OE be parallel to AB and BC and they intersect BC and CD at F and E respectively. Let $\angle COF=\theta$.

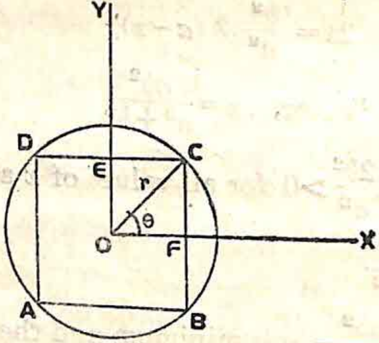


Fig. 3.3

$$\therefore \quad AB=2OF=2OC\cos\theta=2r\cos\theta.$$
$$BC=2CF=2OC\sin\theta=2r\sin\theta.$$

So, if S denote the area of the rectangle $ABCD$, then $S=AB.BC=2r\cos\theta.2r\sin\theta=2r^2\sin 2\theta$

[Here for different rectangles inscribed in the circle, one will get different values of θ and so different values of S]

$$\therefore \quad \frac{dS}{d\theta}=4r^2\cos 2\theta.$$

For maximum or minimum values of S, $\dfrac{dS}{d\theta}=0$

or, $4r^2 \cos 2\theta = 0$ or, $\cos 2\theta = 0$ or $2\theta = \frac{\pi}{2}$ i.e., $\theta = \frac{\pi}{4}$.

Again $\frac{d^2S}{d\theta^2} = -8r^2 \sin 2\theta$;

when $\theta = \frac{\pi}{4}$, $\frac{d^2S}{d\theta^2} = -8r^2 \sin \frac{\pi}{2} = -8r^2 < 0$

So when $\theta = \frac{\pi}{4}$, S is maximum. Again, when $\theta = \frac{\pi}{4}$,

$AB = 2r \cos \frac{\pi}{4} = 2r.\frac{1}{\sqrt{2}} = \sqrt{2}\, r.$ and $BC = 2r \sin \frac{\pi}{4}$

$= 2r.\frac{1}{\sqrt{2}} = \sqrt{2}r.$ $\therefore$ $AB = BC$ and the rectangle $ABCD$ is a square.

Example 22. If AB is a chord of a circle and C is any point on the circumference of the circle, then the area of $\triangle ABC$ is maximum when it is isosceles. [c. f. I. I. T. 1983]

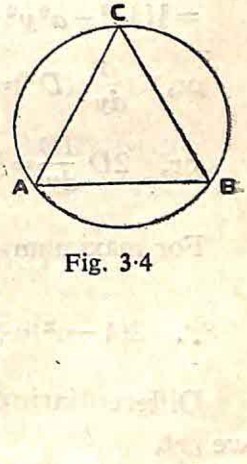


Fig. 3·4

As $\angle ACB$ is an angle of arc ACB, so for any position of the point C on this arc $\angle ACB$ is constant.

Let $\triangle$ denote the the area of the triangle ABC. $\therefore$ $\triangle = \triangle ABC = \frac{1}{2} AC.BC \sin C$

$= \frac{1}{2}.\ 2R \sin B.2R \sin A.\sin C$

$= 2R^2 \sin A \sin B \sin C$

$= R^2 \sin C\{\cos(A-B) - \cos(A+B)\}$

$= R^2 \sin C\{\cos(A-B) - \cos(\pi - C)\}$

$= R^2 \sin C\{\cos(2A - \pi + C) + \cos C\}$

[Here $B = \pi - A - C$; $A - B = 2A - \pi + C$]

$\therefore$ $\frac{d\triangle}{dA} = -R^2 \sin C \sin(2A - \pi + C).2$

For maximum and minimum values of $\triangle$,

$\therefore$ $\frac{d\triangle}{dA} = 0$ or, $\sin(2A - \pi + C) = 0$.

or, $\sin(A-B) = 0$ $\therefore$ $A - B = 0$ or, $A = B$.

Again $\frac{d^2\triangle}{dA^2} = -R^2 \cos(2A - \pi + C) = -4R^2 \cos(A-B)$

$= -4R^2$ [when $A = B$] >0.

So when $A = B$, the area of the triangle is maximum. Also when $A = B$, then $BC = AC$ and the triangle is isosceles.

If C is situated on the opposite side of AB, at C' (say), then $\sin C' = \sin(\pi - C) = \sin C = \text{constant}$. So for any position of the point C, the area of the triangle ABC will be maximum when the triangle is isosceles.

Example 23. Determine the point of the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$, nearest to the point $(0, -2)$.

If D be the distance of the point (x, y) on the curve $4x^2 + a^2y^2 = 4a^2$……(i) from the point $(0, -2)$, then

$$D^2 = x^2 + (y+2)^2 = \frac{a^2(4-y^2)}{4} + (y+2)^2$$

$\left[\because (x, y)\right.$ is a point on the curve-(i), so $4x^2 + a^2y^2 = 4a^2$

or, $\left. x^2 = \dfrac{a^2(4-y^2)}{4}\right]$

$$= \tfrac{1}{4}\{4a^2 - a^2y^2 + 4y^2 + 16y + 16\} = \tfrac{1}{4}\{(4-a^2)y^2 + 16y + 4a^2 + 16\}.$$

or, $\dfrac{d}{dy}(D^2) = \tfrac{1}{4}\{2(4-a^2)y + 16\}$

or, $2D\dfrac{dD}{dy} = \tfrac{1}{4}\{2(4-a^2)y + 16\}$……(ii)

For maximum or minimum values of D, $\dfrac{dD}{dy} = 0$

$\therefore\ 2(4-a^2)y + 16 = 0$ or, $y = \dfrac{8}{a^2-4}$

Differentiating both sides of equation-(ii) with respect to y we get,

$$2\left(\frac{dD}{dy}\right)^2 + 2D\frac{d^2D}{dy^2} = \tfrac{1}{4}\{2(4-a^2)\}$$

When $y = \dfrac{8}{a^2-4}$, then $\dfrac{dD}{dy} = 0$ and $D > 0$.

so, then $\dfrac{d^2D}{dy^2} = \dfrac{1}{4D}(4-a^2) < 0$ $[\because a^2 > 4]$

So, when $y = \dfrac{8}{a^2-4}$, then the distance of the point (x, y) from the point $(0, -2)$ is maximum.

Again, when $y=\frac{8}{a^2-4}$, then $4x^2+\frac{64a^2}{(a^2-4)^2}=4a^2$ [From (i)]

or, $x^2=a^2-\frac{16a^2}{(a^2-4)^2}=\frac{a^2(16-8a^2+a^4-16)}{(a^2-4)^2}=\frac{a^2 \,.\, a^2\,(a^2-8)}{(a^2-4)^2}$

$\therefore \quad x=\frac{a^2\sqrt{a^2-8}}{a^2-4}$.

So, the required point is $\left(\frac{a^2\sqrt{a^2-8}}{a^2-4}, \frac{8}{a^2-4}\right)$.

Example 24. The distance of a point moving along a straight line from a fixed point of the straight line is x cm. at time 't', where $x=t^4-5t^3+6t^2+22t+7$.

When will be the velocity of the particle least ? Find the least velocity.

If the velocity of the particle 't' seconds after start be v, then

$$v=\frac{dx}{dt}=4t^3-15t^2+12t+22$$

So, we are to find the least value of v.

Now, $\frac{dv}{dt}=12t^2-30t+12=6\,(2t^2-5t+2)=6\,(2t-1)\,(t-2)$

For maximum and minimum values of v,

$\frac{dv}{dt}=0$ i.e., $t=\frac{1}{2}$ or, $t=2$. Again $\frac{d^2v}{dt^2}=6(4t-5)$

When $t=\frac{1}{2}$, $\frac{d^2v}{dt^2}=6\,(2-3)=-18<0$.

So, when $t=\frac{1}{2}$, v is maximum.

When $t=2$, $\frac{d^2v}{dt^2}=6(8-3)=18>0$.

So, when $t=2$, then the value of v is minimum and least in this case. Putting $t=2$, in the expression for v, we find the least velocity is $4.2^3-15.2^2+12.2+22=32-60+24+22=18$.

Example 25. A window is in the shape of a rectangle surmounted by a semicircle. If the total perimeter of the window be 10 metres what should be the height and width of the rectangular portion, so that maximum light may be admitted through the window ?

Let the width and height of the window be $2x$ metres and y metres. So the radius of the semicircular portion is x metres and its perimeter πx metres. Hence the total perimeter of the window is $2x+2y+\pi.x=10$ (given)

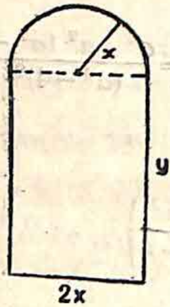


Fig. 3.5

$$\text{or,}\quad y=5-\frac{(\pi+2)x}{2}$$

Now, if A denotes the area of the window,

$$\text{Then } A=2xy+\frac{\pi x^2}{2}$$

$$=2x\left\{5-\frac{(\pi+2)\,x}{2}\right\}+\frac{\pi x^2}{2}=10x-(\pi+2)\,x^2+\frac{\pi x^2}{2}$$

$$\therefore\quad \frac{dA}{dx}=10-2\,(\pi+2)\,x+\pi x=10-(\pi+4)\,x.$$

For the maximum and minimum values of A, $\frac{dA}{dx}=0$

or, $10-(\pi+4)=0$ or $x=\frac{10}{\pi+4}$

Again $\frac{d^2A}{dx^2}=-(\pi+4)<0$ for all x and so, when $x=\frac{10}{\pi+4}$.

So, when $x=\frac{10}{\pi+4}$, then A will be maximum and hence maximum light will be admitted through the window.

In this case the width of the window$=2x=\frac{20}{\pi+4}$ metre

$$\text{Height of the window}=y=5-\frac{(\pi+2)\,x}{2}=5-\frac{\pi+2}{2}\cdot\frac{10}{4+\pi}$$

$$=\frac{10}{\pi+4}\text{ metres.}$$

Also the radius of the semicircular portion is $x=\frac{10}{\pi+4}$ metres.

Example 26. The intensity of light varies inversely as the square of the distance from the source. If two lights are 150m apart and one light is 8 times as strong as the other, where should an object be placed between the lights to have the least illumination.

[If the intensity of light of a source at a distance x from the source be $\frac{k}{x^2}$, then the intensity of a source 8 times stronger than the former will be $\frac{8k}{x^2}$.]

Let the two sources of light be at A and B and the object be placed at C on the straight line AB and $AC = x$.

$\therefore\ BC = 150 - x.$

So if the source at A be weaker, then the total intensity at C is

$$I = \frac{k}{x^2} + \frac{8k}{(150-x)^2}. \quad \text{Now } \frac{dI}{dx} = -\frac{2k}{x^3} + \frac{16k}{(150-x)^3}.$$

For maximum and minimum values of I,

$$\frac{dI}{dx} = 0\,; \quad \text{or,} \quad -\frac{2k}{x^3} + \frac{16k}{(150-x)^3} = 0$$

$$\text{or,} \quad \frac{1}{x^3} = \frac{8}{(150-x)^3} \quad \text{or,} \quad \frac{1}{x} = \frac{2}{150-x}$$

$$\text{or,} \quad 150 - x = 2x \quad \text{or,} \quad 3x = 150 \quad \text{or,} \quad x = 50.$$

Also $\frac{d^2I}{dx^2} = \frac{6k}{x^4} + \frac{48k}{(150-x)^4} > 0$ for all x and so when $x = 50$.

Hence I is minimum when $x = 50$.

So, to be least illuminated the object should be placed at a distance 50 metres from the weaker source.

Example 27. The rate of change of the total cost of production of a commodity due to a change in the quantity of production of the commodity is called the marginal cost of production of the commodity.

The cost of production of x tons of a commodity is Rs. R where $R = \frac{1}{10}x^3 - 5x^2 + 10x + 5$.

What quantity of the commodity should be produced so that

(i) The marginal cost will be least,

(ii) The average variable cost will be least ?

[c.f. C. U. B. Com. '80]

(i) Let the marginal cost of production of x tons be y.

$$\therefore\ y = \frac{dR}{dx} = \frac{d}{dx}\left(\tfrac{1}{10}x^3 - 5x^2 + 10x + 5\right) = \tfrac{3}{10}x^2 - 10x + 10.$$

So, we are to find the least value of y.

Now $\dfrac{dy}{dx}=\frac{8}{5}x-10.$

For maximum and minimum values of $y, \dfrac{dy}{dx}=0$

or, $\frac{8}{5}x-10=0$ or, $x=\frac{50}{8}$.

Again, $\dfrac{d^2y}{dx^2}=\frac{8}{5}>0$, for all values of x and so when $x=\frac{50}{8}$ i.e. marginal cost will be minimum i.e. least when $\frac{50}{8}$ tons of the commodity will be produced.

(ii) Again if u be the average variable cost then

$$u=\frac{\text{total variable cost of production of } x \text{ tons}}{x}$$

$$=\frac{1}{10}x^2-5x+10 \quad \therefore \quad \frac{du}{dx}=\frac{x}{5}-5.$$

For maximum or minimum values of $u, \dfrac{du}{dx}=0.$

or, $\frac{1}{5}x-5=0$ or, $x=25$.

Also $\dfrac{d^2u}{dx^2}=\frac{1}{5}>0$ for all x and so when $x=25$.

So, the average variable cost will be least when $x=25$ i.e., when 25 tons of the commodity will be produced.

Example 28. Show that of all right circular cylinders with given total surface area, the volume of that one will be greatest whose height is equal to the diameter of the base.

Let v, s, r and h denote respectively the volume, total surface area, radius of the base and height of a right circular cylinder.

$\therefore$ $v=\pi r^2h$ and $s=2\pi rh+2\pi r^2=2\pi r(h+r)$.

Here s is given i.e. constant and $h+r=\dfrac{s}{2\pi r}$.

$$\therefore h=\frac{s}{2\pi r}-r=\frac{s-2\pi r^2}{2\pi r}$$

$$\therefore \quad v=\pi r^2 \cdot \left(\frac{s-2\pi r^2}{2\pi r}\right)=\frac{sr}{2}-\pi r^3.$$

Now $\dfrac{dv}{dr}=\dfrac{s}{2}-3\pi r^2.$

For maximum or minimum values of v, $\frac{dv}{dr}=0$.

or, $\frac{s}{2}=3\pi r^2$ or, $\pi r(h+r)=3\pi r^2$ or, $h+r=3r$ or, $h=2r$

Again $\frac{d^2v}{dr^2}=-6\pi r<0$, for all positive values of r

and so when $r=\frac{h}{2}>0$.

So when $h=2r$ i.e., when height of the cylinder is equal to the diameter of the base of the cylinder, then, the volume of the cylinder is maximum i.e. greatest.

EXERCISE 3

1. Find the values of x for which the following functions possess maximum or minimum values. Also find these values.

(i) x^3+3x^2-9x+4 (ii) $x^3+\frac{1}{x^3}$ (iii) $x^3-6x^2+11x-6$.

(iv) $4-x^2+2x$ [C.U. B. Sc. '82] (v) $\frac{x^2-7x+6}{x-10}$

2. Show the function $x^4-8x^3+22x^2-24x+15$ possesses maximum or minimum values when $x=1$, 2 or 3. Find the maximum and minimum values.

3. Show that the function $x^3-6x^2+12x-3$ does not possess any maximum or minimum value.

4. Show that the function $16x^5-30x^4+20x^2+7$ is a maximum at $x=-\frac{1}{2}$ and a minimum at $x=0$; but neither maximum nor minimum at $x=1$.

5. Find the maximum value of $\sin x+\cos x$ and also the value of x for which it is maximum.

[Hints : Put $a=b=1$ in Ex-3, Examples 3]

6. Show that the two functions (i) $\frac{\cos(x+a)}{\cos(x+b)}$ and

(ii) $\frac{ax+b}{a'x+b'}$ have no maximum or minimum value.

7. Find the maximum values of the functions (i) $\sin x+\sin^2 x$ and (ii) $\sin x+\cos^2 x$.

8. Find the maximum or minimum values of the functions (i) $1+2\sin x+3\cos^2 x$ (ii) $\sin^p x\cos^q x$ in the interval $0\leqslant x\leqslant\frac{\pi}{2}$.

9. Show that in the interval $0\leq x\leq\frac{\pi}{2}$, the function $\dfrac{x}{1+x\tan x}$ is maximum when $x=\cos x$.

10. Find the maximum or minimum values of the function $\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x$ in the interval $0\leq x\leq\pi$.

11. Examine whether the function $2x+\cot^{-1}x$ possesses any maximum or minimum value.

12. Examine whether the functions

(i) $\sin x-x+\dfrac{x^3}{6}$ (ii) $\cos x-1+\dfrac{x^2}{2}$ possess any maximum or minimum at $x=0$.

13. Find the maximum value of $\sqrt{3}\sin x+3\cos x$. Show that the function is maximum at $x=\dfrac{\pi}{6}$.

14. $y=a(1-\cos t)$, $x=a(t-\sin t)$; show that when $t=\pi$, then y is maximum.

15. Find the maximum or minimum values of $\sin x+\cos 2x$ in the interval $0<x<2\pi$.

16. Determine when will the function $\sin 3x-3\sin x$ will be greatest or least in the interval $0<x<2\pi$ [C. U. B. Sc. '81]

17. Find the maximum or minimum values of the function $(\sin x)^{\sin x}$ in the interval $0<x<\pi$.

18. If $x>0$, find the maximum value of $\dfrac{\log x}{x}$ and the minimum value of $x\log x$.

19. Show that the maximum value of $\left(\dfrac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$.

20. If $y=\dfrac{ax+b}{(x-1)(x-4)}$ be maximum at $x=-1$, find the values of a and b if the maximum value be 4.

21. Find the maximum and minimum values of $x^4-4x^3-2x^2+12x+1$ in the interval $(-2, 5)$.

22. Show that if the sum of two numbers be constant, then their product will be maximum when the numbers are equal.

23. Divide the number 12 into two parts so that the sum of the square of one part and two times the other part is least.

24. If $xy=c^2$, find the maximum or minimum values of $ax+by$.

25. Find the positive number, so that the sum of the number and its reciprocal is least.

26. If $x+y=c$, find the minimum value of

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=1.$$

27. (i) Show that of all rectangles of given area the perimeter of a square is least [C.U. B.Sc. 1984]

(ii) Show that of all rectangles with a given perimeter the square is of maximum area.

28. Show that the area of a quadrilateral with sides of given length will be maximum if the quadrilateral is concyclic.

29. Show that of all isosceles triangles inscribed within a circle, one with the maximum area is equilateral.

30. The sum of the hypotenuse and another side of a right angled triangle is given. Show that the area of the triangle will be maximum when the angle included between these sides is $\frac{\pi}{3}$.

31. Show that the maximum value of $(x+1)^4\ (x-2)^3$ is maximum when $x=-1$ and minimum when $x=\frac{5}{7}$.

32. The distance x of a particle moving along a straight line from a fixed point of the straight line at time t after start is given by $x=t^4-6t^3+12t^2+4t+7$. When will the velocity of the particle be least ? Find the value of the least velocity.

33. A person being in a boat a miles from the nearest point of a beach, wishes to reach as quickly as possible a point b miles from that point along the shore. The ratio of his rate of walking

to his rate of rowing is sec α. Prove that he should land at a distance $b-a \cot \alpha$ from the place to be reached.

34. Prove that of all isosceles triangles with given radius (r) of the inscribed circle the one with least perimeter is of perimeter $6r\sqrt{3}$.

35. Find the length and breadth of a rectangular plot of ground containing A square ft. which requires the least amount of fencing to enclose it and divide it into two parts by a fence parallel to one side.

36. The response R of a body is found to the dosage x of a drug administered according to the equation,

$$R=kx^2-\frac{x^3}{3}\ (k>0).$$

Show that the body would response so long as $0<x<2k$. Find also the amount of dosage at which the rate of response would be maximum.

37. An orange grower finds that an orange tree produces (on average) 400 oranges per year, if not more than 16 trees are planted in a unit area. For each additional tree planted per unit area, the grower finds that the yield decreases by 20 oranges per tree.

How many trees should the grower plant per unit area to maximize the yield ?

38. The consumption of an engine varies as the square of the velocity ; when velocity is 16 k.m. per hour, then the expense per hour is Rs. 48. If the other expenses per hour be Rs. 300, find the velocity for the least expense.

39. The base of a water tank open at the top is a square. If the volume of the tank be constant, show that the cost of painting the outside of the tank will be least if the height of the tank be equal to the width of the base.

40. A radio manufacturer produces x number of radios per weak and sells each radio at a price $p=2\left(100-\frac{x}{4}\right)$. If the cost

of production of x radios per week be Rs. $\left(120+\frac{x^2}{2}\right)$, then show that the profit will be maximum if 40 radios are manufactured in a week. Also determine the weekly profit.

41. Cans, cylindrical in shape and open at the top are to be manufactured. Show that the least amount of tin will be required if the height of the can be equal to the radius of the base.

42. Show that the volume of the cone with maximum volume that can be placed within a given sphere is $\frac{8}{27}$ of the volume of the sphere.

CHAPTER FOUR

DETERMINATION OF AREA

§ 4·1. *In the Integral Calculus portion of this book it has been shown that the area of the space enclosed by the x-axis, the two ordinates $x=a$ and $x=b$ $(a<b)$ and the curve $y=f(x)$ is*

$$\int_a^b f(x)\,dx \quad \text{or} \quad \int_a^b y\,dx.$$

Similarly, the area enclosed by the y-axis, the ordinates $y=c$ and $y=d$ $(c<d)$ and the curve $x=f(y)$ is

$$\int_c^d f(y)\,dy \quad \text{or,} \quad \int_c^d x\,dy.$$

EXAMPLES 4

Example 1. Determine by integration the measure of the area enclosed by the x-axis, the ordinate $x=2$ and the straight line $y=x$.

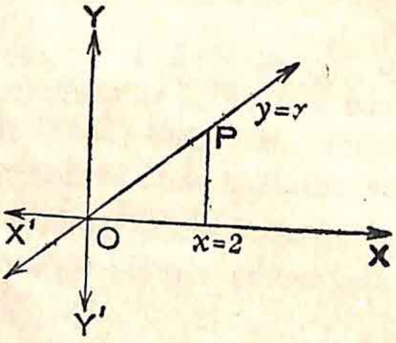


Fig. 4·1

Here, $f(x)=x$. The limits of x are from 0 to 2. Hence the required area

$$=\int_0^2 y\,dx=\int_0^2 f(x)\,dx=\int_0^2 x\,dx=\left[\frac{x^2}{2}\right]_0^2=2 \text{ sq. units.}$$

Example 2. Find the area enclosed by the straight line $\frac{x}{a}+\frac{y}{b}=1$ and the axes of co-ordinates and justify the formula, "the area of a triangle$=\frac{1}{2}\times$base$\times$height sq. units."

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{or,} \quad y=b\left(1-\frac{x}{a}\right)=\frac{b}{a}\,(a-x)$$

The limits of x in the area enclosed by the straight line and the axes of co-ordinates are from 0 to a

Hence the required area

$$=\int_0^a y\,dx=\int_0^a \frac{b}{a}(a-x)dx=\frac{b}{a}\left[ax-\frac{x^2}{2}\right]_0^a=\frac{b}{a}\left[a^2-\frac{a^2}{2}\right]$$

$$=\frac{b}{a}\,\frac{a^2}{2}=\frac{1}{2}ab \text{ sq. units.}$$

Now, if the straight line $\frac{x}{a}+\frac{y}{b}=1$ intersects the axes of co-ordinates at the points A and B, then the area is $\triangle ABO$.

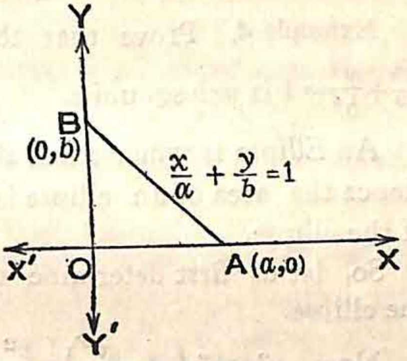


Fig. 4.2

Now from the formula, area of a triangle $=\frac{1}{2}\times$ base $\times$ height, we obtain

$$m\triangle ABC=\tfrac{1}{2}OA.OB$$

The co-ordinates of A are $(a, 0)$ and those of B are $(0, b)$. So, $OA=a$, $OB=b$.

$\therefore\ m\triangle ABC=\frac{1}{2}ab$ sq. units.

Hence the formula is verified.

Example 3. Determine the area enclosed by a parabola and its latus-rectum.

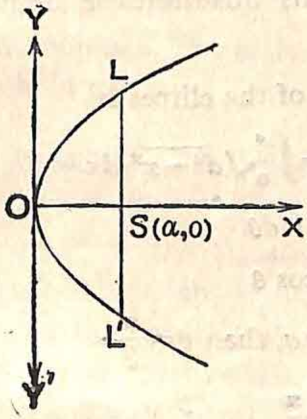


Fig. 4·3

Let the equation of the parabola be $y^2=4ax$.

The equation of the latus-rectum of the parabola is $x=a$ and it intersects the parabola at the points $(a, 2a)$ and $(a, -2a)$ and the x axis at the point $(a, 0)$.

Now we are to determine the area $LOL'SL$. You know that a parabola is symmetrical about its axis (which is the x-axis in this case). Hence the required area $=$twice the area $LOSL$ i.e. $2m$ area $(LOSL)$.

Now, the area $LOSL$ is the area enclosed by x-axis, the parabola $y^2=4ax$ and the ordinates $x=0$ and $x=a$.

Hence m area $(LOSL)=\int_0^a y\,dx$.

Now, for the portion of the parabola above the x-axis, $y=2\sqrt{ax}$.

$$\therefore \text{ Required area}=2\int_0^a y\,dx=2\int_0^a 2\sqrt{ax}\,dx$$

$$=4\sqrt{a}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a=4\sqrt{a}.\tfrac{2}{3}a^{\frac{3}{2}}=\tfrac{8}{3}a^2 \text{ sq. units}$$

Example 4. Prove that the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab sq. units.

An Ellipse is symmetrical about both its major and minor axes. Hence the area of an ellipse is four times the area of a quadrant of the ellipse.

So, let us first determine the area of the first quadrant of the ellipse.

$$\text{Now } \quad y^2=b^2\left(1-\frac{x^2}{a^2}\right)=\frac{b^2}{a^2}(a^2-x^2)$$

$$\therefore \quad y=\pm\frac{b}{a}\sqrt{a^2-x^2}.$$

For, the first quadrant y is positive.

$\therefore \quad y=\frac{b}{a}\sqrt{a^2-x^2}$. Again, for this quadrant the limits of x are 0 and a.

Hence the area of the first quadrant of the ellipes is

$$A=\int_0^a y\,dx=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx.$$

Now, let $x=a\sin\theta$. $\therefore \quad dx=a\cos\theta\,d\theta$.

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=a\cos\theta$$

and when $x=0$ then $\theta=0$; when $x=a$, then $\theta=\frac{\pi}{2}$.

$$\therefore \quad A=\frac{b}{a}\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\,d\theta=ab\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta.$$

$$=ab\int_0^{\frac{\pi}{2}} \tfrac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$=\frac{ab}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\frac{ab}{2}\cdot\frac{\pi}{2}=\frac{\pi ab}{4} \text{ sq. units.}$$

∴ The measure of the whole area enclosed by the ellipse $=4A=4.\ \frac{\pi ab}{4}=\pi ab$ sq. units.

Example 5. The radius of *a* circle is r. Show that the area enclosed by the circle is πr^2.

Let the centre of the circle be the origin and its equation be $x^2+y^2=r^2$.

Now, for the first quadrant of the circle, $y=\sqrt{a^2-x^2}$ and the limits of x are 0 and r.

∴ Required area $=4\int_0^r y\, dx=4\int_0^r \sqrt{r^2-x^2}\, dx$

$$=4\ .\ \pi\,\frac{r^2}{4} \text{ (as in example 4)}=\pi r^2.$$

Note : If we put $b=a$ in the integral $4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx$ for determination of the area of an ellipse, we get the area of the circle of radius a.

Example 6. A plane area is bounded by $y=x^2$, $y=0$ and $x=1$; find its area. [H. S. 1978]

$y=x^2$ is the equation of a parabola whose vertex is the orgin and which opens in the positive direction of the y-axis ; $y=0$ is the x-axis and $x=1$ is the equation of a straight line parallel to the y-axis. The straight line $x=1$ intersects the parabola at the point (1, 1) [obtained by solving the equations]. The shaded region enclosed by the parabola, the x-axis and the straight line $x=1$ is shown in the figure. So our required area is the area enclosed by the curve $y=x^2$, the x-axis and the ordinates $x=0$ and $x=1$.

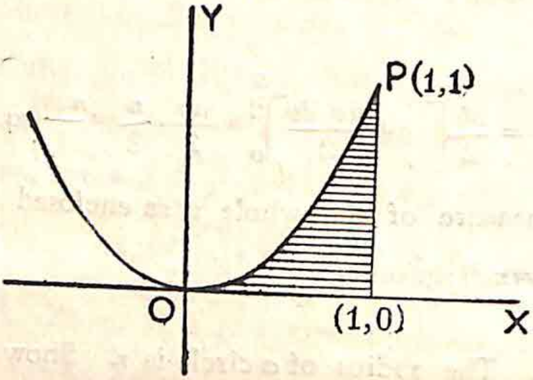


Fig. 4·4

So the required area$=\int_0^1 y\,dx=\int_0^1 x^2\,dx=\left[\frac{x^3}{3}\right]_0^1$

$=\frac{1}{3}$ sq. units.

Example 7. A plane area is bounded by the curve $y=x\,(4-x)$ and the x-axis ; find its area. [H. S. 1979]

The equation of the x-axis is $y=0$.

So putting $y=0$ in the equation $y=x\,(4-x)$ ... ... (i)

We get $x=0$ or 4 i.e., the curve (i) intersects the x-axis at the points (0, 0) and (4, 0).

Hence the required area is the region enclosed by the curve $y=x\,(4-x)$, the x-axis and the ordinates $x=0$ and $x=4$ and is equal to $\int_0^4 y\,dx=\int_0^4 x(4-x)\,dx=\int_0^4 (4x-x^2)\,dx=\left[2x^2-\frac{x^3}{3}\right]_0^4$

$=32-\frac{64}{3}=\frac{32}{3}$ square units.

Example 8. Find the area enclosed between the hyperbola $xy=k^2$, the x-axis and the ordinates $x=a$ and $x=b$.

Here the required area$=\int_a^b y\,dx=\int_a^b \frac{k^2}{x}dx$

$=k^2\Big[\log x\Big]_a^b=k^2\Big(\log b-\log a\Big)=k^2\log\frac{b}{a}$ sq. units.

Example 9. Find the area enclosed by the x-axis and an wave of the curve $y=\sin x$.

The curve $y=\sin x$ intersects the x-axis at the points $(n\pi, 0)$ $[n=0, \pm 1, \pm 2, \ldots\ldots]$; $(0, 0)$ and $(0, \pi)$ are two consecutive points of intersection. It is convenient to find the area of the portion between these points.

Here the required area $=\int_0^\pi y\,dx=\int_0^\pi \sin x\,dx.$

$$=\Big[-\cos x\Big]_0^\pi=-\cos\pi+\cos 0=-(-1)+1=2 \text{ sq. units.}$$

Example 10. Find the measure of the area enclosed by the curves $y=f_1(x)$, $y=f_2(x)$ and the two ordinates $x=a$ and $x=b$.

Let A and B be the two points $(a, 0)$ and $(b, 0)$ and the ordinates at these points intersect the curve $y=f_1(x)$ at the points E and F and the curve $y=f_2(x)$ at the points C and D.

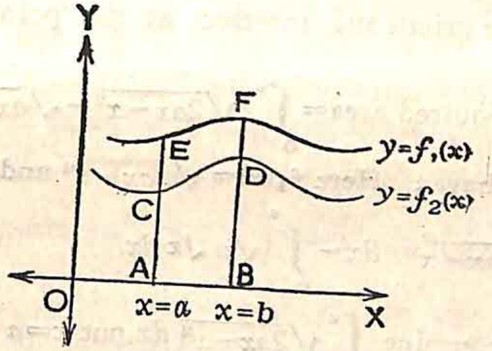


Fig. 4·5

Now, measure of the area $CDFE$

$$=m\,(\text{Area } ABFE)-m\,(\text{Area } ABDC)$$

$$=\int_a^b f_1(x)\,dx-\int_a^b f_2(x)\,dx=\int_a^b \{f_1(x)-f_2(x)\}\,dx.$$

Example 11. Find the area enclosed by the parabola $y^2=4ax$ and the straight line $y=x$.

Putting $y=x$ in the equation of the parabola we get, $x^2=4ax$, $\therefore\ x=0$ or, $4a$. So, $y=0$ or, $4a$.

So, the straight line $y=x$ intersects the parabola $y^2=4ax$ at the points $(0, 0)$ and $(4a, 4a)$.

Hence the required area $=\int_0^{4a}(\sqrt{4ax}-x)\,dx$

[∵ from the equation of the parabola $y=\sqrt{4ax}$.]

$$=\int_0^{4a} 2\sqrt{a}\sqrt{x}\,dx-\int_0^{4a} x\,dx.$$

$$=2\sqrt{a}.\tfrac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{4a}-\left[\frac{x^2}{2}\right]_0^{4a}=\frac{4\sqrt{a}}{3}.(8a\sqrt{a})-8a^2$$

$=\frac{32a^2}{3}-8a^2=\frac{8}{3}a^2$ square units.

Example 12. Find the area enclosed above the x-axis between the circle $x^2+y^2=2ax$ and the parabola $y^2=ax$.

The parabola $y^2=ax$ and the circle $x^2+y^2=2ax$ touch each other at the origin and intersect at the point (a, a) above the x-axis.

Hence the required area $=\int_0^a\{\sqrt{2ax-x^2}-\sqrt{ax}\}\,dx.$

[See Ex. 8 above ; Here $f_1(x)=\sqrt{2ax-x^2}$ and $f_2(x)=\sqrt{ax}$]

$$=\int_0^a \sqrt{2ax-x^2}\;dx-\int_0^a \sqrt{a}\sqrt{x}\,dx.$$

Now, to determine $\int_0^a \sqrt{2ax-x^2}\,dx$ put $x=a\,(1-\cos\theta)$

$\therefore\;\; dx=a\sin\theta\,d\theta.$

$\sqrt{2ax-x^2}=\sqrt{2a^2\,(1-\cos\theta)-a^2\,(1-\cos\theta)^2}$

$=a\sqrt{2-2\cos\theta-1+2\cos\theta-\cos^2\theta}=a\sqrt{1-\cos^2\theta}=a\sin\theta.$

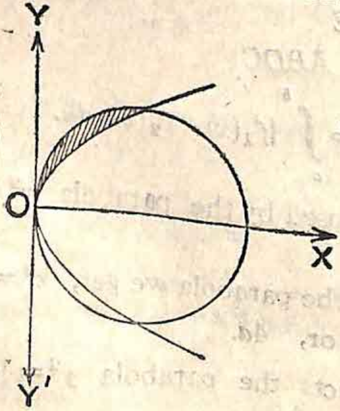


Fig. 4·6

when $x=0$, then $\theta=0$

and when $x=a$, then $\theta=\frac{\pi}{2}$.

$$\therefore\;\int_0^a \sqrt{2ax-x^2}\,dx$$

$$=\int_0^{\frac{\pi}{2}} a^2\sin^2\theta\,d\theta.$$

$$=a^2\int_0^{\frac{\pi}{2}}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta.$$

$$=\frac{a^2}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\frac{\pi a^2}{4}.$$

and $\int_0^a \sqrt{x}\,dx=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a=\frac{2}{3}a^{\frac{3}{2}}$

$\therefore\ \sqrt{a}\int_0^a \sqrt{x}\,dx=\sqrt{a}.\frac{2}{3}a^{\frac{3}{2}}=\frac{2}{3}a^2.$

Hence required area $=\frac{\pi a^2}{4}-\frac{2}{3}a^2=a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$ sq. units.

Example 13. Shade the portion of the area above the x-axis bounded by the parabola $y^2=4x$ and the circle $(x-4)=4\cos\theta$, $y=4\sin\theta$ and obtain this area by integration. [H. S. 1984]

The equation of the circle is $x-4=4\cos\theta$, $y=4\sin\theta$.

$\therefore\ (x-4)^2+y^2=16\cos^2\theta+16\sin^2\theta$

$=16(\cos^2\theta+\sin^2\theta)=16.1=16.$

or, $x^2-8x+16+y^2=16$ or, $x^2+y^2=8x.$

So this example and example 12 are the same (here $a=4$) and so the required area $=a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)=16\left(\frac{\pi}{4}-\frac{2}{3}\right)$ sq. units.

Example 14. Find the measure of the area enclosed by the curve $y^2=x^3$ and the straight line $y=x$.

Putting $y=x$ in the equation $y^2=x^3$, we get $x^2=x^3$

or, $x^3-x^2=0$ or, $x^2(x-1)=0$.

$\therefore\ x=0,\ 0,\ 1$. The corresponding values of y are 0, 0 and ± 1.

But, $x=1$, $y=-1$ does not satisfy the equation $y=x$. Hence the straight line $y=x$ touches the curve $y^2=x^3$ at the point (0, 0) and intersects it at the point (1, 1).

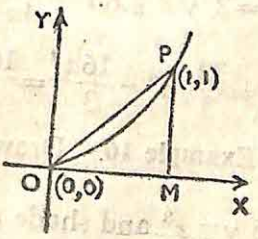


Fig. 4.7

Let P be the point (1, 1) and $\overline{PM}$ be perpendicular on the axis of x.

Hence required area

$=$(Area of $\triangle OPM$) $-$ (the area enclosed by the curve $y^2=x^3$, the x-axis and the ordinates $x=0$ and $x=1$) $=\Delta-A$ (say).

Now $\Delta = \frac{1}{2} OM.PM$ sq. units $= \frac{1}{2}.1.1 = \frac{1}{2}$ sq. units

and $A = \int_0^1 x^{\frac{3}{2}}\, dx$ [for the portion of the curve above the x-axis $y = +x^{\frac{3}{2}}$].

$$\left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 = \frac{2}{5} \cdot 1 \text{ sq. units} = \frac{2}{5} \text{ sq. units.}$$

$\therefore$ required area $= \frac{1}{2} - \frac{2}{5} = \frac{11}{10}$ sq. units.

Example 15. Show that the area enclosed between the curves $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3} a^2$. [C. U.]

The curve $y^2 = 4ax$ and $x^2 = 4ay$ intersect each other at the points $(0, 0)$ and $(4a, 4a)$.

Hence the required area $=$ (Area enclosed by the curve $y^2 = 4ax$, the x-axis and the ordinates $x = 0, x = 4a$) $-$ (Area enclosed by the curve $x^2 = 4ay$, the x-axis and the ordinates $x = 0$, $x = 4a$)

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= 2\sqrt{a}.\tfrac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= 2\sqrt{a}.\tfrac{2}{3}.8a^{\frac{3}{2}} - \frac{1}{4a}\,\frac{64a^3}{3}$$

$$= \frac{32}{3}a^2 - \frac{16a^2}{3} = \frac{16a^2}{3} \text{ square units.}$$

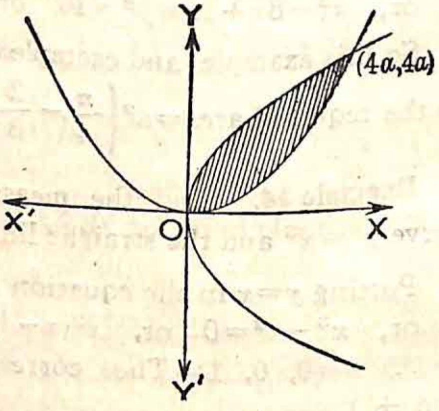


Fig. 4·8

Example 16. Draw the sketch graphs of the functions $y = x^2$ and $y = x^3$ and shade the areas of $\int_0^1 x^2 dx$ and $\int_0^1 x^3 dx$ what will be the value of the area enclosed by these two curves ? [H.S. 1980]

The graph of the function $y = x^2$ is the parabola through the points C, O, B, P. The graph of $y = x^3$ is the curve through the points *P*, *O*, *Q*. The two curves intersect at the origin *O* and the point *P* (1, 1). *PM* is drawn from *P*, perpendicular to the *x*-axis.

$\int_0^1 x^2dx$ is the area enclosed by the curve $y=x^2$, the *x*-axis and the ordinates $x=0$ and $x=1$. It is the region *OBPMO*.

$\int_0^1 x^3dx$ is the area enclosed by the curve $y=x^3$, the *x*-axis and the ordinates $x=0$ and $x=1$. It is the region *PAOMP*.

Hence the area enclosed by the two curves is the shaded region *OAPBO*.

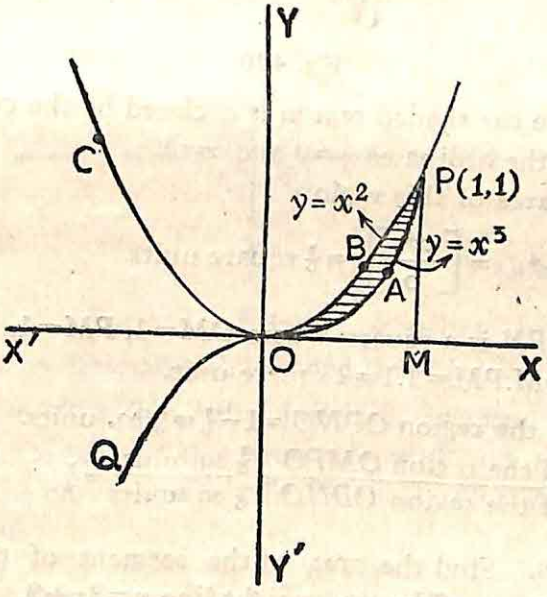


Fig. 4.9

The area of this enclosed region

$$=\int_0^1 x^2dx-\int_0^1 x^3dx=\left[\frac{x^3}{3}\right]_0^1-\left[\frac{x^4}{4}\right]_0^1=\frac{1}{3}-\frac{1}{4}$$

$$=\frac{1}{12} \text{ square unit.}$$

Example 17. Let *PM* and *PN* be the perpendiculars from the point *P*(1, 1) on the curve $y=x^4$ upon the co-ordinate axes *OX*, and

OY respectively. Show that the two areas into which the rectangle *OMPN* is divided by the curve are as 1 : 4.

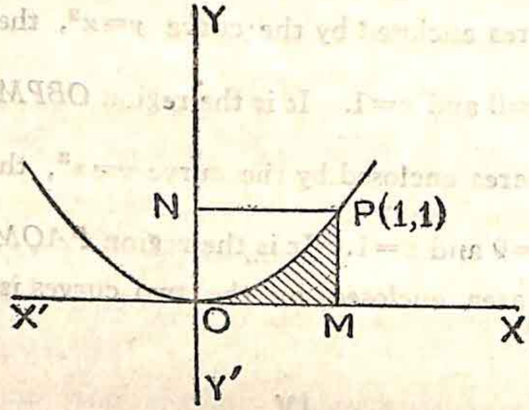


Fig. 4.10

In the figure the shaded region is enclosed by the curve $y=x^4$, the x-axis and the ordinates $x=0$ and $x=1$.

Hence the area of this region

$$=\int_0^1 x^4 dx=\left[\frac{x^5}{5}\right]_0^1=\tfrac{1}{5} \text{ square units.}$$

Again *OMPN* is a square where $OM=1$, $PM=1$. So, area of the square$=OM.PM=1.1=1$ square unit.

So, area of the region $OPNO=1-\frac{1}{5}=\frac{4}{5}$ sq. units.

$$\therefore \frac{\text{Area of the region } OMPO}{\text{Area of the region } OPNO}=\frac{\frac{1}{5}\text{ sq. units}}{\frac{4}{5}\text{ sq. units}}=\tfrac{1}{4}.$$

Example 18. Find the area of the segment of the parabola $y=x^2-5x+15$ cut off by the straight line $y=3x+3$.

[Joint Entrance 1984]

The equation of the parabola is

$y=x^2-5x+15$ or, $y-15+\frac{25}{4}=(x-\frac{5}{2})^2$

or, $y-\frac{35}{4}=(x-\frac{5}{2})^2$ or $Y=X^2$ where $X=x-\frac{5}{2}$, $Y=y-\frac{25}{4}$.

So, the vertex of the parabola is the point ($X=0$, $Y=0$) i.e. ($x=\frac{5}{2}$, $y=\frac{35}{4}$). The latus rectum of the parabola is of length 1 unit. The axis of the parabola is parallel to the y-axis and the parabola opens towards the positive direction of the y-axis.

The straight line $y=3x+3$ is the straight line joining the points $(-1, 0)$ and $(0, 3)$.

Putting $y=3x+3$, in the equation of the parabola we get $3x+3=x^2-5x+15$.

or, $x^2-8x+12=0$ or $(x-2)(x-6)=0$

$\therefore$ $x=2$ or 6 $\therefore$ $y=9$ or 21.

So the straight line $y=3x+3$ intersects the parabola $y=x^2-5x+15$ at the points $P\ (2, 9)$ and $Q\ (6, 21)$.

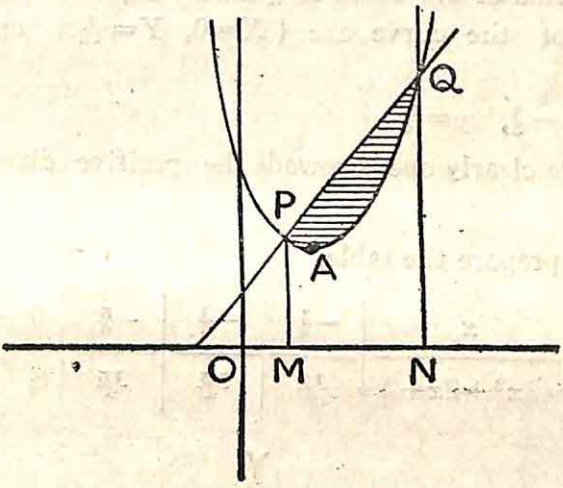


Fig. 4.11

So, we are to determine the shaded are *PAQP*.

The region $PAQP$=(Area bounded by the straight line $y=3x+3$, the x-axis and the ordinates $x=2$ and $x=6$) − (Area bounded by the curve $y=x^2-5x+15$, the x-axis and the ordinates $x=2$ and $x=6$)

$$=\int_2^6 (3x+3)dx-\int_2^6 (x^2-5x+15)\,dx$$

$$=\left[\frac{3x^2}{2}+3x\right]_2^6-\left[\frac{x^3}{3}-\frac{5x^2}{2}+15x\right]_2^6$$

$$=\{(54+18)-(6+6)\}-\{(72-90+90)-(\tfrac{8}{3}-10+30)\}$$

$$=(72-12)-(72-\tfrac{68}{3})=\tfrac{68}{3}-12=\tfrac{32}{3}\text{ square units}$$

Example 19. Draw a graph of the curve $y=3x^2+2x+4$. Shade the area enclosed by the curve, the x-axis and the lines

$x=-1$ and $x=3$. Find the area of the shaded region by the method of integration. [H. S. 1981]

$y=3x^2+2x+4$ is the equation of a parabola. The equation can be written as $y-4=3\left(x^2+\frac{2}{3}x\right)$ or, $y-4+\frac{1}{3}=3\left(x^2+\frac{2}{3}x+\frac{1}{9}\right)$

or, $y-\frac{11}{3}=3\left(x+\frac{1}{3}\right)^2$ or, $Y=3X^2$ where $x+\frac{1}{3}=X$, $y-\frac{11}{3}=Y$.

So, the vertex of the curve is the point $(X=0,\ Y=0)$ i.e.,

$\left(x+\frac{1}{3}=0,\ y-\frac{11}{3}=0\right)$ or, $\left(x=-\frac{1}{3},\ y=\frac{11}{3}\right)$. The length of the latus-rectum of the curve is $\frac{1}{3}$ unit. So, the co-ordinates of the focus of the curve are $\left(X=0,\ Y=\frac{1}{12}\right)$ or, $\left(x+\frac{1}{3}=0,\ y-\frac{11}{3}=\frac{1}{12}\right)$

or, $\left(x=-\frac{1}{3},\ y=\frac{15}{4}\right)$.

The curve clearly opens towads the positive direction of the y-axis.

Now we prepare the table,

x	$-\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{5}{6}$	0	1	-1
$y=3x^2+2x+4$	$\frac{11}{3}$	$\frac{15}{4}$	$\frac{15}{4}$	4	9	5

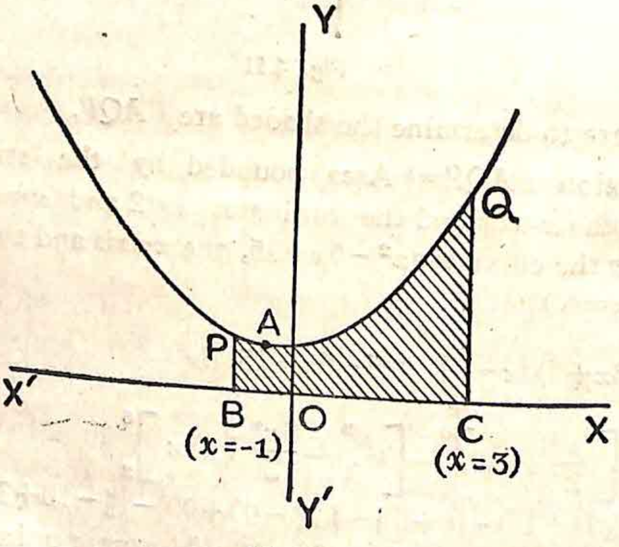


Fig. 4·12

With respect to suitable rectangular axes of co-ordinates points are plotted with corresponding values of x and y from the above table as co-ordinates. These points are joined free hand and we get the parabola as shown in the figure.

Again, when $x=-1$, then $y=3.\ (-1)^2+2(-1)+4=5$

When $x=3$, $y=3\ (3)^2+2.3+4=37$

So, the ordinates $x=-1$ and $x=3$ intersect the parabola at the points $(-1, 5)$ and $(3, 37)$.

So the area bounded by the curve, the x-axis and the ordinates $x=-1$ and $x=3$ is the shaded region in the figure 4·12.

$$\text{The area of this region}=\int_{-1}^{3} y dx=\int_{-1}^{3}(3x^2+2x+4)dx$$

$$=\Big[x^3+x^2+4x\Big]_{-1}^{3}=(27+9+12)-(-1+1-4)=52 \text{ square units.}$$

§ 4·7. Sign of an area.

In examples 4 and 5 of the last section in determining areas of an ellipse and a circle we have first determined the areas lying in the first quadrant. As the areas are situated in the region above the x-axis, the areas have been found positive. As the ellipse or circle are symmetrical about both the axes of co-ordinates, the areas enclosed by the ellipse or circle are obtained by multiplying the corresponding areas in the first quadrant by 4. If an area is completely below the x-axis, then as $y=f(x)$ is negative, the area will be negative. But actually area has no regard to sign. So, though from the algebraic point of view one can obtain negative area, actually we do not attach any sign to areas. Follow the following examples.

Example 1. Find the area enclosed by the parabola

$y=x^2-5x+6$ and the x-axis.

The parabola $y=x^2-5x+6$ intersects the x-axis at points $(2, 0)$ and $(3, 0)$. In the figure the shaded region is the area between the parabola and the x-axis.

The region is situated wholly below the x-axis.

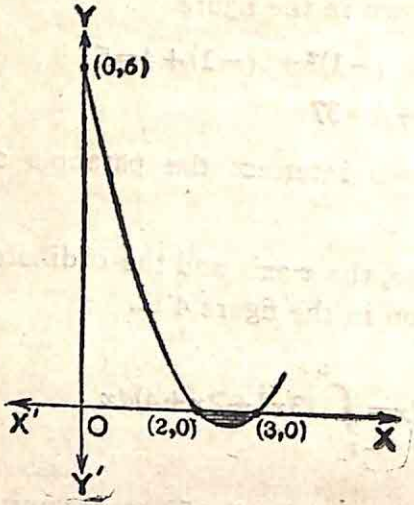


Fig. 4·13

Now the required area

$$= \int_2^3 (x^2 - 5x + 6)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right]_2^3$$

$$= \left(9 - \frac{45}{2} + 18\right)$$

$$- \left(\frac{8}{3} - 10 + 12\right)$$

$$= \tfrac{9}{2} - \tfrac{14}{3} = -\tfrac{1}{6}.$$

Here the area is found to be negative and this is correct according to the algebraic point of view. But actually the area is without sign and it is $\frac{1}{6}$ square units.

Example 2. Determine the area enclosed between the curve $y = 4x(x-1)(x-2)$ and the x-axis.

The curve $y = 4x(x-1)(x-2)$ intersects the x-axis at the points $(0, 0)$, $(1, 0)$ and $(2, 0)$. The total area enclosed between the x-axis and the curve has two portions (i) the portion OAB situated wholly above the x-axis and (ii) the portion BCD situated wholly below the x-axis.

Now the area of the region OAB

$$= \int_0^1 4x(x-1)(x-2)\, dx = 4\int_0^1 (x^3 - 3x^2 + 2x)\, dx$$

$$= 4\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = 4(\tfrac{1}{4} - 1 + 1) = 1 \text{ square unit}$$

The area of the region $BCD = \int_1^2 4x(x-1)(x-2)\, dx$

$$= 4\left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 = -1 \text{ square unit.}$$

If we now neglect the sign of this portion *BCD* below the x-axis then its actual area is 1 square unit and the total required area is $(1+1)=2$ square units.

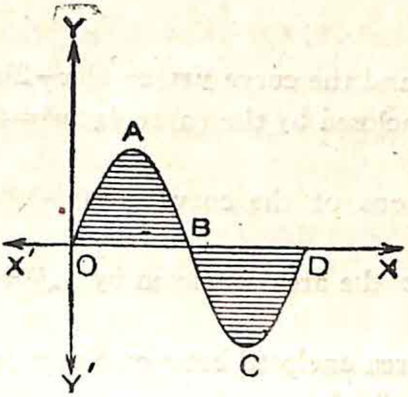


Fig. 4·14

Now integrating from $x=0$ to $x=2$ we get the required area as $\int_0^2 4x(x-1)(x-2)\,dx$

$$=4\left[\frac{x^4}{4}-x^3+x^2\right]_0^2=4(4-8+4)=0.$$

So, though the actual area of the region is 2 square units it is zero from the algebraic point of view.

EXERCISE 4

1. Determine the area enclosed between the straight line $y=3x$, the x-axis and the ordinates $x=1$ and $x=2$.

2. Determine the area enclosed between the straight line $y=-x$, the x-axis and the ordinates $x=1$ and $x=2$.

Determine the area enclosed by the following

3. The curve $y=2x+3x^2$, $y=0$, $x=0$, and $x=4$.

4. $y=\cos x$, $y=0$, $x=0$, $x=\frac{\pi}{2}$.

5. The curve $y=\frac{1}{x^2}$, the x-axis and the ordinates $x=1$ and $x=2$.

6. The curve $y=\sin 2x$ $y=0$, $x=0$, $x=\frac{\pi}{6}$.

7. The curve $y=\sin x+\cos x$, the x-axis and the ordinates $x=0$ and $x=\frac{\pi}{4}$.

8. The x-axis and the curve $y=(x-1)(x-2)$.

9. The area enclosed by the parabola $y^2=4ax$ and the double ordinate $x=h$.

10. The portions of the curve $y=(x-2)(x-3)$ intercepted by the x-axis.

11. Prove that the area enclosed by $\sqrt{x}+\sqrt{y}=\sqrt{a}$ and the x-axis is $\frac{1}{6}a^2$.

12. Find the area enclosed between the parabola $y^2+x-1=0$ and the circle $x^2+y^2=1$.

13. Find the area common to the two ellipses $ax^2+by^2=1$ and $bx^2+ay^2=1$.

14. Find the area enclosed by the parabolas $y^2=4x$ and $y^2=x$ and the ordinates $x=1$ and $x=4$. [Burdwan 1970]

15. (a) Find the measure of the area enclosed by the parabolas $y^2=6x$ and $x^2=6y$. [C.U. 1972]

(b) Prove that the curves $y^2=4x$ and $x^2=4y$ divide the square formed by $y=0$, $y=4$, $x=0$ and $x=4$ into three equal parts.

16. Find the areas enclosed by the curve $y=x^3$ and the straight line $y=2x$.

17. Find the areas enclosed between the curve $y=\cos x$, the x-axis (i) between the ordinates $x=0$ and $x=\frac{\pi}{2}$ (ii) between the ordinates $x=\frac{\pi}{2}$ and $x=\pi$ and (iii) between the ordinates $x=0$ and $x=\pi$.

18. Find the areas enclosed between the curve $y=\sin x$, the x-axis (i) between the ordinates $x=0$ and $x=\pi$ (ii) between the ordinates $x=\pi$ and $x=2\pi$ and (iii) between the ordinates $x=0$ and $x=2\pi$.

19. Shade the area enclosed by the two parabolas $y^2=4x$ and $x^2=4y$ and find, by integration, the area of the shaded region.

[H. S. 1982 ; Joint Entrance 1983]

20. Shade the area bounded by $y^2=8x$ and $y=x$ along positive direction of x-axis and use integration to find the area of that part.

21. Shade the area enclosed by the two parabolas $y^2=8x$ and $x^2=y$ and find, by integration its area. [H. S. 1987]

22. Calculate the area enclosed by the ellipse $4x^2+9y^2=36$ and the x-axis. [Joint Entrance 1985]

23. Find the area enclosed by the two curves $y^2-4x-4=0$ and $y^2+4x-4=0$.

24. Find by integration the area of the figure bounded by $y^2=2x+1$ and $x-y-1=0$. [H. S. 1983]

The equation of the curve is

$$y^2=2x+1\cdots\cdots(\text{i}) \quad \text{or,} \quad y^2=2(x+\tfrac{1}{2}).$$

The vertex of the parabola is the point $(-\frac{1}{2}, 0)$ and the axis is the x-axis ; the curve open towards the positive direction of the x-axis. The focus being the origin. Again the straight line $x-y-1=0\cdots\cdots$(ii) is the straight line joining the points (1, 0) and $(-1, 0)$ intersecting the x-axis at the point R (1, 0). From the equation (ii) we get $x=y+1$.

Putting this value of x in equation (i) we get, $y^2=2y+3$

or, $y^2-2y-3=0$ or, $(y-3)(y+1)$ $\therefore$ $y=3$ and -1

$\therefore$ $x=4$ and 0. So the parabola (i) and the straight line (ii) intersect at the points $P(4, 3)$ and $Q(0, -1)$.

From P, PM is drawn perpendicular on the x-axis.

So, the required area = Area of the region $PAQP$ = (Area of the region $PAMP$) + (Area of region $OAQO$) − (Area of $\triangle PRM$) + (Area of $\triangle QOR$) = (Area enclosed by the curve $y^2=2x+1$, the x-axis, and the ordinates $x=-\frac{1}{2}$, $x=4$) + (Area enclosed by the curve $y^2=2x+1$, the x-axis and the ordinates $x=-\frac{1}{2}$, $x=0$) $-\frac{1}{2}\,RM.PM+\frac{1}{2}\,OR.OQ$

$$=\int_{-\frac{1}{2}}^{4}\sqrt{2x+1}\,dx+\int_{-\frac{1}{2}}^{0}\sqrt{2x+1}\,dx-\tfrac{1}{2}.(4-1).3+\tfrac{1}{2}.1.1.$$

$$=\left[\frac{1}{2}\cdot\frac{2}{3}(2x+1)^{3/2}\right]_{-\frac{1}{2}}^{4}+\left[\frac{1}{2}\cdot\frac{2}{3}(2x+1)^{3/2}\right]_{-\frac{1}{2}}^{0}-\tfrac{1}{2}.33+\tfrac{1}{2}.1.1$$

$$=\tfrac{1}{3}(27-0)+\tfrac{1}{3}(1-0)-\tfrac{9}{2}+\tfrac{1}{2}=9+\tfrac{1}{3}-\tfrac{9}{2}+\tfrac{1}{2}=5\tfrac{1}{3} \text{ square units.}$$

Note : Actually the area $OAQO$

$$=\int_{-\frac{1}{2}}^{0}-y\,dx \quad [\text{ As } y \text{ is below the } x\text{-axis }]$$

But we have considered the magnitude of the area and taken $\int_{-\frac{1}{2}}^{0} y\,dx.$

25. Shade the the area bounded by $x^2=2y$ and $x^2+y^2=8$ above the x-axis and use the method of integration to find the area of that part.

[H.S. 1986]

[Solving the equations $x^2=2y$ and $x^2+y^2=8$ we find that the parabola intersects the circle at the points $(-2, 2)$ and $(2, 2)$. The required area is the portion of the circle above the x-axis bounded by the parabola between the points $(-2, 2)$ and $(2, 2)$. This portion is symmetrical about the y-axis and so the required area

$$=2\left(\int_0^2\sqrt{8-x^2}\,dx-\int_0^2\frac{x^2}{2}dx\right)=\tfrac{2}{3}(3\pi+2) \text{ square units.}$$

CHAPTER V

APPLICATION OF CALCULUS IN DYNAMICS

§ 5.1. *It has been already said that there are vast and organised uses of calculus in different branches of science, Economics and other social sciences. In fact there are different branches of science whose basis is calculus. Dynamics is one such branch. In this chapter we shall discuss some applications of calculus in Dynamics. But before we proceed the students should be introduced with some elementary concepts.*

1. **Rest and motion :** A body is said to be at rest when it does not change its position ; a body is said to be in motion when it changes its position. In this universe every body is changing its position ; so nobody is at actual rest. By rest we actually mean relative rest ; a body is at rest (we generally omit the term relative) if it does not change its position with respect to its surroundings. Motion of a body is also studied with respect to its surroundings.

2. **Displacement** : The displacement of a body is its change of position in a definite direction and is measured by the length of the line segment joining its initial and final positions.

§ 5.2. *Speed and velocity.*

The rate of change of position of a body is its *speed* ; the rate of displacement of a body is its *velocity.*

When a body undergoes equal displacements in equal intervals of time, however small the equal intervals may be, i.e., if the body changes its position in the same direction through the same distance in equal intervals of time (however small the equal intervals may be), then the velocity of the body is said to be *uniform velocity.*

Again if a body traverses equal distances in equal intervals of time, however small the equal intervals may be, then the speed of the body is uniform. Let a particle starting from a point A on the circumference of a circle of radius r moves along the circumference of the circle with uniform speed. Let the particle reaches the extremity B of the diameter AB of the circle drawn through A after time 't'. Then the distance traversed by the particle in time t is πr and as the speed is uniform, so its speed is $\frac{\pi r}{t}$. In this case the displacement of the particle is $2r$ (length of the diameter AB). In this case the velocity of the particle is not uniform; for uniform velocity requires maintaining the same direction. In this case the direction of the body is changing at every instant. In this case $\frac{2r}{t}$ is the *average velocity* of the particle. Here if the speed of the particle be not uniform, then $\frac{\pi r}{t}$ is the average speed of the body.

Let a particle moves in the direction $\overrightarrow{AB}$ along the straight line AB. In this case the velocity of the body is positive in the direction $\overrightarrow{AB}$ and negative in the opposite direction $\overrightarrow{BA}$. As speed has no direction, so any question of positive or negative speed cannot arise.

§ 5·3. *Acceleration :*

The rate of change of velocity of a particle at any instant is called the acceleration of the particle at the instant. A particle moving along a curved path with uniform speed possesses acceleration; for the direction of the velocity of the particle is changing in every moment. If the acceleration of a particle remains unchanged during any interval of time, i.e. if the rate of change of velocity of the particle be the same in equal intervals of time (however small the equal intervals may be), then the acceleration of the particle is uniform acceleration. If the acceleration of a particle be not uniform, then it is non-uniform. The acceleration of a particle moving along a curved path is non-uniform as the direction of the acceleration also changes with the direction of velocity. Again the velocity of a particle moving along a straight line may be uniform or non-uniform.

If the rate of change of velocity be equal in equal intervals of time, however small the equal intervals may be, then the acceleration is uniform otherwise the acceleration is non-uniform.

When the velocity of a particle decreases, then the rate of change or decrease of the velocity is called retardation. Acceleration is the rate of change of velocity of a particle, no matter whether the velocity increases or decreases. So retardation is also acceleration ; acceleration is the general term where as retardation is a particular type of acceleration or in otherwords retardation is negative acceleration.

Let u and v be the initial and final velocities of a particle in an interval of time 't' ; $\frac{v-u}{t}$ is the average rate of change of velocity of the particle or average acceleration of the particle in the interval.

§ 5·4. *Units of velocity and acceleration* :

Velocity is the rate of displacement. Again the displacement of a body or particle is measured by length ; so the unit of velocity is related to length. Again the rate of displacement is measured with respect to time. So the unit of velocity is the length of displacement in unit time or displacement length/unit time. For example, 10 cm./second.

Again acceleration is the rate of change of velocity per unit time ; so in the unit of acceleration another unit of time is associated over and above the unit of time associated with the unit of velocity. For example, if the velocity of a particle at any instant be 10 cm. / second and the rate of change of velocity at the instant be 2 cm. per second, then acceleration is the rate of change of velocty per second at the rate of 2 cm. per second i.e., 2 cm. per second per second. The first per second relates to the velocity per second. i.e., rate of change of displacement and the second relates to the rate of change of velocity measured per second. The acceleration 2 cm. per second per second is frequently written as 2 cm./per sec^2.

We now list the units of velocity and acceleration in different systems of measurement.

In the M.K.S. (Metre-kilogram-second) system the unit of velocity is metre per second and the unit of acceleration is metre per second per second.

In the C.G.S. system the units of velocity and acceleration are centimetre per second and centimetre per second^2.

Foot per second and Foot per second per second are the F. P. S. units of velocity and acceleration.

Now a days M.K.S units are generally used ; but C. G. S and F. P. S. units are not obsolete.

§ 5·5. *Expressions for velocity and acceleration in terms of derivatives.*

In § 1·3. We have shown that the velocity v of a particle moving along a straight line at time t after start is $\frac{dx}{dt}$ or $\frac{ds}{dt}$ where x or s is the displacement of the particle during the interval. The corresponding expressions for acceleration are $\frac{dv}{dt}=\frac{d^2x}{dt^2}=v\frac{dv}{dx}$. Students are advised to study § 1·3 once again so that they can understand the subsequent discussions easily.

§ 5·6. *Formulas relating to the motion of a particle moving along a straight line with uniform acceleration.*

Let a particle starts moving along the straight line OX in the direction $\overrightarrow{OX}$ from the fixed point O of the straight line with initial velocity u and uniform acceleration f. At time 't' after start let the position of the particle be P so that $OP=s$. Then at this instant the acceleration of the particle is $\frac{d^2s}{dt^2}=v\frac{dv}{ds}=\frac{dv}{dt}$ (where v is the velocity of the particle at P).

$$\text{So, } \frac{dv}{dt}=f \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

Here f is constant as the particle moves with uniform acceleration.

or, $dv=f\,dt$ or, $\int dv=\int f\,dt$ or, $v=ft+c_1$.

Now when $t=0$, $v=u$ $\therefore$ $u=c_1$

$\therefore$ $v=u+ft$ ... ... (ii)

Again at time 't' the velocity of the particle is $\dfrac{ds}{dt}$.

$\therefore$ $\dfrac{ds}{dt}=u+ft$ [From (ii)] or, $ds=(u+ft)\,dt$

or, $\int ds=\int (u+ft)\,dt$ or, $s=ut+\frac{1}{2}ft^2+c_2$

when $t=0$, $s=0$. $\therefore$ $c_2=0$.

$\therefore$ $s=ut+\frac{1}{2}ft^2$ ... ... (iii)

Again if the acceleration at P is written as $v\dfrac{dv}{ds}$, then $v\dfrac{dv}{ds}=f$

or, $v\,dv=f\,ds$ or, $\int v\,dv=\int f\,ds$ or, $\dfrac{v^2}{2}=fs+c_3$.

when $s=0$, then $v=u$.

$\therefore$ $\dfrac{u^2}{2}=c_3$ $\therefore$ $\dfrac{v^2}{2}=fs+\dfrac{u^2}{2}$ or, $v^2=u^2+2fs$ ... ... (iv)

Again the distance moved by the particle in the t-th second of its motion is

$s_t=$ {distance traversed in the first t seconds}
$-$ {distance traversed in the first $(t-1)$ seconds}
$=\{ut+\frac{1}{2}ft^2\}-\{u(t-1)+\frac{1}{2}f(t-1)^2\}$
$=ut+\frac{1}{2}ft^2-ut+u-\frac{1}{2}ft^2+ft-\frac{1}{2}f$
$=u+ft-\frac{1}{2}f=u+\frac{1}{2}f(2t-1)$ ... ... (v)

The equations (ii), (iii), (iv) and (v) should be remembered as formulas. The formulas are listed below.

If u be the initial velocity of a particle moving along a straight line with uniform velocity f, v be the final velocity after t seconds, s, the displacement in the first t seconds and s_t be the distance moved in the t-th second of motion, then

$v=u+ft$; $s=ut+\frac{1}{2}ft^2$; $v^2=u^2+2fs$.
and $s_t=u+\frac{1}{2}f(2t-1)$.

Corollary : If in the above discussion instead of uniform acceleration f, the particle moves with uniform retardation f, then writing $-f$ in place of f in the above formulas we get

$v=u-ft$; $s=ut-\frac{1}{2}ft^2$; $v^2=u^2-2fs$ and $s_t=u-\frac{1}{2}f(2t-1)$.

EXAMPLES 5A

Example 1. The equation of motion of a particle moving along a straight line is given by $x=16t+5t^2$. Prove that the particle moves with uniform acceleration. [Cf. H. S. 1985]

Here $x=16t+5t^2$. $\therefore\ \frac{dx}{dt}=16+10t$ and $\frac{d^2x}{dt^2}=10$.

Hence the acceleration is 10 units which is a constant. So the particle moves with a uniform acceleration.

Example 2. The velocity of a particle moving in a straight line is given by $v=(1+t)$ km./second. Find the distance moved by the particle in the first 9 seconds of its motion.

[C. U. B. Sc. 1982]

If s be the distance traversed in t seconds after start, then the velocity at the t-th second, is by question

$$\frac{ds}{dt}=v=(1+t) \quad \text{or,} \quad ds=(1+t)\,dt.$$

$$\text{or,} \quad \int ds=\int(1+t)\,dt \quad \text{or,} \quad s=t+\tfrac{1}{2}t^2+c.$$

Now when $t=0$, then $s=0$ $\therefore\ c=0.$ $\therefore\ s=t+\frac{1}{2}t^2$.

$\therefore$ when $t=9$ then $s=9+\frac{1}{2}\cdot 9^2=9+\frac{81}{2}=49\frac{1}{2}$.

Hence the particle moves through $49\frac{1}{2}$ km. in the first 9 seconds.

Example 3. The distance x traversed by a particle in t seconds is given by $x=2+6t-3t^2$

Find its velocity 1 second after start. [C. U. B. Sc. 1984]

Here distance moved in t seconds is $x=2+6t-3t^2$.

Hence velocity in the t-th second $=\frac{dx}{dt}=6-6t$.

Putting $t=1$, we get velocity after 1 second is $6-6.1=0$.

Example 4. When the distances of a particle moving along a straight line, from a fixed point of the straight line is x ft., then the velocity of the particle is v ft./sec. If x and v are connected by the relation $v^2=x^2+2x+3$, then find the acceleration of the particle at a point 2 ft. from the fixed point.

Here the relation between x and v are $v^2=x^2+2x+3$.

Differentiating both sides with respect to x we get

$$2v\frac{dv}{dx}=2x+2 \quad \text{or,} \quad v\frac{dv}{dx}=x+1.$$

So, when $x=2$ (i.e., the distance of the particle from the fixed point is 2 ft., distance of unit being ft), the acceleration of the particle is $v\dfrac{dv}{dx}=2+1=3$. i.e., 3 ft./sec^2.

Example 5. A particle moves along a straight line according to the law $x=a\sin(\mu t+b)$ where x is the distance moved in t seconds. If v be the velocity at time 't' show that

$$v^2=(a^2-x^2)\,\mu^2.$$

$$x=a\sin(\mu t+b). \quad \therefore \quad \frac{dx}{dt}=a\mu\cos(\mu t+b).$$

i.e., the velocity of the particle at time 't' is $v=a\mu\cos(\mu t+b)$

$$\text{or,} \quad \frac{v}{\mu}=a\cos(\mu t+b)$$

$$\therefore \quad \frac{v^2}{\mu^2}+x^2=a^2\cos^2(\mu t+b)+a^2\sin^2(\mu t+b)$$

$$=a^2\{\cos^2(\mu t+b)+\sin^2(\mu t+b)\}=a^2.1=a^2.$$

$$\text{or,} \quad \frac{v^2+\mu^2x^2}{\mu^2}=a^2 \quad \text{or,} \quad v^2+\mu^2x^2=\mu^2a^2.$$

$$\text{or,} \quad v^2=\mu^2a^2-\mu^2x^2.=\mu^2(a^2-x^2).$$

Example 6. A particle moves from rest with an initial velocity and a uniform acceleration. Its distance after 4 seconds from start is 56 cms. If it goes through a distance of 88 cms. in the next 4 secs., find its initial velocity and acceleration.

Let u be the initial velocity and f be the constant acceleration.

According to the problem, the particle moves through distances 56 cms and $(56+88)=144$ cms in 4 seconds and 8 seconds respectively.

$$\therefore \quad 56=u\,.\,4+\tfrac{1}{2}f\,.\,4^2=4u+8f \quad \cdots \quad \cdots\text{(i)}$$

$$144=u\,.\,8+\tfrac{1}{2}f\,.\,8^2=8u+32f \quad \cdots \quad \cdots\text{(ii)}$$

Solving equations (i) and (ii) we get $u=10$ and $f=2$.

Hence the required initial velocity is 10 cm./sec. and 2 cms./sec^2.

Example 7. If a, b, c be the spaces described in the pth, qth, rth seconds by a particle starting with a given velocity and moving with uniform acceleration in a straight line, show that

$$a(q-r)+b(r-p)+c(p-q)=0.$$

Let u and f be the initial velocity and uniform acceleration of the particle. Now a=space described by the particle in the pth second.

$$\therefore \quad a=u+\tfrac{1}{2}f(2p-1)$$

Similarly $b=u+\frac{1}{2}f(2q-1)$ and $c=u+\frac{1}{2}f\ (2r-1)$

$$\begin{aligned}
\therefore \quad & a(q-r)+b(r-p)+c(p-q) \\
&=\{u+\tfrac{1}{2}f(2p-1)\}\ (q-r)+\{u+\tfrac{1}{2}f(2q-1)\ \}\ (r-p) \\
&\qquad +\{u+\tfrac{1}{2}f(2r-1)\ \}\ (r-p) \\
&=u\ (q-r+r-p+p-q)+\tfrac{1}{2}f\ \{(2p-1)\ (q-r) \\
&\qquad +(2q-1)(r-p)+(2r-1)\ (p-q)\} \\
&=u\ .\ 0+\tfrac{1}{2}f\ \{2\ (pq-pr+qr-pq+rp-pq) \\
&\qquad -(q-r+r-p+p-q)\} \\
&=0+\tfrac{1}{2}\ f\{\ 2\ .\ 0-0\ \}=0.
\end{aligned}$$

Example 8. A bullet fired into a target with a given velocity loses half its velocity after penetrating 3 cms. What further distance will it penetrate till it comes to rest. [H. S. 1981]

Let u be the velocity of the bullet at the instant, when it strikes the target and f be the retardation due to the resistance offered by the target.

After penetrating 3 cms, the bullet loses half of its velociry i.e. the velocity becomes $\frac{u}{2}$.

$$\therefore \quad \frac{u^2}{4}=u^2-2f.3 \qquad \text{or,} \quad 6f=\frac{3u^2}{4} \quad \text{or,}\ f=\frac{u^2}{8}$$

Let the bullet penetrate a distance s more before it comes to rest.

$$\therefore \quad 0=\frac{u^2}{2}-2f.s$$

[considering the motion of the bullet from the instant when its velocity is $\frac{u}{2}$ to the instant when it comes to rest]

$$\text{or,}\quad s=\frac{\frac{u^2}{4}}{2f}=\frac{u^2}{8f}=\frac{u^2}{8\cdot\frac{u^2}{8}}=1.$$

Hence the bullet will penetrate 1 cm. more till it comes to rest.

Example 9. A rifle bullet loses $\frac{1}{20}$ th of its velocity while penetrating an wooden plank. Find how many such planks shall the bullet penetrate till it comes to rest. [Assume that the resistance of the planks are same] [Joint Entrance 1987]

Let the thickness of each plank be s and u be the velocity of the bullet at the instant it strikes the first plank. As the resistances of the planks are the same, so the retardation of the bullet while penetrating the planks are the same and let the uniform retardation be f.

As the bullet loses $\frac{1}{20}$ th of its velocity, while penetrating each plank, so its velocity after penetrating the first plank is

$$u-\frac{u}{20}=\frac{19}{20}u.$$

$$\text{So}\quad \left(\frac{19}{20}u\right)^2=u^2-2f.s.\quad\text{or,}\quad 2fs=u^2-\frac{361}{400}u^2=\frac{39}{400}u^2.$$

Let the bullet comes to rest after penetrating n planks.

$$\therefore\quad 0=u^2-2f.ns=u^2-n.2fs.=u^2-n\cdot\frac{39}{400}u^2$$

$$\therefore\quad n=\frac{400}{39}=10\frac{10}{39}.$$

Hence the bullet will penetrate 10 planks and $\frac{10}{39}$ part of the 11 th plank, till it comes to rest.

Example 10. A particle starting from rest moves with alternate acceleration f and retardation f' during equal intervals of time t. Assuming f and f' uniform, show that the distance described at the end of 4 such intervals is $t^2(5f-3f')$

Let the distances moved by the particle during the four successive intervals be s_1, s_2, s_3, s_4 respectively.

$\therefore$ distance moved by the particle during the first interval is $s_1=\frac{1}{2}ft^2$ [As the particle starts from rest]. Also velocity of the particle at the end of this interval i.e., at the beginning of the second interval is ft.

$\therefore\ s_2=ft.t-\frac{1}{2}f't^2=t^2(f-\frac{1}{2}f')$.

Also the velocity at the end of the second interval i.e., at the beginning of the third interval$=ft-f't=(f-f')t$.

So the distance moved through in the third interval is

$$s_3=(f-f')t.t+\tfrac{1}{2}ft^2=t^2(\tfrac{3}{2}f-f')$$

Velocity of the particle at the end of this interval i.e., at the beginning of the 4th interval is $(f-f')t+ft=(2f-f')t$.

So, distance moved by the particle during the 4th interval is

$$s_4=(2f-f')t.t-\tfrac{1}{2}f't^2=t^2(2f-\tfrac{3}{2}f')$$

Hence total distance moved by the particle from start to the end of the fourth interval is $s_1+s_2+s_3+s_4$

$$=\tfrac{1}{2}ft^2+t^2(f-\tfrac{1}{2}f')+t^2(\tfrac{3}{2}f-f')+t^2(2f-\tfrac{3}{2}f')$$
$$=t^2\{f(\tfrac{1}{2}+1+\tfrac{3}{2}+2)-f'(\tfrac{1}{2}+1+\tfrac{3}{2})\}=t^2(5f-3f')$$

Example 11. A particle moving with uniform acceleration, describes in the last second of its motion $\frac{9}{25}$th of the whole distance. It started from rest, find how long it was in motion and through what distance did it move, if it described 6 cms. in the first second.

Let f be the uniform acceleration of the particle. As it describes 6 cms. in the first second, so $6=\frac{1}{2}f.1^2$ (as the particle starts from rest) or, $f=12$.

Let the particle was in motion for t seconds and s be the total distance traversed during this time.

$$\therefore\ s=\tfrac{1}{2}\,.\,12\,.\,t^2=6t^2 \qquad \cdots \qquad \cdots\text{(i)}$$

and as it describes $\frac{9}{25}$th of the total distance in the last second i.e., the t-th second

$$\text{so}\quad \frac{9}{25}\,s=\frac{1}{2}\cdot 12\,(2t-1)=6(2t-1)\cdots\cdots\text{(ii)}$$

From equation-(i) and equation-(ii) we get.

$$\frac{s}{\frac{9}{25}\,s}=\frac{6t^2}{6\,(2t-1)}\ \text{or,}\ \frac{25}{9}=\frac{t^2}{2t-1}$$

or, $9t^2=50t-25$ or, $9t^2-50t+25=0$

or, $9t^2-45t-5t+25=0$ or, $9t(t-5)-5(t-5)=0$
or, $(9t-5)(t-5)=0$ $\therefore$ $t=\frac{5}{9}$ or, 5.
But $t \geqslant 1$ $\therefore$ $t \neq \frac{5}{9}$. So $t=5$ i.e., the particle was in motion for 5 seconds and the distance moved by the particle during this time$=6.5^2$ [From (i)]$=150$ cms.

Example 12. A particle moving with uniform acceleration successively passes through the points P, Q and R. If $PQ=QR=b$ and times taken by the particle to move from P to Q and from Q to R be a and c respectively, then prove that the acceleration$=\dfrac{2b(a-c)}{ac(a+c)}$ [Joint Entrance 1987]

Let f be the uniform acceleration and u be the velocity at P. Considering motion from P to Q and P to R we get

$$b=PQ=ua+\frac{1}{2}fa^2 \cdots\cdots \text{(i)}$$

$$2b=PR=u\,.\,(a+c)+\tfrac{1}{2}f\,(a+c)^2 \cdots\cdots \text{(ii) respectively.}$$

[Time from P to R=(Time from P to Q)+(Time from Q to R) $=a+c$]

From (i) we get, $u=\dfrac{b-\frac{1}{2}fa^2}{a}=\dfrac{b}{a}-\frac{1}{2}fa.$

$\therefore$ From (ii) we get,

$$2b=\left(\frac{b}{a}-\tfrac{1}{2}fa\right)(a+c)+\tfrac{1}{2}f(a+c)^2$$

$$=\frac{b(a+c)}{a}+\tfrac{1}{2}f(a+c)(a+c-a)=(a+c)\left(\frac{b}{a}+\tfrac{1}{2}cf\right)$$

or, $\dfrac{2b}{a+c}=\dfrac{b}{a}+\dfrac{cf}{2}$ or, $\dfrac{2b}{a+c}-\dfrac{b}{a}=\dfrac{cf}{2}$

or, $\dfrac{b(2a-a-c)}{(a+c)a}=\dfrac{cf}{2}$ or, $f=\dfrac{2b(a-c)}{(a+c).ac}$.

Example 13. A train starting from rest moves from a station A and stops at another station B at a distance s. The train moves the first part with uniform acceleration u and the remaining portion with uniform retardation v. Show that the time taken to move from A to B is $\left\{2S\left(\dfrac{1}{u}+\dfrac{1}{v}\right)\right\}^{\frac{1}{2}}$

[Joint Entrance 1980]

Let the train move from A to C with uniform acceleration u and $AC=s_1$ and the velocity at C be ω. Also let $CB=s_2$

$\therefore\ s_1=\frac{1}{2}ut_1^2$……(i) and $\omega=ut_1$……(ii)

[As the train starts from rest from A, so its initial velocity is 0.]

Again the velocity of the train at B is 0. So considering motion of the train from C to B, we get,

$s_2=\omega t_2-\frac{1}{2}vt_2^2$……(iii) [where t_2 is the time from C to B], and $0=\omega-vt_2$……(iv)

From (iv) we get $v=\dfrac{\omega}{t_2}$.

$\therefore$ From (iii) we get $s_2=\omega t_2-\frac{1}{2}\dfrac{\omega}{t_2}\cdot t_2^2=\frac{1}{2}\omega t_2$

Again, from (ii) we get $u=\dfrac{\omega}{t_1}$.

From (i) we get $s_1=\dfrac{1}{2}\dfrac{\omega}{t_1}\cdot t_1^2=\frac{1}{2}\omega t_1$

Total distance $s=s_1+s_2=\frac{1}{2}\omega(t_1+t_2)=\frac{1}{2}\omega t$

where $t=t_1+t_2$ is the total time from A to B. $\therefore\ \omega=\dfrac{2s}{t}$.

Again from $u=\dfrac{\omega}{t_1}$ and $v=\dfrac{\omega}{t_2}$ we get $t_1=\dfrac{\omega}{u}$, $t_2=\dfrac{\omega}{v}$

$\therefore$ Total time $t=t_1+t_2=\omega\left(\dfrac{1}{u}+\dfrac{1}{v}\right)$

or, $t=\dfrac{2s}{t}\left(\dfrac{1}{u}+\dfrac{1}{v}\right)$ or, $t^2=2s\left(\dfrac{1}{u}+\dfrac{1}{v}\right)$

$\therefore\ t=\left\{2s\left(\dfrac{1}{u}+\dfrac{1}{v}\right)\right\}^{\frac{1}{2}}$.

Example 14. A point moving with uniform acceleration describes spaces s_1 and s_2 in successive intervals of time t_1, t_2. Prove that the acceleration is $\dfrac{2(s_2t_1-s_1t_2)}{t_1t_2(t_1+t_2)}$

Let the velocity of the particle at the beginning of the interval t_1 be u and f be the uniform acceleration.

Now considering the motion of the particle during the intervals t_1 and t_2 we get,

$$s_1 = ut_1 + \tfrac{1}{2}ft_1^2 \quad \dots \quad \dots \quad \text{(i)}$$

$$s_2 = (u+ft_1)t_2 + \tfrac{1}{2}ft_2^2 \quad \dots \quad \dots \quad \text{(ii)}$$

[velocity of the particle at the beginning of the interval t_2 is $(u+ft_1)$]

From (i) $\dfrac{s_1}{t_1} = u + \tfrac{1}{2}ft_1$ and

From (ii) $\dfrac{s_2}{t_2} = u + ft_1 + \tfrac{1}{2}ft_2$

$$\therefore \ \frac{s_2}{t_2} - \frac{s_1}{t_1} = \frac{1}{2}f(t_1+t_2)$$

$$\text{or,} \quad \frac{s_2t_1 - s_1t_2}{t_1t_2} = \frac{1}{2}f(t_1+t_2) \quad \therefore \ f = \frac{2(s_2t_1 - s_1t_2)}{t_1t_2(t_1+t_2)}.$$

Example 15. If a particle moving in a straight line with uniform acceleration describes equal distances in succesive times t_1, t_2, t_3 then prove that

$$\frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} = \frac{3}{t_1+t_2+t_3}.$$

Let f be the uniform acceleration of the particle and u be its velocity at the beginning of the interval t_1. Then its velocity at the beginning of the intervals t_2 and t_3 are $(u+ft_1)$ and $u+f(t_1+t_2)$ respectively. Let each equal distance be s. So, considering motions of the particle in the three successive intervals we get,

$$s = ut_1 + \tfrac{1}{2}ft_1^2 \quad \text{or,} \quad \frac{s}{t_1} = u + \tfrac{1}{2}ft_1$$

$$s = (u+ft_1)t_2 + \tfrac{1}{2}ft_2^2 \quad \text{or,} \quad \frac{s}{t_2} = u + ft_1 + \tfrac{1}{2}ft_2$$

$$\text{and } s = \{u + f(t_1+t_2)\}t_3 + \tfrac{1}{2}ft_3^2 \quad \text{or,} \quad \frac{s}{t_3} = u + f(t_1+t_2) + \tfrac{1}{2}ft_3$$

$$\therefore \ \frac{s}{t_1} - \frac{s}{t_2} + \frac{s}{t_3} = u + \tfrac{1}{2}ft_1 - u - ft_1 - \tfrac{1}{2}ft_2 + u + ft_1 + ft_2 + \tfrac{1}{2}ft_3$$

$$\text{or,} \quad s\left(\frac{1}{t_1} - \frac{1}{t_2} - \frac{1}{t_3}\right) = u + \tfrac{1}{2}f(t_1+t_2+t_3) \quad \dots \quad \dots \quad \text{(i)}$$

Again considering the motion of the particle during the total interval $(t_1+t_2+t_3)$ we get

$$3s=u(t_1+t_2+t_3)+\tfrac{1}{2}f(t_1+t_2+t_3)^2$$

$$\text{or,}\quad \frac{3s}{t_1+t_2+t_3}=u+\tfrac{1}{2}f\,(t_1+t_2+t_3)\quad \dots\quad \dots\qquad \text{(ii)}$$

So from (i) and (ii) we get

$$\therefore\quad s\left(\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}\right)=\frac{3s}{t_1+t_2+t_3}$$

$$\text{or,}\quad \frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}=\frac{3}{t_1+t_2+t_3}.$$

Example 16. A particle is moving with uniform acceleration along a straight line. The distances of the particle from a fixed point of the straight line at times t_1, t_2 and t_3 are x_1, x_2 and x_3 respectively. Prove that the acceleration of the particle is

$$2\left\{\frac{t_1(x_2-x_3)+t_2(x_3-x_1)+t_3(x_1-x_2)}{(t_2-t_3)\,(t_3-t_1)\,(t_1-t_2)}\right\}$$

Let the particle be moving along the straight line OX, O be the fixed point of the line so that $OP=x$ where P is the point of start and f be the uniform acceleration.

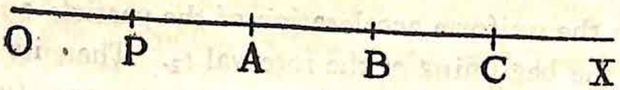


Fig. 5·1

$\therefore\ OA=x_1,\ OB=x_2,\ OC=x_3$

So if u be the initial velocity of the particle, then

$$PA=ut_1+\tfrac{1}{2}ft_1^2\quad\text{or,}\quad OA-OP=ut_1+\tfrac{1}{2}ft_1^2$$

$$\text{or,}\quad x_1-x=ut_1+\tfrac{1}{2}ft_1^2\quad \dots\quad \dots\qquad \text{(i)}$$

$$\text{Similarly,}\quad x_2-x=ut_2+\tfrac{1}{2}ft_2^2\quad \dots\quad \dots\qquad \text{(ii)}$$

$$\text{and}\quad x_3-x=ut_3+\tfrac{1}{2}ft_3^2\quad \dots\quad \dots\qquad \text{(iii)}$$

From (ii) − (iii), (iii) − (i) and (i) − (ii) we get respectively,

$$x_2-x_3=u(t_2-t_3)+\tfrac{1}{2}f(t_2^2-t_3^2)\quad \dots\quad \dots\qquad \text{(iv)}$$

$$x_3-x_1=u(t_3-t_1)+\tfrac{1}{2}f(t_3^2-t_1^2)\quad \dots\quad \dots\qquad \text{(v)}$$

$$\text{and}\quad x_1-x_2=u(t_1-t_2)+\tfrac{1}{2}f(t_1^2-t_2^2)\quad \dots\quad \dots\qquad \text{(vi)}$$

$$\therefore\quad t_1\,(x_2-x_3)+t_2\,(x_3-x_1)+t_3\,(x_1-x_2)$$

$$
\begin{aligned}
&= u\,\{t_1(t_2-t_3)+t_2(t_3-t_1)+t_3(t_1-t_2)\}\\
&\quad+\tfrac{1}{2}f\{t_1(t_2{}^2-t_3{}^2)+t_2(t_3{}^2-t_1{}^2)+t_3(t_1{}^2-t_2{}^2)\}\\
&= u\,.\,0+\tfrac{1}{2}f(t_2-t_3)(t_3-t_1)(t_1-t_2)\\
&= \tfrac{1}{2}f(t_2-t_3)(t_3-t_1)(t_1-t_2)
\end{aligned}
$$

$$\therefore\quad f=2\left\{\frac{t_1(x_2-x_3)+t_2(x_3-x_1)+t_3(x_1-x_2)}{(t_2-t_3)(t_3-t_1)(t_1-t_2)}\right\}.$$

[**Note :** Factorising the expression

$$t_1(t_2{}^2-t_3{}^2)+t_2(t_3{}^2-t_1{}^2)+t_3(t_1{}^2-t_2{}^2)$$

in cyclic order one gets the expression$=(t_2-t_3)(t_3-t_1)(t_1-t_2)$. This is remembered as a formula]

Example 17. A train starting from rest from a station stops in another station at a distance 2 kilometres in 4 minutes. The train moves first with uniform acceleration and then with uniform retardation y. Taking kilometre and minute as the units of distance and time respectively, prove that $\frac{1}{x}+\frac{1}{y}=4$.

Let the train move a distance s k.m. in t minutes with acceleration x km./min^2 and moves a distance $(2-s)$ km. in $(4-t)$ mins with retardation y km./(min^2). Let also the maximum velocity of the train be v km./min.

$$\therefore\quad v=xt \;\cdots\;\cdots\text{ (i) and } 0=v-y(4-t) \text{ or, } v=y(4-t)\;\cdots\;\cdots\text{ (ii)}$$

$$\therefore\quad \frac{1}{x}=\frac{t}{v}\text{ and }\frac{1}{y}=\frac{4-t}{v}$$

$$\therefore\quad \frac{1}{x}+\frac{1}{y}=\frac{t}{v}+\frac{4-t}{v}=\frac{4}{v}\quad\cdots\quad\cdots\quad\text{(iii)}$$

$$\text{Again } v^2=2xs\;\cdots\;\cdots\text{ (iv) and } 0=v^2-2y(2-s)\;\cdots\;\cdots\text{ (v)}$$

$$\therefore\quad \frac{1}{x}=\frac{2s}{v^2}\text{ and }\frac{1}{y}=\frac{2(2-s)}{v^2}$$

$$\therefore\quad \frac{1}{x}+\frac{1}{y}=\frac{2s}{v^2}+\frac{2(2-s)}{v^2}=\frac{4}{v^2}\quad\cdots\quad\cdots\quad\text{(vi)}$$

From (iii)$\div$(vi) we get $v=1$.

$\therefore$ From (iii) we get $\frac{1}{x}+\frac{1}{y}=4$.

Example 18. A particle moves along a straight line with initial velocity u. It successively passes the two halves of a given distance with accelerations f and f' respectively. Prove that the final velocity is the same as if the whole distance were traversed with uniform acceleration $\frac{1}{2}(f+f')$.

Let the given distance be $2s$ and v be the velocity of the particle after traversing the first half.

$$\therefore \quad v^2=u^2+2fs.$$

If ω be the final velocity, then the particle attains the velocity ω after traversing a distance s, starting with a velocity v.

$$\therefore \quad \omega^2=v^2+2f's=u^2+2fs+2f's=u^2+2(f+f')s$$

Next let the particle starts with the same initial velocity u and moves the whole distance $2s$ with acceleration $\frac{1}{2}(f+f')$. Then the final velocity in this case will be ω_1 where

$$\omega_1{}^2=u^2+2\cdot\tfrac{1}{2}(f+f')\cdot 2s=u^2+2(f+f')s=\omega^2$$

$$\therefore \quad \omega_1{}^2=\omega^2 \qquad \text{or,} \quad \omega_1=\omega$$

i.e., the final velocities are the same in both the cases.

Example 19. A particle moves with uniform acceleration along a straight line from A to B and its velocities at A and B are u and v respectively. Find the velocity of the particle at the middle point C of AB. If the time taken by the particle to move from A to C is twice the time taken to move from C to B, show that $v=7u$.

Let the distance $AB=2s$ and the constant acceleration be f.

$\therefore \; AC=BC=s$. As the velocity of the particle at B is v.

$$\therefore \quad v^2=u^2+2f\cdot 2s=u^2+4fs. \quad \therefore \quad f=\frac{v^2-u^2}{4f}.$$

Let the time taken by the particle to move from A to C be $2t$, then the time taken to move from C to B is t.

$\therefore$ Time taken to move from A to B is $3t$

$$\therefore \quad v=u+f\cdot 3t \quad \therefore \quad t=\frac{v-u}{3f}.$$

Hence if ω be the velocity at C, then

$$\omega=u+2ft=u+2f\cdot\frac{v-u}{3f}=\frac{u+2v}{3}$$

Again considering motions from A to C and A to B we get

$$s=u\,.\,2t+\tfrac{1}{2}f(2t)^2=2ut+2ft^2 \quad \cdots \quad \cdots \quad \text{(i)}$$

and $$2s=u\,.\,3t+\tfrac{1}{2}f(3t)^2=3ut+\tfrac{9}{2}ft^2 \quad \cdots \quad \cdots \quad \text{(ii)}$$

Subtracting equation-(i) from equation-(ii) we get, $s=ut+\frac{5}{2}ft^2$

or, $2ut+2ft^2=ut+\frac{5}{2}ft^2$ [From (i)]

or, $ut=\frac{1}{2}ft^2$ or, $ft=2u$. $\therefore$ $v=u+3ft=u+3\,.\,2u=7u$.

Example 20. The velocity of a train increases from 0 to v at the uniform rate f_1; then the velocity remains uniform for an interval and finally decreases to 0 at a uniform rate f_2. If s be the total distance described then show that the total time taken is

$$\frac{s}{v}+\frac{v}{2}\left(\frac{1}{f_1}+\frac{1}{f_2}\right).$$

Let the train starting from A with uniform acceleration f_1 comes to the place B after time t_1, the velocity of the train at B beving v. Also let $AB=s_1$.

$$\therefore\ v=f_1t_1 \quad \cdots \quad \cdots \quad \text{(i)}$$

and $$s_1=\tfrac{1}{2}f_1t_1^2=\tfrac{1}{2}vt_1 \quad [\because f_1t_1=v] \quad \cdots \quad \cdots \quad \text{(ii)}$$

Let now the velocity remains uniform from B to C and the time taken from B to C be t_2 and $BC=s_2$

$$\therefore\ s_2=vt_2 \quad \cdots \quad \cdots \quad \text{(iii)}$$

Now if the train moves from C to D in time t_3 with uniform retardation f_2 so that its velocity at D becomes 0 and $CD=s_3$, then

$$0=v-f_2t_3 \quad \text{or,} \quad v=f_2t_3 \quad \cdots \quad \cdots \quad \text{(iv)}$$

and $$s_3=vt_3-\tfrac{1}{2}f_2t_3^2=vt_3-\tfrac{1}{2}vt_3=\tfrac{1}{2}vt_3 \quad \cdots \quad \text{(v)}$$

$$\therefore\ s=s_1+s_2+s_3=\tfrac{1}{2}vt_1+vt_2+\tfrac{1}{2}vt_3=\tfrac{1}{2}v\,(t_1+t_3)+vt_2$$

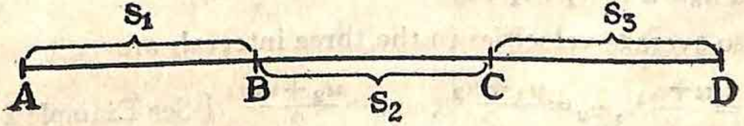


Fig. 5·2

Again $t_1=\dfrac{v}{f_1}$ and $t_3=\dfrac{v}{f_2}$ $\therefore$ $s=\frac{1}{2}v\left(\dfrac{v}{f_1}+\dfrac{v}{f_2}\right)+vt_2$

$$\therefore\ t_2=\frac{s}{v}-\frac{1}{2}\left(\frac{v}{f_1}+\frac{v}{f_2}\right)$$

$\therefore$ Required total time$=t_1+t_2+t_3$

$$=\frac{v}{f_1}+\frac{s}{v}-\frac{1}{2}\left(\frac{v}{f_1}+\frac{v}{f_2}\right)+\frac{v}{f_2}$$

$$=\frac{s}{v}+\frac{1}{2}\frac{v}{f_1}+\frac{1}{2}\frac{v}{f_2}=\frac{s}{v}+\frac{1}{2}v\left(\frac{1}{f_1}+\frac{1}{f_2}\right).$$

Example 21. Show that the average velocity of a particle moving along a straight line with a uniform acceleration during any interval of time is equal to the mean of the initial and final velocities of the particle in that interval.

Let a particle traverses a distances s in time t and its initial and final velocities during this time be u and v respectively.

$\therefore$ $v=u+ft$ and $s=ut+\frac{1}{2}ft^2$.

Now the average velocity of the particle during this time is

$$\frac{s}{t}=\frac{ut+\frac{1}{2}ft^2}{t}=u+\frac{1}{2}ft=\frac{1}{2}(2u+ft)=\frac{1}{2}(u+u+ft)=\frac{1}{2}(u+v)$$

=mean of the initial and final velocities.

Example 22. Average velocities of a particle moving along a straight line with uniform acceleration in successive intervals t_1, t_2, t_3 are v_1, v_2 and v_3 respectively.

Prove that $\dfrac{v_1-v_2}{v_2-v_3}=\dfrac{t_1+t_2}{t_2+t_3}$.

Let u be the velocity of the particle at the beginning of the first interval u and f be the uniform acceleration. Also let u_1, u_2, u_3 be the velocities at the end of the three intervals.

$\therefore$ $u_1=u+ft_1$ ... ... (i) $u_2=u+f(t_1+t_2)$ ... ... (ii)

and $u_3=u+f(t_1+t_2+t_3)$

Also average velocities in the three intervals are

$$v_1=\frac{u+u_1}{2},\ v_2=\frac{u_1+u_2}{2},\ v_3=\frac{u_2+u_3}{2}\quad \text{[See Example 21]}$$

$$\therefore\ v_1-v_2=\frac{u+u_1}{2}-\frac{u_1+u_2}{2}=\frac{u-u_2}{2}=\frac{-f(t_1+t_2)}{2}\quad \text{[From (ii)]}$$

$$v_2-v_3=\frac{u_1+u_2}{2}-\frac{u_2+u_3}{2}=\frac{u_1-u_3}{2}=\frac{(u+ft_1)-\{u+f(t_1+t_2+t_3)\}}{2}$$

[From (i) and (iii)]

$$=\frac{-f(t_2+t_3)}{2}\quad\therefore\quad\frac{v_1-v_2}{v_2-v_3}=\frac{\dfrac{-f(t_1+t_2)}{2}}{\dfrac{-f(t_2+t_3)}{2}}=\frac{t_1+t_2}{t_2+t_3}.$$

Example 23. A train starts from Howrah and stops at Kharagpur. Its velocity uniformly increases and reaches to v. The velocity then decreases uniformly. Time taken from Howrah to Kharagpur by the train is t. Prove that the distance between the stations is $\frac{1}{2}vt$.

Let the train attains the maximum velocity v at a place O between Howrah and Kharagpur and the distance of O from Howrah and Khargpur be s_1 and s_2 respectively. Hence the required distance is s_1+s_2. Again if the times taken by the train to reach O from Howrah be t_1, and t_2 be the time from O to Khargpur, then the total time is t_1+t_2. $\therefore$ $t=t_1+t_2$.

Now considering the motion of the train from Howrah to the place O we get,

$v=f_1t_1$ and $v^2=2f_1s_1$ where f_1 is the uniform acceleration of the train between Howrah and O.

$$\therefore\ f_1=\frac{v}{t_1}\ \text{and}\ s_1=\frac{v^2}{2f_1}=\frac{v^2}{2\dfrac{v}{t_1}}=\frac{v}{2}t_1.$$

Again considering the motion of the train between the place O and Kharagpur we get $0=v-f_2t_2$ and $0=v^2-2f_2s_2$, where f_2 is the uniform retardation of the train between O and Kharagpur.

$$\therefore\ f_2=\frac{v}{t_2}\ \text{and}\ s_2=\frac{v^2}{2f_2}=\frac{v^2}{2\dfrac{v}{t_2}}=\frac{v}{2}t_2.$$

$\therefore$ Distance between Howrah and Kharagpur

$$=s_1+s_2=\frac{v}{2}t_1+\frac{v}{2}t_2=\frac{v}{2}(t_1+t_2)=\tfrac{1}{2}vt.$$

Example 24. Two motor cars on the same line approach each other with velocities u_1 and u_2 respectively. When each is seen from the other, the distance between them is x. Prove that if f_1

and f_2 be the maximum retardations of the two cars, then a collision can be just avoided if.

$$u_1{}^2f_2+u_2{}^2f_1=2f_1f_2x$$ [Joint Entrance 1982]

Since the two motor cars approach each other along the same line, in order to avoid a collision they will apply the maximum retardations, the brakes can produce. The collision is just avoided means that they will stop at the same instant and when the cars will stop, then the distance between then will be zero.

Let the cars stop at time t after they see each other and during this time they have moved through distances x_1 and x_2.

$$\therefore \quad x_1+x_2=x.$$

Considering the motion of the first car we get,

$$0=u_1-f_1t \quad \text{or,} \quad t=\frac{u_1}{f_1} \text{ and } x_1=u_1t-\tfrac{1}{2}f_1t_1{}^2$$

$$=u_1.\frac{u_1}{f_1}-\tfrac{1}{2}.f_1.\frac{u_1{}^2}{f_1{}^2}=\frac{u_1{}^2}{f_1}.-\tfrac{1}{2}\frac{u_1{}^2}{f^1}=\tfrac{1}{2}\frac{u_1{}^2}{f_1}.$$

Considering the motion of the second car we get

$$0=u_2-f_2t \quad \text{or,} \quad t=\frac{u_2}{f_2}.$$

Also $x_2=u_2t-\frac{1}{2}f_2t^2=u_2.\frac{u_2}{f_2}-\frac{1}{2}f_2.\frac{u_2{}^2}{f_2{}^2}=\frac{u_2{}^2}{f_2}-\frac{1}{2}\frac{u_2{}^2}{f_2}=\frac{1}{2}\frac{u_2{}^2}{f_2}.$

$$\therefore \quad x=x_1+x_2=\tfrac{1}{2}\frac{u_1{}^2}{f_1}+\tfrac{1}{2}\frac{u_2{}^2}{f_2}=\frac{u_1{}^2f_2+u_2{}^2f_1}{2f_1f_2}$$

$$\therefore \quad u_1{}^2f_2+u_2{}^2f_1=2f_1f_2x.$$

Example 25. Two trains A and B were moving along the same line in the same direction with velocities u_1 and u_2 $(u_1>u_2)$. If x be the distance between the trains when each is seen from the other, show that collision can be just avoided if $(u_2-u_1)^2=2(f_1+f_2)x$ where f_1 and f_2 are the greatest retardation and the greatest acceleration that the trains can produce.

When the trains see each other, then in order to avoid collision, the train coming from behind will apply the greatest possible retardation f_1 and the train in the front will apply the greatest possible acceleration.

Collision can be just avoided means when A will be just behind B i.e., their distance will be 0, then their velocities will be equal. Let at time t after each is seen from the other A is just behind B. Let during this time B moves through a distance x_1. Then during this time the distance moved by A is $x+x_1$ and at this time the velocity of both be v.

Now considering the motion of the train A we get

$$v=u_1-f_1t \quad \cdots \quad \cdots \text{(i)} \qquad x+x_1=u_1t-\tfrac{1}{2}f_1t^2 \quad \cdots \quad \cdots \text{(ii)}$$

Again considering the motion of the train B we get

$$v=u_2+f_2t \quad \cdots \quad \cdots \text{(iii) and } x_1=u_2t+\tfrac{1}{2}f_2t^2 \quad \cdots \quad \cdots \text{(iv)}$$

From (i)—(iii) we get $0=u_1-u_2-(f_1+f_2)t$ or, $t=\dfrac{u_1-u_2}{f_1+f_2}$

Again from (iii)—(iv) we get, $x=(u_1-u_2)\,t-\frac{1}{2}(f_1+f_2)\,t^2$

$$=(u_1-u_2)\left(\frac{(u_1-u_2)}{f_1+f_2}\right)-\frac{1}{2}(f_1+f_2)\cdot\left(\frac{(u_1-u_2)^2}{(f_1+f_2)^2}\right)^2$$

$$=\frac{(u_1-u_2)^2}{f_1+f_2}-\frac{(u_1-u_2)^2}{2(f_1+f_2)}=\frac{1}{2}\frac{(u_1-u_2)^2}{f_1+f_2}$$

$$\therefore\ (u_1-u_2)^2=2\,(f_1+f_2)\,x.$$

Example 26. Two particles P and Q move along a straight line AB. Particle P starts from A in the direction AB with velocity u_1 and acceleration f_1 at the same time Q starts from B in the direction BA with velocity u_2 and acceleration f_2. If they pass each other at the mid point of AB and arrive at the other ends of AB with equal velocity, then prove that

$$(u_1+u_2)\,(f_1-f_2)=8\,(f_1u_2-f_2u_1).$$

[Joint Entrance 1988]

Let the distance between A and B be s and they meet at time t after start at the middle point C of AB.

$$\therefore\ AB=BC=\frac{s}{2}.$$

So, $\dfrac{s}{2}=u_1t+\frac{1}{2}f_1t^2$ and $\dfrac{s}{2}=u_2t+\frac{1}{2}f_2t^2$

$$\therefore\ u_1t+\tfrac{1}{2}f_1t^2=u_2t+\tfrac{1}{2}f_2t^2$$

$$\text{or, } (u_1-u_2)t=\tfrac{1}{2}(f_2-f_1)t^2 \qquad \therefore\ t=\frac{2\,(u_1-u_2)}{f_2-f_1}$$

Again the particles reach B and A with the same velocity v (say). $\therefore\ v^2=u_1{}^2+2f_1s$ and $v^2=u_2{}^2+2f_2s$

$$\therefore\ u_1{}^2+2f_1s=u_2{}^2+2f_2s.$$

$$\text{or,}\quad u_1{}^2-u_2{}^2=2(f_2-f_1)s \quad \therefore\ s=\frac{u_1{}^2-u_2{}^2}{2(f_2-f_1)}.$$

So, from the relation $\frac{s}{2}=u_1t+\frac{1}{2}f_1t^2$ or, $\frac{s}{2t}=u_1+\frac{1}{2}f_1t_2$

we get,

$$\frac{\dfrac{u_1{}^2-u_2{}^2}{2(f_2-f_1)}}{\dfrac{4(u_1-u_2)}{f_2-f_1}}=u_1+\frac{1}{2}f_1\,\frac{2(u_1-u_2)}{f_2-f_1}$$

$$\text{or,}\quad \frac{u_1+u_2}{8}=\frac{u_1f_2-u_1f_1+f_1u_1-f_1u_2}{f_2-f_1}$$

$$\text{or,}\quad \frac{u_1+u_2}{8}=\frac{u_2f_1-u_1f_2}{f_1-f_2}$$

$$\text{or,}\quad (u_1+u_2)(f_1-f_2)=8(f_1u_2-f_2u_1).$$

Example 27. A bus starts from rest with uniform acceleration, and a man 100 metres behind the bus starts at the same instant to run after it with 10m/second velocity and just catches the bus. Find the acceleration of the bus, the distance the bus runs and the time the man takes to catch the bus.

Find with what speed the man would run if he were originally 200m behind the bus. [Joint Entrance 1981]

Let the acceleration of the bus be f metre/sec². and the man catches the bus at a distance s metres at time t after start.

So the distance traversed by the bus in t seconds is

$$s=\tfrac{1}{2}ft^2 \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

So, the distance moved by the man during the same time is $s+100$.

$$\therefore\ s+100=10t \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

Again as the man just catches the bus so at the instant the velocity of the bus is equal to the velocity 10 m./second of the man [In the next instant the velocity of the bus will be greater than the velocity of the man and it will not be possible for the man to catch the bus].

$\therefore\ ft=10 \quad \cdots \quad \cdots \quad$ (iii) [The bus starts from rest]

From (i) and (ii) we get $100=10t-\frac{1}{2}ft^2=10t-\frac{1}{2}ft.t$

or, $100=10t-\frac{1}{2}.10t$ [From (iii) $ft=10$] $=5t$.

$$\therefore\ t=20. \qquad \therefore\ f=\frac{10}{t}=\frac{10}{20}=\frac{1}{2}.$$

$$s+100=10t=20.10=200 \qquad \therefore\ s=100.$$

Hence the acceleration of the bus is $\frac{1}{2}$ m/sec^2 and the man will catch the bus 20 seconds after start after the bus has run 100 metres.

If the man could just catch the bus starting from a place 200 metres behind the bus at a distance s' from the starting place of the bus and the uniform velocity of the man were v, then the previous equations (i), (ii) and (iii) would be

$$s'=\frac{1}{2}ft'^2=\frac{1}{2}.\frac{1}{2}t'^2 \quad [\because\ f=\frac{1}{2}]=\frac{1}{4}t'^2 \qquad \cdots \quad \cdots \quad \cdots \text{(iv)}$$

$$s'+200=vt' \quad \cdots \quad \cdots \quad \text{(v) and } ft'=v \quad \text{or, } \frac{1}{2}t'=v \quad \cdots \quad \cdots \text{(vi)}$$

From equations (iv) and (v) we get,

$$\frac{1}{4}t'^2+200=vt'=\frac{1}{2}t'^2 \quad [\text{ From (vi) }]$$

$$\text{or, } \frac{1}{4}t'^2=200 \quad \text{or, } t'^2=800 \quad \therefore\ t'=\sqrt{800}=20\sqrt{2}.$$

$$v=\frac{1}{2}t'=10\sqrt{2}.$$

So the man would have run at the rate of $10\sqrt{2}$ metres per second.

Example 28. Two particles start together from the same point along a given straight line, the first with a constant velocity u and the second with a uniform acceleration f starting from rest. Prove that before the second particle catches the first, the greatest distance between them is $\frac{u^2}{2f}$ at time $\frac{u}{f}$ from the start. [H. S. 1978 ; Joint Entrance 1983]

Let the distance between the particles at time t after start be s. By question, $u<f$ (for otherwise, any question of the second particle catching the first cannot arise) and $f>0$.

Now distance traversed by the two particles in time t are ut and $\frac{1}{2}ft^2$.

$$\therefore\ s=ut-\frac{1}{2}ft^2 \qquad \therefore\ \frac{ds}{dt}=u-ft.$$

For maximum or minimum values of s, $\frac{ds}{dt}=0$. or, $u-ft=0$.

or, $t=\frac{u}{f}$.

$$\text{Also } \frac{d^2s}{dt^2}=-f<0 \quad [\text{as } f>0].$$

So, the maximum value of s will be when $t=\frac{u}{f}$ and this maximum value of s is $u \cdot \frac{u}{f}-\frac{1}{2}f.\frac{u^2}{f^2}=\frac{u^2}{f}-\frac{1}{2}\frac{u^2}{f}=\frac{u^2}{2f}$.

Example 29. The velocities of two particles moving along the same straight line in the same direction are u and u' when they are at distances a and a' respectively from a fixed point of the straight line. Prove that the particles cannot meet each other more than twice and if they do so, then they will do it at an interval $\frac{2}{f-f'}[(u-u')^2-2(a-a')(f-f')]^{1/2}$ where f and f' are the uniform acceleration of the particles.

Let the particles meet each other at a distance x from the fixed point O, t seconds after the instant when they were at distances a and a' from O. So in t seconds they traverse distance $x-a$ and $x-a'$ respectively.

$\therefore \quad x-a=ut+\frac{1}{2}ft^2 \quad \cdots \quad \cdots$ (i)

and $x-a'=u't+\frac{1}{2}f't^2$. Subtracting (ii) from (i) we get

$$a'-a=(u-u')t+\tfrac{1}{2}(f-f')t^2$$

or, $\frac{1}{2}(f-f')t^2+(u-u')t+a-a'=0 \cdots\cdots$ (iii)

Equation—(iii) is a quadratic equation in 't' and cannot have more than two roots. So the particles cannot meet more than twice. If they meet twice, then the roots of equation—(iii) will be real and unequal and let them be t_1 and t_2.

$$t_1+t_2=-\frac{u-u'}{\frac{1}{2}(f-f')}=-\frac{2(u-u')}{f-f'}$$

$$\text{and } t_1t_2=\frac{a-a'}{\frac{1}{2}(f-f')}=\frac{2(a-a')}{f-f'}.$$

Now, $(t_1-t_2)^2=(t_1+t_2)^2-4t_1t_2$

$$=\frac{4(u-u')^2}{(f-f')^2}-\frac{8(a-a')}{f-f'}=\frac{4}{(f-f')^2}\{(u-u')^2-2(a-a')(f-f')\}$$

Hence the difference between the two times is $t_1 \sim t_2$

$$=\frac{2}{f-f'}\{(u-u')^2-2(a-a')(f-f')\}^{1/2}$$

Example 30. The velocity at time t after start of a particle starting to move along a straight line with initial velocity u is $ue^{a(t+x)}$, where a is a positive constant and x is the distance of the particle at time t from a fixed point of the straight line. Prove that the time taken by the particle to attain a velocity $2u$ per second is $\frac{1}{a}\log\frac{2u+2}{2u+1}$ and during this time the particle would move through a distance $\frac{1}{a}\log\frac{2u+1}{u+1}$.

Let O be the fixed point of the path OX of the particle and the distance of the particle from O at time t after start be x. Given velocity of the particle at time 't' is

$$v=u.e^{a(t+x)} \text{ or, } \frac{dv}{dt}=ue^{a(t+x)}.\,a\left(1+\frac{dx}{dt}\right)$$

$$\text{or, } \frac{dv}{dt}=av(1+v) \quad \left[\because \; v=ue^{a(t+x)}=\frac{dx}{dt}\right]$$

$$\text{or, } \frac{dv}{v(1+v)}=adt \text{ or, } \int\left\{\frac{1}{v}-\frac{1}{1+v}\right\}dv=\int adt$$

$$\text{or, } \log\frac{v}{v+1}=at+c_1$$

Now, when $t=0$, then $v=u$.

$$\therefore \; c_1=\log\frac{u}{u+1} \quad \therefore \; \log\frac{v}{v+1}=at+\log\frac{u}{u+1}.$$

So, when the velocity of the particle will be $2u$, i.e. $v=2u$,

$$\text{then } \log\frac{2u}{2u+1}=at+\log\frac{u}{u+1}$$

$$\text{or, } at=\log\frac{2u}{2u+1}-\log\frac{u}{u+1}=\log\left\{\frac{2u}{2u+1}\div\frac{u}{u+1}\right\}$$

$$\text{or, } at=\log\frac{2u+2}{2u+1} \text{ or, } t=\frac{1}{a}\log\frac{2u+2}{2u+1}.$$

So to attain the velocity $2u$, the particle will take time

$$\frac{1}{a}\log\frac{2u+2}{2u+1}$$

Again, from the relation $v=ue^{a(t+x)}$ we get

$\log v=\log u+a(t+x)$ or, $a(t+x)=\log v-\log u$

$$\text{or,}\quad t+x=\frac{1}{a}\log\frac{v}{u}\quad\text{or,}\ x=\frac{1}{a}\log\frac{v}{u}-t$$

when $v=2u$, then $t=\dfrac{1}{a}\log\dfrac{2u+2}{2u+1}$

$$\text{So then}\quad x=\frac{1}{a}\log\frac{2u}{u}-\frac{1}{a}\log\frac{2u+2}{2u+1}$$

$$=\frac{1}{a}\log\frac{2}{\frac{2u+2}{2u+1}}=\frac{1}{a}\log\frac{2u+1}{u+1}$$

So to attain the velocity $2u$, the particle shall have to traverse the path $\dfrac{1}{a}\log\dfrac{2u+1}{u+1}$.

Exercise 5A

1. The distance of a particle moving along a straight line from a fixed point of the straight line at time t after start is given by $t=as^2+bs+c$ where a, b, c are constants.

Show that the velocity of the particle at the instant is $\dfrac{1}{2as+b}$.

2 The distance s of a particle moving along a straight line from a fixed point of the straight line at time t after start is given by $s=\frac{1}{2}t-\frac{1}{4}t^2$.

Prove that the particle is moving with constant retardation.

3. The distance x of a particle moving along a straight line from a fixed point of the straight line at time t after start is given by $x=t^3-3t^2+4$. When will the particle come to rest? What will be its acceleration at that time. Find the velocity and acceleration of the particle after 4 seconds.

4. The distance x of a particle moving along a straight line from a fixed point of the straight line at time t after start is given by $x=3t^2$ when $0\leqslant t\leqslant 2$; $=6t$ when $t>2$.

Discuss the motion of the particle.

5. The distance s of a particle moving along a straight line from a fixed point of the straight line at time t after start is given by $s^2=at^2+2bt+c$. Show that the acceleration of the particle varies inversely as s^3.

6. The distance s traversed by a particle moving along a straight line in time t from start is given by $s=63t-6t^2-t^3$. Find the velocity of the particle 2 seconds after start and also the distance traversed by it before coming to rest. [C. U.]

7. The distance x of a particle moving along a straight line from a fixed point O of the straight line and the velocity at the instant are connected by the relation $v^2=1-x^2$. Show that the acceleration of the particle then is given by $f=-x$.

8. The distance traversed by a particle moving along a straight line in time t after start is given by

$s=a+bt+ct^2+dt^3$ (a, b, c, d are constants)

Find the signitfiance of a, b, c, d

9. The distance s of a particle moving along a straight line from a fixed point O of the straight line at time t seconds after start is given by $s=(t-1)^2(t-2)$. Find the distance of the particle from O when its velocity is 0. [C. U. B. Sc. 1984]

10. When the distance of a particle moving along a straight line from a fixed point of the straight line is x, then if v be the velocity of the particle, then $v^2=6a(\, x\sin x+\cos x\,)$. Find the acceleration of the particle.

11. A man starts on a long walk at 4 miles per hour and his speed at any instant is inversely proportional to $x+10$ where x is the mumber of miles he has walked. Find (i) the time he takes to walk 20 miles and (ii) the distance he goes in 3 hrs. 45 minutes.

[C.U. B. Sc. 1984]

12. The distance x of a particle moving along a straight line from a fixed point of the straight line at time 't' after start is given by $t=ax^2+bx+c$ (a, b, c are positive constants). Show that the particle is moving with acceleration $2av^3$, v being the velocity of the particle at the instant. If u $(\neq 0)$ be the initial velocity of the particle show that $\frac{1}{v^2}-\frac{1}{u^2}=4at$.

13. When the distance of a particle moving along a straight line from a fixed point of the straight line is x, then the velocity and acceleration of the particle are v and f and $v^2=ax^2+2bx+c$. Show that $f^2-av^2=b^2-ac$.

14. In the following two questions u, v, f, s and t have usual meanings.

(i) If $u=5$ cm. ; $v=29$ cm. and $t=8$ second find f and s.

(ii) If $u=100$ m, $f=5$ m, $s=3000$ m find v and t.

15. A motor car travelling at the rate of 40 km/hour is stopped by its brakes in 4 seconds. How long will it go from the point at which the brakes are first applied ?

[Joint Entrance 1984]

16. A particle moving along a straight line with a given initial velocity and uniform acceleration goes through distances x ft and y ft in the t-th and $(t+n)$-th seconds of its motion. Prove that the uniform acceleration of the particle is $\frac{y-x}{n}$ ft/sec^2.

17. A body moves through distances 15 cm., 63 cm. and 111 cm. in the first, second and fifth second respectively. Is it consistent with uniform acceleration ?

18. A train takes 50 minutes to move from a station X to a station Y. The train attains a maximum velocity of 50 miles an hour at a place Z between X and Y. The train starting from rest from X goes to Z with uniform acceleration and then goes with uniform retardation from Z to Y and stops at Y. What is the distance between the two stations X and Y.

19. A particle moves through 396·9 metres in the first 3 seconds and in the next 4 seconds through 392 cms. and 269·5 metres in its next 5 seconds. Prove that it is consisent with uniform acceleration. What more time will it take to come to rest.

20. The initial velocity and uniform retardation of a particle moving along a straight line are mn ft/second and m ft/sec^2. Show that the particle will return at time $(n+k)$ seconds after start at the same place which it passed at time $(n-k)$ seconds after start with the same velocity.

21. A bullet after penetrating a wall 9·6 inches thick has its velocity reduced to 800 ft./second from 1200 ft./second. Find the time taken by the bullet to penetrate the wall; also find its velocity when it penetrated half of the wall.

22. A bullet comes to rest after penetrating x cm. into some sand. v and u are the velocities of the bullet at the instant of entering the sand and after penetrating y cm. into the sand $(x>y)$. If the retardation due to the resistance of sand be uniform, show that $u : v=\sqrt{x-y} : \sqrt{x}$.

23. A train travels from rest at one station to rest at another, in the same straight line. The train moves the first part of the journey with an acceleration a ft./sec^2 and the rest with a retardation b ft./sec^2. Show that it will accomplish the journey in $\sqrt{\dfrac{2(a+b)d}{ab}}$ secs. where d is the distance between the two stations.

24. A particle moving along a straight line describes three distances, $AB=153$ metres, $BC=320$ metres and $CD=135$ metres in three successive intervals 3 seconds, 8 seconds and 5 seconds. Show that the particle is moving with uniform acceleration and comes to rest at a distance 729 metres from A.

25. A particle moving in a straight line with a uniform acceleration is observed to be at distances a, b, c, d from a fixed point of the line at time $t=0$, n seconds, $2n$ seconds, $3n$ seconds respectively.

[Prove that (1) $d-a=3(c-b)$. (2) initial velocity$=\dfrac{4b-3a-c}{2n}$ and (3) acceleration$=\dfrac{c+a-2b}{n^2}$].

26. A train moving along a straight road from rest again comes to rest after describing a certain distance. If the train moved with uniform acceleration for the first $\frac{1}{3}$rd of the distance and with uniform retardation for the last $\frac{1}{4}$th of the distance, prove that the ratio of its greatest velocity to its average velocity is 19 : 12.

27. A particle moves from rest with uniform acceleration f along a given straight line. Its acceleration becomes $2f$, $3f$ etc.

after times t, $2t$ etc. Show that after time nt the distance described by the particle is $\frac{n(n+1)\,(2n+1)}{12}\,ft^2$.

28. A particle moving with uniform acceleration along a straight path, describes equal distances s in the first t seconds and in the next $\frac{t}{2}$ seconds. Prove that during the next $\frac{3t}{2}$ seconds the particle will describe a distance $5s$.

29. A train starting from rest at a station A stops at another station B at a distance c. The train moves with uniform acceleration for the first t seconds, with uniform velocity for the next t seconds and with uniform retardation for the next t' seconds before coming to rest at B. Show that the uniform acceleration in the first part of the journey is $\frac{2c}{t(3t+t')}$.

30. A train moving along a straight course takes time T to perform a journey from rest to rest; it travels for time $2T/n$ with uniform acceleration attaining a velocity V, with uniform velocity for the next time $(n-3)T/n$ and finally for time T/n with constant retardation, prove that its average velocity is $(2n-3)V/2n$.

31. Two particles start from the same place at the same time in the same direction. The first moves with uniform velocity 40 ft./sec. and the second with initial velocity 16 ft./sec. and uniform acceleration 6 ft./sec^2. When will the distance of the particles be maximum. Also find the maximum distance.

32. A distance s is divided into n equal parts; A particle starts from rest with uniform acceleration f. At the end of each part the acceleration increases by f/n. Show that the velocity of the particle after describing the whole distance is $\sqrt{fs(3-1/n)}$.

33. A particle moves along a straight line with uniform acceleration f. From the same place after time T another particle starts in the same direction with uniform velocity u. Show that if $u>2fT$, then the second particle will be in front of the first for an interval $\frac{2}{f}\sqrt{u(u-2fT)}$.

34. A constable seeing a thief at a distance x starts with velocity u and uniform acceleration α. The thief at the same instant starts from rest with uniform acceleration β. Prove that

it will be possible for the constable to catch the thief if $\alpha > \beta$, or $\alpha < \beta < \alpha + \dfrac{u^2}{2x}$.

35. A cat chases a mouse at a distance 15 ft. from rest with uniform acceleration 2 ft./sec^2. On seeing the cat, the mouse also begins to run with uniform velocity 14 ft./sec. When and where will the cat catch the mouse.

36. If v denotes the velocity at time t of a particle moving along a straight line, then the uniform retardation of the particle at that time is kv^{n+1}, where k is a constant. Show that if u be the initial velocity of the particle and s be the distance described in time t, then $ks = \dfrac{1}{n-1}\left(\dfrac{1}{v^{n-1}-1} - \dfrac{1}{u^{n-1}-1}\right)$.

37. A train starting from rest at one station stops at another. The train moves the first $\dfrac{1}{m}$ part with uniform acceleration and the last $\dfrac{1}{n}$ part with uniform retardation. Show that ratio of maximum velocity and average velocity of the train is

$$\left(1 + \frac{1}{m} + \frac{1}{n}\right) : 1.$$

38. 2 straight roads inclined at an angle α with each other intersect at the point O. Two particles starting from two points A and B of the two roads move towards O with uniform velocities u and v respectively. If $OA = a$ and $OB = b$, show that the least distance between the particles is

$$\frac{(av - bu)\sin\alpha}{\sqrt{u^2 + v^2 + 2uv\cos\alpha}}.$$

39. When the velocity of a particle moving along a straight line is v, then its acceleration $f = kv^2$ (k is a constant). If u be the velocity after time t and s be distance described during this time then show that $v = \dfrac{u}{1 + ksu}$ and $t = \frac{1}{2}ks^2 + \dfrac{s}{u}$.

40. O is a fixed point of the straight line OX. A particle is moving along this straight line with uniform acceleration. The distances of the particle from O after times t_1, t_2, t_3 after start

are s_1, s_2, s_3 respectively. If t_1, t_2, t_3 be in *A.P* and s_1, s_2, s_3 be in *G.P.*, then show that the acceleration of the particle is

$$\left(\frac{\sqrt{s_3}-\sqrt{s_1}}{t_2-t_1}\right)^2.$$

§ 5·7. *Acceleration due to gravity.*

Newton's Law of Gravitation :

In this universe every body attracts any other body with a force which varies directly as the product of the masses of the two bodies and inversely as the square of the distance between them ; the force of attraction acts along the straight line joining the two bodies.

If the masses of two bodies placed at the points A and B be m and m' and d be the distance between them, then the bodies attract each other with a force P so that $P \propto mm'$ and $P \propto \frac{1}{d^2}$; so $P \propto \frac{mm'}{d^2}$ or, $P = G\frac{mm'}{d^2}$ where G is a constant of variation. This constant of variation G is called the gravitational constant. The mass 'm' attracts the mass 'm'' along the direction $\overrightarrow{BA}$ and the mass 'm'' attracts the mass m along the direction $\overrightarrow{AB}$.

Gravity : The force of attraction between two bodies, as discussed above, is called the *Force of Gravitation.* When of the two bodies discussed in the law of gravitation, one body is the earth, then the force with which earth attracts any other body towards the centre of the earth is called *gravity.* So gravity is a special type of force of gravitation.

According to Newton's second law of motion, if a force is applied on a body, then the body produces an acceleration of the body and the direction of acceleration is the direction of the force. If the mass of the earth be M and m be the mass of a body at a distance d from the centre of the earth, then the second body will be attracted towards the centre of the earth with a force $G\frac{Mm}{d^2}$. The acceleration produced in the body due to this force of attraction is $\frac{GM}{d^2}$ (according to the formula

$P=mf$) in the direction towards the centre of the earth. This acceleration of the body due to the earth's attraction is called acceleration due to gravity and it is denoted by g. So a body of mass m is attracted towards the centre of the earth with a force mg. This force of attraction mg is called the *weight* of the body. So the weight of a body is that force by which it is attracted towards the centre of the earth.

Note : The mass m also produces an acceleration $\frac{Gm}{d^2}$ of the earth in the direction of the body. But as the mass m is negligible in comparison with the mass M of the earth, so this acceleration produces little influence on the motion of the body.

As the heights of different places on the surface of the earth are different and the acceleration due to gravity is towards the centre of the earth, so at different places the value of $g=\frac{Gm}{d^2}$ varies with d. In the present discussion if, otherwise is not stated, we shall take the value of g as 980 or, 981 cm./sec^2 in the C.G.S. system, 9·81 metres/sec^2 or 9·81 metres/sec^2 in the M. K. S system or 32 ft / sec^2 or 32·2 ft/sec^2 in the F. P. S system.

From the above discussion we understand that when a heavy body falls from a height h, its motion during fall, is subjected to an acceleration g (acceleration due to gravity). We conclude this section by stating the Laws of Falling Bodies formulated by the great scientist Galileo.

Law 1. In any place the acceleration due to gravity during the motion of a falling body is constant.

Law 2. All Bodies fall freely from rest in the downward direction with the same rapidity.

§ 5·8. *Motion of a body falling freely under gravity.*

A body is falling freely means that the only force acting on the body is gravity and the only acceleration of the body is that due to gravity.

Let a body of mass m be falling freely from a point O at a height h above the surface of the earth with an initial velocity u and at time t after start the distance of the body from O be x.

Now let us take O as the origin, the vertically downward direction through O as the positive direction of the x-axis. Now the distance of the body along the x-axis at time t (seconds) after start from the origin is x. So its acceleration at the instant is $\frac{d^2x}{dt^2}$ in the vertically downward direction. Also the only force acting on the body is the force of gravity or weight mg of the body acting vertically downwards.

So according to the formula $P=mf$,

$$m\frac{d^2x}{dt^2}=mg \text{ or, } \frac{d^2x}{dt^2}=g. \text{ or, } \frac{dv}{dt}=g \quad \cdots \quad \cdots \quad \text{(i)}$$

[where v is the velocity of the falling body]

or, $dv=gdt$ or, $\int dv=\int gdt$ or, $v=gt+c_1$

Now when $t=0$, then $v=u$. $\therefore$ $u=g.0+c_1$

$\therefore$ $c_1=u$. $\therefore$ $v=u+gt \quad \cdots \quad \cdots \quad$ (ii)

or, $\frac{dx}{dt}=u+gt$ $\left[\text{as at time } t \text{ the velocity } v \text{ of the falling body is } \frac{dx}{dt}\right]$

or, $dx=u\,dt+gt\,dt$ or, $\int dx=\int u\,dt+\int gt\,dt$

or, $x=ut+\frac{1}{2}gt^2+c_2$. Now when $t=0$, then $x=0$.

$\therefore$ $0=u\,.\,0+\frac{1}{2}g\,.\,0+c_2$ $\therefore$ $c_2=0$.

So $x=ut+\frac{1}{2}gt^2 \cdots \quad \cdots \quad \cdots$(iii)

Again from equation-(i) we get

$$v\frac{dv}{dx}=g \quad \left[\because \frac{dv}{dt}=\frac{d}{dt}(v)=\frac{d}{dx}(v)\frac{dx}{dt}=v\frac{dv}{dx}\right]$$

or, $vdv=g\,dx$. or, $\int vdv=\int gdx$

or, $\frac{v^2}{2}=gx+c_3$.

Now when $t=0$, then $x=0$, $v=u$.

$\therefore$ $\frac{u^2}{2}=g\,.\,0+c_3$ $\therefore$ $c_3=\frac{u^2}{2}$

So $\frac{v^2}{2}=gx+\frac{u^2}{2}$ or, $v^2=u^2+2gx\cdots \quad \cdots$(iv)

Again if h_t be the height descended by the body in the t-th second of motion, then $h_t=$[height descended in the first t seconds]

$$\begin{aligned}&-[\text{ height descended in the first } (t-1) \text{ seconds }]\\&=[ut+\tfrac{1}{2}gt^2]-[u(t-1)+\tfrac{1}{2}g(t-1)^2]\\&=ut+\tfrac{1}{2}gt^2-ut+u-\tfrac{1}{2}gt^2+gt-\tfrac{1}{2}g\\&=u+\tfrac{1}{2}g(2t-1)\cdots \quad \cdots \quad \cdots(\text{v})\end{aligned}$$

So, if a body falls freely under gravity with an initial velocity u per second then

(i) The velocity of the body in the t-th second of motion is $v=u+gt$. (ii) The height descended in the first t seconds is $x=ut+\frac{1}{2}gt^2$. (iii) $v^2=u^2+2gx$. and (iv) the distance h_t descended in the t-th second is $h_t=u+\frac{1}{2}g(2t-1)$.

Note. If the particle falls freely vertically downwards from a height h with initial velocity u, then it will reach the ground with velocity v, given by $v^2=u^2+2gh$.

§ 5·9. *Discussion on the motion of a body falling freely from rest.*

If a body falls freely from rest then its initial velocity is 0 and the only acceleration is g vertically downwards. So putting $u=0$ in the formulas (ii), (iii), (iv) and (v) of § 5·8 we get that if a particle falls freely from rest from a height h and v be the velocity in the t-th second, x be the height descended in the first t seconds and h_t be the height descended in the t-th second of motion, then

(i) $v=gt$ (ii) $x=\frac{1}{2}gt^2$ (iii) $v^2=2gx$ and $h_t=\frac{1}{2}g(2t-1)$.

Again if the body reaches the ground with velocity v, then $v^2=2gh$ or, $v=\sqrt{2gh}$. Also if t_1 be the time of fall to the ground then $h=\frac{1}{2}gt_1^2$ or, $t_1=\sqrt{\frac{2h}{g}}$.

§ 5·10. *Motion of a body projected vertically upwards.*

A body is projected freely vertically upwards with velocity u from a point O of the ground. Let O be the origin, the vertical line through O, the x-axis and the vertically upward direction be the positive direction of the x-axis. As the body is projected freely from O in the vertically upward direction, so its only acceleration is the acceleration due to gravity $-g$. As the direction of the acceleration due to gravity is the vertically downward direction, so the sign of the acceleration is negative. Actually the body undergoes a retardation g.

If the body ascends a height x in t seconds, then the differential equation of motion of the body is

$$\frac{d^2x}{dt^2}=-g \quad \text{or,} \quad \frac{dv}{dt}=-g \quad \therefore \quad dv=-gdt$$

or, $\int dv=-g\int dt$ or, $v=-gt+c_1$.

Now when $t=0$, then $v=u$. $\therefore$ $u=c_1$

$\therefore$ $v=u-gt \cdots \quad \cdots \quad \cdots$(ii)

Again the equation-(ii) can be written as

$$\frac{dx}{dt}=u-gt\left(\because \text{ the velocity at time } t \text{ is } v=\frac{dx}{dt}\right)$$

or, $dx=(u-gt)\,dt$ or, $\int dx=\int(u-gt)\,dt$

or, $x=ut-\frac{1}{2}gt^2+c_2$.

Now when $t=0$, then $x=0$. $\therefore$ $c_2=0$.

$\therefore$ $x=ut-\frac{1}{2}gt^2 \ldots \quad \cdots \quad \cdots$(iii)

Again the differential equation-(i) can be written as

$$v\,\frac{dv}{dx}=-g \quad \text{or,} \quad vdv=-gdx$$

or, $\int vdv=-g\int dx$ or, $\dfrac{v^2}{2}=-gx+c_3$

Now when $x=0$, then $v=u$.

$\therefore$ $\dfrac{u^2}{2}=c_3$ So, $\dfrac{v^2}{2}=-gx+\dfrac{u^2}{2}$ or, $v^2=u^2-2gx \cdots \quad \cdots$(iv)

Also if h_t be the height ascended in the t-th second of motion, then h_t={ height ascended in the first t seconds }

−{ height ascended in the first $(t-1)$ seconds }

$=(ut-\frac{1}{2}gt^2)-\{u(t-1)-\frac{1}{2}g(t-1)^2\}=u-\frac{1}{2}g(2t-1)\cdots \quad \cdots$(v).

So, if a body is projected freely vertically upwards with velocity u and if the velocity in the t-th second of ascent be v, height ascended in first t seconds be x, then

(i) $v=u-gt$; (ii) $x=ut-\frac{1}{2}gt^2$; (iii) $v^2=u^2-2gx$.

Also if h_t be the height ascended in the t-th second of motion, then $h_t=u-\frac{1}{2}g(2t-1)$.

§ 5·11. *Greatest height, Time of ascent and time of flight of a body projected freely vertically upwards.*

Let a body be projected freely vertically upwards from a point O with velocity u. Then the body will have only a retardation $-g$. So, at some time t after projection the velocity of the particle will be 0.

So, $0=u-gt$ or, $t=\frac{u}{g}$ ... ... (i)

Hence time of ascent to the greatest height (i.e. the height where its velocity is 0) is $\frac{u}{g}$.

Let the greatest height be H.

Then $0=u^2-2gH$ or, $H=\frac{u^2}{2g}$.

After ascending to the greatest height H, the body will descend with acceleration g. If t_1 be the time to reach the point of projection i.e. to descend the height H, then.

$H=\frac{1}{2}gt_1^2$ or, $\frac{u^2}{2g}=\frac{1}{2}gt_1^2$ or, $t_1^2=\frac{u^2}{g^2}$ or, $t_1=\frac{u}{g}$.

So, $t_1=t$.

Hence time of ascent to the greatest height from the point of projection = Time of fall from the greatest height to the point of projection

The sum of the two times of ascent and descent is called the *time of flight* of the body.

Hence the time of flight of the body

is $t+t_1=\frac{u}{g}+\frac{u}{g}=\frac{2u}{g}$.

Let now the velocity of the body at the instant of reaching the point of projection be V. Then considering the motion of the body from the highest point to the point of projection, we get

$V^2=2gH=2g\,\frac{u^2}{2g}=u^2$ $\therefore$ $V=u$. (in magnitude)

So the velocity of projection = velocity at the instant of fall to the point of projection (in magnitude)

Though these two velocities have equal magnitude, their directions are opposite.

Note: In the result that time of ascent to the greatest height from the point of projection = time of fall to the point of projection, taking any point in the path of the body as the point of projection one can say,

time of rise from a point of the path to the greatest height = time of fall from the greatest height to the point.

§ 5.12. *Time to reach a given height and velocity at that height.*

Let h be a given height and a particle be projected with a given velocity of projection (from the bottom of the height). Let the particle reaches the height h at time t after projection. So h is the height ascended by the particle in time t.

$$\therefore\ h=ut-\tfrac{1}{2}gt^2 \quad \text{or,}\quad t^2-\frac{2u}{g}t+\frac{2h}{g}=0.$$

This is a quadratic equation in t and so from the equation we get two and only two values of t. These two values are

$$\frac{\frac{2u}{g}\pm\sqrt{\frac{4u^2}{g^2}-\frac{8h}{g}}}{2}=\frac{u}{g}\pm\frac{\sqrt{u^2-2gh}}{g}.$$

Now if $u^2>2gh$, i.e., $h<\frac{u^2}{2g}$ i.e., h is less than the greatest height attained by the particle, then the two values of t are real and unequal; $\frac{u}{g}-\frac{\sqrt{u^2-2gh}}{g}$ is the time of ascent to the height h and $\frac{u}{g}+\frac{\sqrt{u^2-2gh}}{g}$ is the time taken by the particle to ascend to the greatest height and then descending to the height.

If $u^2=2gh$, then the two values of t will be equal and this equal value is $\frac{u}{g}$ and we know that a particle ascends to the greatest height once and only once.

If $u^2<2gh$ i.e., $h>\frac{u^2}{2g}$, then the two values of t will be imaginary i.e., the particle cannot ascend to a height higher than the greatest height.

Again the velocity v of the particle at the height h during ascent is given by

$$v^2=u^2-2gh \quad \text{or,}\quad v=\sqrt{u^2-2gh}.$$

If v_1 be the velocity of the particle during descent from the greatest height $\frac{u^2}{2g}$, i.e. velocity after descending a distance $\frac{u^2}{2g}-h$ from the greatest height, then

$$v_1^2=2g\left(\frac{u^2}{2g}-h\right)=u^2-2gh. \quad \text{or,} \quad v_1=\sqrt{u^2-2gh}$$

$$\therefore \quad v=v_1,$$

Hence the velocity of the particle at a height during ascent to the height and the velocity at the height during descent from the greatest height are equal in magnitude. Of course they are opposite in direction.

EXAMPLES 5B

Example 1. A particle projected vertically upwards returns to its point of projection after 7 seconds. Find the greatest height attained by the particle, its velocity of projection and its height from the point of projection after four seconds.

[$g=32$ ft. / sec^2] [H. S. 1984]

Let the velocity of projection be u ft / second.

So, its time of flight$=\frac{2u}{g}$.

$\therefore$ By question, $\frac{2u}{g}=7 \quad \therefore \quad 2u=7g=7\times 32$

or, $u=\frac{7\times 32}{2}=112$ ft.

So, the velocity of projection is 112 ft. / second.

So, the greatest height attained by the particle

$$=\frac{u^2}{2g}=\frac{112\times 112}{2\times 32}=196 \text{ ft.}$$

Also the height of the particle from the point of projection after four seconds

$$=u.4-\tfrac{1}{2}g.4^2=112.4-\tfrac{1}{2}.32.16=448-256=192 \text{ ft.}$$

Example 2. A particle is projected vertically upwards, the greatest height attained by it is 144 feet. Find when it will be at height 80 feet after projection. [H. S. 1979]

Let the velocity of projection of the particle be u ft. / second.

$\therefore$ Greatest height attained by the particle is

$$\frac{u^2}{2g}. \quad \text{So,} \quad \frac{u^2}{2g}=144. \quad \text{or,} \quad u^2=144.2.32$$

or, $u=96$. So the velocity of projection is 96 feet/second.

Let the particle was at a height 80 feet above the point of projection t seconds after projection

$\therefore$ 80=height attained in t seconds$=ut-\frac{1}{2}gt^2$

or, $80=96t-16t^2$ or, $t^2-6t+5=0$

or, $(t-5)(t-1)=0$. $\therefore$ $t=5, 1$.

So, the particle was at a height 80 feet 1 second and 5 seconds after projection.

Note : The particle reached the height 80 ft. at first during ascent and then once again during descent. That is why we get the double answer.

Example 3. A ball is projected vertically upwards. Show that the ratio of the two times taken by the particle to reach half of the greatest height is $(3+2\sqrt{2}) : 1$.

Let u be the velocity of projection and h be the greatest height. $\therefore$ $h=\frac{u^2}{2g}$.

So, half of the greatest height is $\frac{u^2}{4g}$.

Let the particle attain this height $\frac{u^2}{4g}$ at time t after projection.

$$\therefore \quad \frac{u^2}{4g}=ut-\tfrac{1}{2}gt^2 \qquad \text{or,} \quad gt^2-2ut+\frac{u^2}{2g}=0 \qquad \cdots \qquad \text{(i)}$$

The two roots of the quadratic equation—(i) are the two times taken by the particle to reach the height $\frac{u^2}{4g}$.

The roots of equation (i) are,

$$\frac{2u\pm\sqrt{4u^2-4\dfrac{u^2}{2g}\cdot g}}{2g}=\frac{2u\pm\sqrt{2u^2}}{2g}=\frac{2u\pm\sqrt{2}u}{2g}$$

So, if t_1 and t_2 be the two times, then

$$t_1=\frac{2u+\sqrt{2}u}{2g} \quad \text{and} \quad t_2=\frac{2u-\sqrt{2}u}{2g}.$$

$$\therefore \quad \frac{t_1}{t_2}=\left(\frac{2u+\sqrt{2}u}{2g}\right)\bigg/\left(\frac{2u-\sqrt{2}u}{2g}\right)=\frac{u(2+\sqrt{2})}{u(2-\sqrt{2})}$$

$$=\frac{(2+\sqrt{2})^2}{(2+\sqrt{2})(2-\sqrt{2})}=\frac{6+4\sqrt{2}}{4-2}=\frac{2(3+2\sqrt{2})}{2}=\frac{3+2\sqrt{2}}{1}.$$

Hence the ratio of the two times is $(3+2\sqrt{2}) : 1$.

Example 4. Show that the velocity at height h of a particle projected vertically upwards from the ground will be the same in magnitude as that due to fall from rest from maximum height H to the same height. [c.f. H. S. 1983]

Let the velocity of projection be u (in magnitude)

$\therefore$ Greatest height attained by the particle is $H=\frac{u^2}{2g}$.

If v be the velocity of the particle at a height h,

then $v^2=u^2-2gh$.

Again if v_1, be the velocity of the particle at the height h due to fall from rest from the greatest height $H=\frac{u^2}{2g}$, then

$$v_1{}^2=2g(H-h) \quad [\because \text{ initial velocity is } 0]$$

$$=2g\left(\frac{u^2}{2g}-h\right)=u^2-2gh=v^2$$

$\therefore$ $v_1=v$ (in magnitude)

Hence the two velocities are equal.

Example 5. A particle is projected vertically upwards under gravity and t secs. afterwards another particle is projected vertically upwards with the same initial velocity as the first. Show that their velocities when they meet will be each $\frac{1}{2}gt$ in magnitude (g acceleration due to gravity) [H. S. 1988]

Let the initial velocity of projection be u in magnitude and they meet at a height h, t_1 seconds after the second particle was projected.

So the height of the first particle $(t+t_1)$ seconds after it was projected is h and so $h=u(t+t_1)-\frac{1}{2}g(t+t_1)^2$

Also height of the second particle t_1 seconds after its projection is $h=ut_1-\frac{1}{2}gt_1{}^2$.

$$\therefore \; u(t+t_1)-\tfrac{1}{2}g(t+t_1)^2=ut_1-\tfrac{1}{2}gt_1{}^2 \quad \text{or,} \quad ut-\tfrac{1}{2}g(t^2+2tt_1)=0$$

$$\text{or,} \quad gtt_1=ut-\tfrac{1}{2}gt^2 \;\therefore\; t_1=\frac{ut-\frac{1}{2}gt^2}{gt}=\frac{u}{g}-\tfrac{1}{2}t.$$

At this instant of meeting the velocity of the first particle is

$$u-g\,(t+t_1)=u-g\left(t+\frac{u}{g}-\tfrac{1}{2}t\right)=u-g\left(\frac{1}{2}\,t+\frac{u}{g}\right)$$

$$=u-\tfrac{1}{2}gt-u=-\tfrac{1}{2}gt.$$

Again at this time the velocity of the second particle is

$$u-gt_1=u-g\left(\frac{u}{g}-\frac{1}{2}t\right)=u-u+\frac{1}{2}gt=\frac{1}{2}gt.$$

Hence the magnitude of the velocities of both the particles at the time of meeting are equal and each is $\frac{1}{2}gt$.

Example 6. Find the height ascended by a particle thrown vertically upwards in the last second of its ascent.

Let O be the highest point ascended by the particle and P be the position of the particle at the beginning of the last second. So PO is the height ascended by the particle in the last second of ascent. Again $PO=OP=$distance traversed by the particle in the first second of descent (as time of ascent from a given point to the highest point O is equal to the time of descent from the highest point to the given point)$=\frac{1}{2}.g.1^2=\frac{1}{2}g=\frac{1}{2}.32=16$ ft.

or, $\frac{1}{2}.9{\cdot}81$ m.$=4{\cdot}95$ metres.

Example 7. A stone, dropped from the top of a tower (freely under gravity) covers $\frac{9}{25}$th of the height of the tower in the last second of its motion. Find the height of the tower. [H.S. 1985]

Let h be the height of the tower. Here the vertically downward velocity of projection of the stone is 0.

If the particle takes t seconds to reach the foot of the tower, then $h=\frac{1}{2}gt^2$ ... ... (1)

Again distance descended by the particle in $(t-1)$ seconds after projection is $\frac{1}{2}g(t-1)^2$.

So distance covered in the last second of its motion is

$=\frac{1}{2}gt^2-\frac{1}{2}g(t-1)^2 \quad \therefore \quad \frac{9}{25}h=\frac{1}{2}gt^2-\frac{1}{2}g(t-1)^2$

or, $\frac{9}{25}\cdot\frac{1}{2}gt^2=\frac{1}{2}gt^2-\frac{1}{2}g(t-1)^2$

or, $\frac{1}{2}g(t-1)^2=\frac{1}{2}gt^2(1-\frac{9}{25})=\frac{1}{2}gt^2\cdot\frac{16}{25}$

or, $25\,(t-1)^2=16t^2$ or, $9t^2-50t+25=0$

or, $(9t-5)(t-5)=0 \quad \therefore \quad t=\frac{5}{9}$ or, 5.

But $t\neq\frac{5}{9}$ as by question the time of descent of the particle is greater than 1 second. Hence $t=5$.

So, the height of the tower is $\frac{1}{2}gt^2=\frac{1}{2}.32.25=400$ ft.

or, $\frac{1}{2}.9{\cdot}8\times25=122{\cdot}5$ metre.

Example 8. Three particles are simultaneously projected vertically upwards from heights x, y, z above the ground with velocities u, v, w respectively and all of them reach the ground at the same instant. Prove that $u(y-z)+v(z-x)+w(x-y)=0$.

[Joint Entrance 1986]

According to the question all the three particles have taken equal times to reach the ground from heights x, y, z respectively. So if the equal time be 't', then taking the vertically downward direction as the positive direction, we get

$$x=-ut+\tfrac{1}{2}gt^2 \cdots \cdots \text{(i)} \quad y=-vt+\tfrac{1}{2}gt^2 \cdots \cdots \text{(ii)}$$

and $z=-wt+\frac{1}{2}gt^2 \cdots \cdots$ (iii)

[As the particles have been projected upwards so the signs of u, v, w are negative].

From (ii)—(iii), (iii)—(i) and (i)—(ii) we get respectively

$$y-z=(w-v)t, \; z-x=(u-w)t \text{ and } x-y=(v-u)t.$$

$$\therefore \; u(y-z)+v(z-x)+w(x-y)$$
$$=t\{u(w-v)+v(u-w)+w(v-u)\}=t.0=0.$$

Example 9. A particle projected vertically upwards reaches a height h in t seconds and takes t' seconds more to reach the ground. Prove that $h=\frac{1}{2}gt\,t'$. [C.U.]

Let the velocity of projection be u.

$$\therefore \; h=ut-\tfrac{1}{2}gt^2.$$

Also by question the time of flight of the particle is $t+t'$.

$$\therefore \; \frac{2u}{g}=t+t' \text{ or, } u=\frac{g}{2}(t+t')$$

$$\therefore \; h=\frac{g}{2}(t+t')t-\tfrac{1}{2}gt^2=\tfrac{1}{2}gt^2+\tfrac{1}{2}gt\,t'-\tfrac{1}{2}gt^2=\tfrac{1}{2}gt\,t'.$$

Example 10. A particle projected vertically upwards, reaches a given height in t seconds and returns to ground after t_1 seconds more. Prove that the greatest height ascended by the particle is $\frac{g}{8}(t+t_1)^2$.

By question the time of flight of the particle is $t+t_1$.

So, if u be the velocity of projection of the particle then

$$t+t_1=\frac{2u}{g} \quad \text{or,} \quad u=\tfrac{1}{2}g\,(t+t_1).$$

Hence the greatest height attained by the particle is

$$\frac{u^2}{2g}=\frac{\{\frac{1}{2}g(t+t_1)\}^2}{2g}=\frac{1}{8}g\,(t+t_1)^2$$

Example 11. A mass of 10 gm falls freely from rest through 10 metres and is then brought to rest after penetrating 5 cm of sand. Find the resistance of the sand (assumed constant) in gm-wt. [Jt. Entrance 1979]

Let the velocity of the particle be v after fall through 10 metres=1000 cm.

$\therefore\ v^2=2\times980\times1000$ or, $v=1400$ cm/sec.

Let now f be the resultant of the acceleration due to gravity vertically downwards and the uniform acceleration f_1 vertically upwards due to the uniform resistance of the sand. As the mass is brought to rest so f is retardation of the mass and

$f=f_1-g$ or, $f_1=f+g$

As the particle is brought to rest after penetrating 5 cms in the sand,

so $0=v^2-2f.5$ or, $f=\frac{v^2}{10}=\frac{2\times980\times1000}{10}=196000.$

$\therefore\ f_1=f+g=196000+980=196980\text{cm/sec}^2.$

So, the resistance of the sand$=P=mf_1=10\times196980$

$=1969800$ dyne$=\frac{1969800}{980}$ gm wt$=2010$ gm wt.

Example 12. A stone is dropped into a well and the sound of the splash is heard in $3\frac{9}{10}$ seconds, if the velocity of sound is 1120 ft/sec, find the depth of the well. [Joint Entrance 1980]

Let the stone takes t_1 seconds to reach the bottom of the well and sound takes t_2 seconds to reach the ground.

$\therefore\ t_1+t_2=3\frac{9}{70}=\frac{219}{70}$ ... ... (i)

Let h ft be the depth of the well.

So considering the fall of the stone we get $h=\frac{1}{2}gt_1^2=16t_1^2$

Again considering the motion of sound we get

$h=1120t_2=1120(\frac{219}{70}-t_1)$ [From (i) $t_2=\frac{219}{70}-t_1$]

$\therefore\ 16t_1^2=1120(\frac{219}{70}-t_1)$

or, $t_1^2=70(\frac{219}{70}-t_1)=219-70t_1$ or, $t_1^2+70t_1-219=0$

or, $(t_1+73)(t_1-3)=0$ $\therefore\ t_1=-70$ or, 3.

But time cannot be negative. So $t_1=3$.

So, the height of the well$=16t_1^2=16.9=144$ ft.

Example 13. A piece of stone is let fall from rest into an empty pit and the sound of the stone striking the bottom of the pit is heard after t seconds. Prove that if v be the velocity of sound per second and the depth of the pit be h, then $2h\left(1+\frac{gt}{v}\right)=gt^2$, where the magnitude of v is so large in comparison with h so that $\left(\frac{h}{v}\right)^2$ can be neglected. [C. U.]

Let the time taken by the stone to reach the bottom of the pit be t_1 $\therefore$ $h=\frac{1}{2}gt_1{}^2$ The time taken by sound to reach the ground from the bottom of the pit is $t-t_1$.

$$\therefore\ h=v(t-t_1)=v\left(t-\sqrt{\frac{2h}{g}}\right).\ \left[\because\ h=\tfrac{1}{2}gt_1{}^2,\ \therefore\ t_1=\sqrt{\frac{2h}{g}}\right]$$

$$\text{or,}\ \frac{h}{v}=t-\sqrt{\frac{2h}{g}}\ \ \text{or,}\ \sqrt{\frac{2h}{g}}=t-\frac{h}{v}$$

$$\text{or,}\ \left(\frac{2h}{g}\right)=\left(t-\frac{h}{v}\right)^2=t^2-\frac{2ht}{v}\qquad\left[\text{neglecting}\left(\frac{h}{v}\right)^2\right]$$

$$\text{or,}\ 2h=gt^2-\frac{2htg}{v}$$

$$\text{or,}\ gt^2=2h+\frac{2hgt}{v}=2h\left(1+\frac{gt}{v}\right).$$

Example 14. From a lift ascending with an acceleration f ft/sec², a man throws a ball vertically upwards with a velocity v ft/sec, relatively to the lift. If the man catches the ball t seconds afterwards, show that $f+g=\frac{2v}{t}$. [C. U. 1964]

Let the velocity of the ball be u when the ball is thrown. So the ball is thrown vertically upwards with velocity $u+v$.

The man catches the ball again t seconds afterwards. So in these t seconds, height ascended by the lift=height ascended by the ball.

$$\text{or,}\ ut+\tfrac{1}{2}ft^2=(u+v)t-\tfrac{1}{2}gt^2$$

$$\text{or,}\ \tfrac{1}{2}(f+g)t^2=vt.\ \text{or,}\ f+g=\frac{2v}{t}\ \text{ft.}$$

Example 15. From an aeroplane rising straight upwards with a uniform acceleration f, a stone is dropped; 8 seconds after this another stone is dropped from it. Show that the distance between the two stones t seconds after the second stone is dropped is $8(f+g)(t+4)$.

Let the two stones be dropped from A and B respectively and when the second stone is dropped then the position of the first be at C. Let t seconds after the second stone is dropped, their positions be at E and D respectively. The point C is below the point B, as in the 8 seconds before the second stone is dropped, the aeroplane was ascending with acceleration f and initial velocity v (say) and the first stone was ascending with the same inital velocity v and acceleration $-g$.

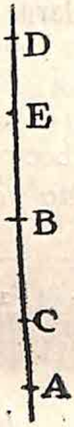


Fig. 5·3

$\therefore$ AE=displacement of the first stone in $(t+8)$ seconds$=v(t+8)-\frac{1}{2}g(t+8)^2$

and AB=displacement of the aeroplane in 8 seconds$=8v+\frac{1}{2}.f.8^2=8v+32f$.

BD=displacement of the second stone in t seconds

$$=(v+8f)t-\tfrac{1}{2}gt^2.$$

[The initial velocity of projection of the second stone =velocity of the aeroplane at $B=v+8f$].

So the required distance $DE=AD-AE=AB+BD-AE$

$$=8v+32f+(v+8f)t-\tfrac{1}{2}gt^2-\{v(t+8)-\tfrac{1}{2}g(t+8)^2\}$$
$$=8v+32f+vt+8ft-\tfrac{1}{2}gt^2-vt-8v+\tfrac{1}{2}gt^2+8gt+32g$$
$$=32f+8ft+8gt+32g=8(4f+ft+gt+4g)$$
$$=8\{4(f+g)+t(t+g)\}=8(f+g)(t+4).$$

Example 16. A stone falling from the top of a vertical tower has descended x ft. when another is let fall from a point y ft. below the top. If they fall from rest and reach the ground together, show that the height of the tower is $\dfrac{(x+y)^2}{4x}$.

Let the tower be OA and C and B are the two points x ft and y ft below the top of the tower. If the velocity of the first particle at c be v then $v^2=2gx$ or, $v=\sqrt{2gx}$. So, when the second particle is lef fall from the point B from rest, then the velocity of the first particle at C is $v=\sqrt{2gx}$. Hence as the two particles reach the ground together, so the point C is above the point B. Let the two particles reach the ground t seconds after the second particle is released. Let h be the height of the tower.

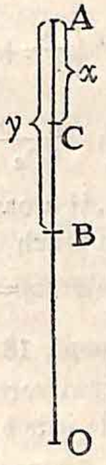


Fig. 5.4

$\therefore\ CO=h-x$, and $BO=h-y$.

$\therefore\ h-x=$ displacement of the first particle during these t seconds $=\sqrt{2gx}\,t+\frac{1}{2}gt^2$.

and $h-y=$ displacement of the second particle during thest t seconds $=\frac{1}{2}gt^2$

$\therefore\ (h-x)-(h-y)=\sqrt{2gx}\,.\,t+\frac{1}{2}gt^2-\frac{1}{2}gt^2$.

$$\text{or,}\quad y-x=\sqrt{2gx}\cdot t \quad \therefore\quad t=\frac{y-x}{\sqrt{2gx}}.$$

$$\text{So, } h=y+\tfrac{1}{2}gt^2=y+\tfrac{1}{2}g\,.\frac{(y-x)^2}{2gx}$$

$$=y+\frac{(y-x)^2}{4x}=\frac{4xy+(y-x)^2}{4x}=\frac{(x+y)^2}{4x}.$$

Example 17. From the same height two particles are projected with the same velocity, one vertically upwards and the other vertically downwards. They reach the ground t_1 and t_2 seconds after projection. Prove that if a particle is let fall from rest from the same height, then it will reach the ground $\sqrt{t_1t_2}$ seconds.

Let each of the two particles be projected from height h above the ground and magnitude of each velocity of projection be u. If the downward direction is taken as the positive direction, then the velocity of projection of the first particle is $-u$ and that of the second is $+u$.

Since they reach the ground in t_1 and t_2 seconds respectively,

$$\text{So,}\quad h=-ut_1+\tfrac{1}{2}gt_1^2 \quad\text{or,}\quad \frac{h}{t_1}=-u+\tfrac{1}{2}gt_1 \quad \cdots\ \cdots\text{(i)}$$

$$\text{and}\quad h=ut_2+\tfrac{1}{2}gt_2^2 \quad\therefore\quad \frac{h}{t_2}=u+\tfrac{1}{2}gt_2 \quad \ldots\ \ldots\ \text{(ii)}$$

Adding equations (i) and (ii) we get $\frac{h}{t_1}+\frac{h}{t_2}=\frac{1}{2}g\,(t_1+t_2)$

or, $h\left(\frac{1}{t_1}+\frac{1}{t_2}\right)=\frac{1}{2}g\,(t_1+t_2)$

or, $h\,\frac{(t_1+t_2)}{t_1t_2}=\frac{1}{2}g\,(t_1+t_3)$ or, $h=\frac{1}{2}gt_1t_2$.

Now, if a particle is let fall from rest from the height h, takes time t to reach the ground, then $h=\frac{1}{2}gt^2$

or, $\frac{1}{2}gt_1t_2=\frac{1}{2}gt^2$ or, $t_1t_2=t^2$ or, $t=\sqrt{t_1t_2}$

Example 18. From a place A above the surface of the earth is projected vertically upwards. The velocity of the particle when a particle it is h below A, is double the velocity when it was at a height h above A. Show that the greatest height attained by the particle above A is ^5h.

Let the two points B and C are vertically h above and below A and the velocity of the particle at A and B are u and v respectively. So the velocity of the particle at c is $2v$ in magnitude.

Let us take the vertically upword direction as the positive direction.

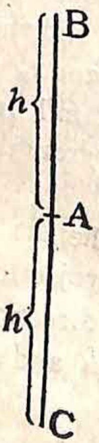


Fig. 5.5

So the velocities at B and C are v and $-2v$ respectively and

$$v^2=u^2-2gh \qquad \ldots \quad \ldots \text{ (i)}$$

$$\text{and } (-2v)^2=u^2+2(-g)(-h)$$
$$=u^2+2gh$$
$$=v^2+2gh+2gh \quad [\text{From (i) } u^2=v^2+2gh\,]$$
$$=v^2+4gh. \quad \text{or, } 4v^2=v^2+2gh$$

$$\text{or, } 3v^2=4gh \quad \text{or, } v^2=\frac{4gh}{3}.$$

$$\therefore \quad u^2=v^2+2gh=\frac{4gh}{3}+2gh=\frac{10gh}{3}.$$

So the greatest height of the particle above A is

$$\frac{u^2}{2g}=\frac{10gh}{2g.3}=\frac{5}{3}h$$

Exercise 5B

1. The greatest height attained by a particle projected vertically upwards is 5886 cm ; find at what time after projection the particle will be at a height of 3270 cm. Explain the double answer. [H.S. 1973]

2. From the bottom of a cliff 78·4 metres high a stone is thrown vertically upwards with a velocity which is just sufficient to carry it to the top. After two seconds another stone is dropped from the top. Where do they meet (Take $g=980$ cms/sec^2). [Madhyamik '76]

3. A stone is thrown vertically upwards with a velocity of 60ft/sec. After what time will its velocity be 20ft./sec. and at what height will it be then ? [C.U.]

4. A particle falls from rest and in the last second of its fall passes through 224 ft. Find the height from which it fell and the time of its fall.

5. Two particles are projected upwards, one with a velocity of 64 ft./sce. and the other with a velocity of 32 ft./sec. Find the distance between their highest points [C.U.]

6. The height of the lowest storey of a sky scraper is 50ft. A stone dropped from the top of the building was observed to cross the lowest storey in a quarter of a second. Find the height of the sky scraper.

7. A body falling from rest under gravity passes a certain point with a velocity of 120 ft./sec. where was it 2 secs. previously and where will it be 2 secs. later ? [B.U.H.]

8. A piece of stone falling freely under gravity from rest passes two points at a distance of 10 metres from each other at an interval of $\frac{1}{5}$ second. From what height above the higher point was the particle dropped ?

9. Three particles are projected vertically upwards at the same time from three places at heights h_1, h_2, h_3 above the ground with velocities v_1, v_2, v_3 respectively. The particles reach the ground at the same time. Show that

$$\frac{h_1-h_2}{v_1-v_2}=\frac{h_2-h_3}{v_2-v_3}=\frac{h_3-h_1}{v_1-v_2}.$$

10. A stone is projected vertically upwards from a place at height h above the ground with a velocity that a particle dropped

from rest acquires after falling a height $\frac{h}{2}$. Prove that the stone will reach the ground after $(1+\sqrt{3})\sqrt{\frac{h}{g}}$ seconds.

11. A particle falling freely from rest under gravitational attraction describes 49 metres in a particular second. Find the total distance traversed by the particle upto that second.

12. A ball thrown up is caught by the thrower 7 secs. afterwards. With what speed was it thrown ?

13. A bomb is let fall from an aeroplane ascending vertically upwards with uniform velocity reaches the ground in 5 seconds. At what height was the aeroplane, when the bomb reached the ground ?

14. A body is let fall from rest freely under gravity from a height 117·6 metres above the ground. 4 seconds afterwards another body is projected vertically upwards with a velocity of 39·2 metres per second. When and where will the two bodies meet ?

15. A particle falls freely from the top of a tower. During the last second of its motion, it falls $\frac{5}{9}$th of the whole height. What is the height of the tower ?

16. A particle is projected vertically upwards with a velocity of u ft. per sec. and after t seconds another particle is projected upwards from the same point with the same initial velocity. Prove that the particles will meet after a lapse of time $\left(\frac{t}{2}+\frac{u}{g}\right)$ secs. from the starting of the first at a height $\frac{4u^2-g^2t^2}{8g}$. [H. S. 1960]

17. A stone falling freely from the top of a tower takes t secs. to describe the last h ft. Show that the total time of fall is $\left(\frac{t}{2}+\frac{h}{gt}\right)$ secs.

18. Two points A and B are in the same vertical line and A is above B. A body is projected vertically upwards from B with a velocity just sufficient to carry it to A. At the same time another body is dropped from rest under gravity from A. Prove that the two bodies will meet with equal and opposite velocities and at the time of meeting the distances described by them are in the ratio 3 : 1.

19. A rocket ascending vertically from the ground explodes when it reaches the greatest height. The initial velocity of the rocket was $\sqrt{2gy}$ ft. per sec. and the interval between the sound of explosion reaching the place of starting of the rocket and a place at a distance x ft. from it is $\frac{1}{n}$th of a second. Prove that the velocity of sound is $n\,(\sqrt{x^2+y^2}-y)$ ft. / sec.

20. A particle dropped from a height h after falling $\frac{2}{3}h$ passes another particle thrown vertically upwards at the same time when the first was dropped. Find the greatest height of the later particle.

21. A stone is dropped into a well and the sound of its striking the water is heard. in $2\frac{7}{12}$ seconds. If the stone strikes the water with a velocity of 80 ft./sec., find the velocity of sound.

22. A body is thrown up in a lift with a velocity u relative to the lift and returns to the lift in time t. Show that the lift's upward acceleration is $\frac{2u-tg}{t}$. [B. U. '65]

23. A piece of stone released from a balloon rising vertically at the rate of 980 cm./sec reaches the ground in 17 seconds. Find the height from which the stone was released from the balloon.

24. A piece of stone was released from a balloon ascending vertically upwards with an acceleration 11 ft./sec^2, when the height of the balloon was 352 ft. How long will the piece of stone be ascending? When will it reach the ground? What was the greatest height of the piece of stone? (Here initial velocity of the balloon is 0).

25. A stone is dropped from rest from the top of a tower 294 meters high. The stone after describing 76·4 metres meets another particle projected vertically upwards at the instant when the first was dropped. Find the greatest height of the 2nd particle.

26. From a lift ascending upwards with a velocity 5 ft /second a screw dropped all on a sudden. Find the velocity of the screw and the distance of the lift and screw after $1\frac{1}{2}$ seconds.

27. A particle is projected vertically upwards with a velocity u. After n seconds another particle is projected vertically upwards from the same place with velocity v and the two particles meet at the greatest height of the first. Prove that

$$2(v-u)\,(u-ng)=n^2g^2.$$

28. A man in a lift, ascending with an acceleration 4 ft/sec^2 throws a ball vertically upwards with a velocity of 36 ft. per second relative to the lift. After what time will the man again catch the ball ?

29. From an aeroplane ascending vertically upwards with acceleration f a body is dropped and after 4 more seconds another particle is let fall from the aeroplane. Show that 6 seconds after the first was released, the distance between the two particles will be $16(g+f)$.

30. A stone is let fall freely from rest into a well h ft. deep and the sound of the stone striking the bottom of the well is heard after t secs. If the velocity of sound be v, then prove that

$$gh^2-2(v^2+gtv)h+gv^2t^2=0.$$

31. A particle is projected vertically upwards and after t seconds another particle is projected in the same direction with the same velocity. Prove that the particles will meet each other after $\frac{1}{2}gt$ seconds.

32. B, C, D are three points on the same verticle line through a fixed point A so that $AB=BC=CD$. Prove that the times that a particle falling freely from rest from A will take to describe the distances AB, BC, CD are in the ratio

$$1:(\sqrt{2}-1):(\sqrt{3}-\sqrt{2}).$$

33. Two particles projected from the same height with same velocity vertically upwards and vertically downwards reach the ground after t_1 and t_2 seconds respectively. Prove that the particles were projected from the height $\frac{1}{2}gt_1t_2$ above the ground.

34. Two particles are projected vertically upwards from two places on the surface of the earth with the same velocity. If h_1 and h_2 be the greatest heights of the particle, then show that $\dfrac{h_1}{h_2}=\dfrac{g_2}{g_1}$ where g_1 and g_2 are the values of acceleration due to gravity in the two places.

35. A particle acquires a velocity v more and takes t times less at one place on the surface of the earth than at another in falling freely through the same height. Prove that the geometric mean of the numerical values of g in the two places is $\dfrac{v}{t}$.

ANSWERS

Exercise 1

1. 11 sq. cm./second. 2. will decrease at the rate of $\frac{1}{14}$ cm./sec.
3. $12\frac{1}{2}$ cu. ft./min. 6. (3, 6). 7. 2·5 metre/sec.
8. ·36 cm./min. 9. $\frac{5}{4}$ sq. cm./sec.
11. 11·25 sq. metre/unit time. 12. $82\frac{6}{17}$ ft./sec.
13. 0·6 metre/sec. 14. 3 miles per hour.
15. 12 sq. inches/second. 16. 6 sq. cm./second.
17. $4\frac{4}{5}$ km./hour. 18. $1\frac{1}{3}$ metre/second. 19 $1\frac{1}{7}$ ft./min.
20. 120π sq. ft./sec. 22. $4\frac{1}{4}$ unit/second ; $\frac{31}{32}$ unit/sec^2.
25. velocity$=Ae^{-2t}\cos(5t+\alpha)$ where $A=\sqrt{29}$, $\alpha=\tan^{-1}\frac{2}{5}$.
When the velocity will be zero, then $t=\frac{1}{5}\left(\frac{\pi}{2}-\alpha\right)$.
27. decreasing if $x>3$ or, $x<1$, increasing if $1<x<3$.
29. (a) $-2<x<4$ (b) $0<x<1$ or, $x>2$.
38. $f(x)$ is a decreasing function of x when $x<1$.
39. $-\frac{1}{2}\leqslant x<0$, $x\geqslant\frac{1}{2}$; $x\leqslant-\frac{1}{2}$, $0\leqslant x\leqslant\frac{1}{2}$.
40. (i) 1·261 ; $3x^2\,dx$ (ii) 3·000 3001 ; $4\pi r^2\,dr$.
(iii) 0·0011 ; ·4343 $\left(\frac{dx}{x}\right)$. 41. (i) $-\frac{3\Delta x}{x(x+\Delta x)}$, $-\frac{3\,dx}{x^2}$
(ii) $-2\sin x\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}$; $-\sin x\,dx$.
42. (i) $-3\cos^2 x\sin x\,dx$, (ii) ·4343 $\frac{dx}{x}$, (iii) $2e^{2x}\,dx$
45. (i) 9·1 (ii) 2·502 46 (i) 96·7 (ii) 30·44
47. (i) 2·313 (ii) 1·004343. 48. 2·606344
49. (i) 1·93 (ii) ·848 (iii) 1·034.
50. 251·3 sq. cm. 52. ·02 kg./cm^2. 53. 8 metre (nearly)
54. ·00217 (nearly). 58. 4·4 sq. cm. and $2\frac{6}{7}$%.
60. (i) 2% decrease ; (ii) 1% increase.
61. 1·4% increase.

Exercise 2

1. (i) $x+2y=5$ (ii) $3x+4y=32$
(iii) $x+y=3$. (iv) $x-4y+24=0$ (v) $y=\pm x$.
(vi) $\sqrt{7}x+4y=16$, $\sqrt{7}x-4y=16$, $\sqrt{7}x-4y+16=0$,
$\sqrt{7}x+4y+16=0$.
(vii) $2x+3y=12$ (viii) $3x-2y=4$.

(ix) $\dfrac{xx_1^{-1/3}}{a^{2/3}}+\dfrac{yy_1^{-1/3}}{b^{2/3}}=1.$ (x) $x\sin\frac{t}{2}-y\cos\frac{t}{2}=a\sin\frac{t}{2}.$

(xi) $axx_1+h(xy_1+x_1y)+byy_1=1.$

(xii) $y-y_1=a(\cot x_1)(x-x_1)$

2. (i) $x+y-3=0$ (ii) $x+y\pm3=0.$

(iii) $4x-7y=1,\ 4x+7y=1.$ (iv) $4x-2y=1.$

(v) $3x-4y-6=0.$ (vi) $x\cos t-y\sin t=a\cos 2t.$

(vii) $\dfrac{xy_1^{m-1}}{b^m}-\dfrac{yx_1^{m-1}}{a^m}=\dfrac{x_1y_1^{m-1}}{b^m}-\dfrac{y_1x_1^{m-1}}{a^m}$

3. $20y-x+7=0,\quad y+20x=140.$

5. $x+2y\pm10=0$; 6. $5x-12+17=0,\ 5x-12y-152=0.$

8. $(\frac{5}{3}, 10), (15, 0).$ 9. $m=-\frac{3}{5}, c=-5.$

10. $x+2y+4=0.$ 11. $(-1, 1).$ 12. $\pm\frac{5}{12}.$

13. $3x+4y\pm15=0.$ 14. $3x+4y\pm25=0.$

15. (i) $3x+4y\pm25=0$ (ii) $5x\pm12y=65.$

16. $4x+3y+19=0,\ 4x+3y-31=0.$

17. $x+2y+11=0,\quad x+2y-9=0.$

18. $\left(\dfrac{\sqrt{2}+1}{\sqrt{2}}, \dfrac{\sqrt{2}+1}{\sqrt{2}}\right).$ 19. $c=2(1\pm\sqrt{1+m^2})$

20. (i) $y=mx\pm a\sqrt{1+m^2}$ (ii) $x+my\pm a\sqrt{1+m^2}=0$
(iii) $x-y=\pm\sqrt{2}a.$

21. $x^2+y^2-4x-6y-11=0,\ y=x+3, y=x-1$

26. $64x+16y+7=0.$ 27. $9x+6y+8=0$; $(\frac{8}{9}, -\frac{8}{3}).$

28. 12. 29. $(am^2, -2am).$ 30. $(-\frac{16}{5}, \frac{9}{5})$

31. $\left(-\dfrac{2\sqrt{21}}{21}, \dfrac{\sqrt{2}}{14}\right).$ 33. ±1; $(-1, 2), (1, 2).$

34. $y=3x\pm\frac{1}{2}\sqrt{\dfrac{155}{3}}.$ 35. $x-3y+2=0$; $x-3y=2.$

$(-1, \frac{1}{3}), (1, -\frac{1}{3}).$ 36. $x+y=5.$ 37. (i) $24y-30x\pm\sqrt{161}=0$

(ii) $x+2y\pm2\sqrt{2}=0.$ 38. $(-\frac{8}{3}, -\frac{7}{3}).$ 39. $(-2, 1).$

40. $y=\sqrt{3}x\pm2\sqrt{3}.$ 41. $(\frac{1}{2}, 2).$ 42. $y=2x-12.$

43. $y=2x-9$; $(3, -3).$ 44. $(2, 1).$ 46. $\left(\dfrac{a}{3}, \dfrac{-2a}{\sqrt{3}}\right).$

47. $y=\sqrt{3}(x+1)$; $(1, 2\sqrt{3}).$ 48. $(6, -4\sqrt{3}).$

49. $x-3y+18=0,\ 9x+3y+2=0.$ 90°.

51. $2x+y+1=0$; $(\frac{1}{2}, -2)$; $2y-x-8=0$; $(8, 8).$

53. $(3a, 2\sqrt{3}a)$. 65. (i) $(gl+fm-n)^2=(l^2+m^2)(g^2+f^2-c)$.
(ii) $gl+fm-n=0$. 66. $p=0$ or, $2a$. 69. $am^2=ln$.
70. $a=5,\ b=4$. 71. $a^2l^2+b^2m^2=n^2$

72. $\dfrac{a^2}{l^2}+\dfrac{b^2}{m^2}=\dfrac{(a^2-b^2)^2}{n^2}$ 73. $x^2+y^2=\dfrac{1}{a^2}$.

74. $al^3+2alm=m^2n$. 78. (a) (i) $c^2=a^2m^2+b^2$
(ii) $c^2=a^2m^2-b^2$ (b) (i) $a^2l^2-b^2m^2=n^2$
(ii) $a^2l^2+b^2m^2=n^2$. 81. $2x+3y+36=0$.

83. $y=\pm x\pm\sqrt{a^2+b^2}$.

84. (i) $\dfrac{\pi}{4}$ at every point of intersection.

(ii) $\tan^{-1}\left[\dfrac{3a^{1/3}b^{1/3}}{3(a^{2/3}+b^{2/3}}\right]$

85. $\tan^{-1}\left(\frac{12}{41}\right)$.

86. $\tan^{-1}\left(\dfrac{3}{2^{2/3}+2^{4/3}}\right)$; $\tan^{-1}\left(\dfrac{1}{\sqrt[3]{4}}\right)$ and $\tan^{-1}(\sqrt[3]{16})$.

95. $1\frac{4}{5}$. 100. $(\frac{5}{3}, 5)$. 103. $x+2y=0$; $2x+y=0$.
105. $4x-5y+25=0$, $(-4, \frac{9}{5})$; $x-4y-13=0$, $(\frac{25}{13}, \frac{-36}{13})$.
109. $x^2+y^2=2a^2$. 110. $y^2-4ax=(x+a)^2\tan^2\alpha$.
113. $x^2+y^2=a^2$.
115. $(x^2+y^2-a^2-b^2)=4\cot^2\theta\,(b^2x^2+a^2y^2-a^2b^2)$.
116. $(x^2+y^2)^2=a^2x^2+b^2y^2$. 120. $x^2+y^2=a^2$.

122. $(\frac{33}{4}, 2)$, $\frac{1}{4}$.

124. Length of tangent$==2a\sin\dfrac{\theta}{2}$; length of normal $=2a\sin\dfrac{\theta}{2}\tan\dfrac{\theta}{2}$; length of subtangent$=a\sin\theta$; length of subnormal$=2a\sin^2\dfrac{\theta}{2}\tan\dfrac{\theta}{2}$.

126. $\frac{1}{4}a\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2$; $\frac{1}{4}a\left(e^{\frac{2x}{a}}-e^{-\frac{2x}{a}}\right)$.

Exercise 3

1. (i) Max. value 31 at $x=-3$; Min. value -1 at $x=1$.
(ii) Max. value -2 at $x=-1$; Min. value 2 at $x=1$.

(iii) Max. value $(-\frac{2}{9}\sqrt{3})$ at $x=2+\frac{1}{\sqrt{3}}$.

Min. value $(\frac{2}{9}\sqrt{3})$ at $x=2-\frac{1}{\sqrt{3}}$.

(iv) Max. value 5 at $x=1$.

(v) Max. value 0 at $x=1$; Min. value $-\frac{14}{9}$ at $x=-1$.

2. Max. value 7 at $x=2$; At $x=1$ and 3, Min. value 6, 6.

5. $\sqrt{2}$; $\frac{\pi}{4}$. 7. (i) Max. value 2 ; Min. value 0.

(ii) Max. value 1 and -1 ; Min. value $\frac{5}{4}$ and $\frac{1}{4}$.

8. (i) Max. value $4\frac{1}{3}$, Min. value 3.

(ii) when $\theta=\tan^{-1}\sqrt{\frac{p}{q}}$ then Max. value $\left\{\frac{p^p q^q}{(p+q)^{p+q}}\right\}^{\frac{1}{2}}$.

10. Max. value $\frac{4}{3\sqrt{2}}+\frac{1}{2}$ at $x=\frac{\pi}{4}$ and $\frac{4}{3\sqrt{2}}-\frac{1}{2}$ at $x=\frac{3\pi}{4}$. Min value $\frac{\sqrt{3}}{4}$ at $x=\frac{2\pi}{3}$.

11. No. Max. or Min. value.

12. (i) No. max. or min. value. (ii) Min. value. 13. $2\sqrt{3}$.

15. At $x=\frac{\pi}{2}$, Min value 0 ; at $x=\sin^{-1}\frac{1}{4}$ Max. value $\frac{9}{8}$.

16. Max. at $x=\frac{3\pi}{2}$; Min. at $x=\frac{\pi}{2}$.

17. Max. value 1; at Min. value $\left(\frac{1}{e}\right)^{1/e}$

18. at $x=e$ Max. value $\frac{1}{e}$; at $x=\frac{1}{e}$. Min. value $\left(\frac{-1}{e}\right)$

20. $a=-28$ $b=12$. 21. greatest value 136 ; smallest value -8.

23. 1 and 11 (sum of square of 1 and two times of 11 is least).

24. Max. value $-2(ab)^{\frac{1}{2}}c$; Min. value, $2(ab)^{\frac{1}{2}}c$.

25. 1. 26. $\frac{(a^{\frac{2}{3}}+b^{\frac{2}{3}})^3}{c^2}$.

32. least velocity12 units/second at $t=2$.

33. length $\frac{\sqrt{6A}}{2}$, width $\frac{\sqrt{6A}}{2}$; least length of fence $2\sqrt{6A}$ (each in foot).

36. $x=2k$. 37. 18 38. 40 k.m. / hour. 40. Rs. 1600.

Exercise 4

1. $\frac{9}{2}$ sq. units 2. $\frac{3}{2}$ sq. units 3. 80 sq. units
4. 1 sq. unit. 5. (a) $\frac{1}{2}$ sq. unit. (b) 8 sq. units.
9. (a) $\frac{8}{3}\sqrt{a}h^{3/2}$ sq. units (b) $\frac{a^2}{3}$ sq. units 10. $\frac{1}{6}$ sq. units.
12. $(\frac{1}{2}\pi+\frac{4}{3})$ sq. units. 13. $\frac{2}{\sqrt{ab}}\tan^{-1}\frac{2\sqrt{ab}}{b-a}$ sq. units.
14. $\frac{28}{3}$ sq. units. 15. (a) 12 sq. units. 16. 2 sq. units.
17. (i) 1 sq. unit, (ii) 1 sq. unit, (iii) 2 sq. units.
18. (i) 2 sq. units, (ii) 2 sq. units, (iii) 4 sq. units.
19. $\frac{16}{3}$ sq. units. 20. $\frac{32}{3}$ sq. units. 21. $\frac{8}{3}$ sq. units.
22. 3π sq. units. 23. $\frac{16}{3}$ sq. units.

Exercise 5A

3. After 2 seconds ; 6 units/sec^2 ; 24 units/sec ; 18 units/sec^2.

4. The particle moves with uniform acceleration if $0 \leq t \leq 2$. If $t > 2$, the particle moves with uniform velocity 6 units/sec.

8. a is the displacement of the particle at time $t=0$; b is the velocity of the particle at time $t=0$; c is $\frac{1}{2}$ of the acceleration of the particle at time $t=0$; d is $\frac{1}{6}$ th of the rate of change of acceleration of the particle at time $t=0$.

9. 0 and $-\frac{4}{27}$ units 10. $3ax \cos x$. 11. 10 hours ; 10 miles.

14. (i) $f=3$ cm. / sec^2 ; $s=136$ cm.

(ii) $v=50\sqrt{10}$ cm. / second ; $t=10(\sqrt{10}-2)$ second.

15. $\frac{200}{9}$ metres. 17. No. 18. $20\frac{5}{6}$ miles.

19. 3 more seconds. 21. ·0008 seconds ; 1020 ft. (nearly).

31. The particles will meet 8 seconds after start ; the maximum distance between the particles will be 48 ft. at time 4 seconds after start. 36. At a distance 225 ft. after 15 seconds.

Exercise 5B

1. After $\frac{2\sqrt{3}}{3}$ and $\frac{10\sqrt{3}}{3}$ seconds.

2. 3920 cm./second. 3. After $\frac{5}{4}$ seconds at a height 50 ft.

4. 900 ft ; $\frac{15}{2}$ seconds. 5. 48 ft. 6. $650\frac{1}{4}$ ft.

7. At a height 176 ft. above the fixed point two seconds before passing it. At a height 304 ft. below the fixed point 2 seconds after passing it.

8. 122·5 ft. (nearly) 11. 148·25 metres.

12. 112 ft./second. 13. 400 ft.

14. At a height 39·2 ft. above the ground 1 second after the projection of the second particle.

15. 144 ft. 20. $\frac{3}{8}h$. 21. 1200 ft. / sec. 22. 1200 ft /sec.

23. 4080 ft. 24. $2\frac{3}{4}$ seconds ; After 8·2 seconds (nearly) ; 473 ft. 25. 122·5 metres.

26. 43 ft / second downwards ; 36ft. 28. 2 seconds.

Objective and Short Answer Type Questions

[Below is a list of objective and short answer type questions on the topics discussed in this volume. This list is not complete. In the different Examples and Exercises these types of questions have been discussed. Students are advised to go through these examples and exercises. This list may be called a sample set.]

1. (i) Show that if the sum of the digits of a number of 3 digits be divisible by 3, then the number is divisible by 3.

(ii) Show that the sum or product of three consecutive natural numbers is divisible by 3.

(iii) Show that the square of an odd integer can be written in the form $8m+1$ where m is a natural number or 0.

2. The decimal expression of a number is non-terminating and recurring. Determine whether the number is rational or irrational ?

3. Which of the following numbers are rational and which are irrational ?

$$\tfrac{5}{8},\ \sqrt{2}+1,\ \pi-2,\ \frac{e}{3},\ \sqrt{3}+\sqrt{2}+1.$$

4. $x\sqrt{3}+y\sqrt{5}=0$ and x and y are rational. Show that $x=y=0$.

5. "The sum of two irrational numbers is always irrational" —Comment with reasons or examples on the validity of the statement.

6. If $c\neq 0$ be a real number, then show that

$$|\,f(c)-f(-c)\,|=2, \text{ where } f(x)=\frac{|\,x\,|}{x}.$$

[Joint Entrance 1983]

7. (i) If $f(x+y)=f(x)+f(y)$, show that $f(0)=0$.

(ii) If $f(x+y)=f(x)\,.\,f(y)$ and $f(2)\neq 0$, then show that $f(0)=.1$.

8. If $f(x)$ be an odd function, show that $f(0)=0$.

9. (i) Show that the product of r consecutive natural numbers is divisible by $\lfloor r$.

(ii) Show that if the sum of p positive integers be even, then the number of odd integers among them cannot be odd.

[Cf. I. I. T. 1985]

10. Determine the domain of definition of the following functions.

(i) $f(x)=\sqrt{6-x}$; [Joint Entrance 1984]

(ii) $f(x)=\sqrt{4-x^2}$; (iii) $f(x)=\sqrt{x^2-9}$;

(iv) $\cos^{-1}\sqrt{x-1}$.

11. Find the domain of definition of the following functions.

(i) $\dfrac{x-2}{x-2}$, (ii) $\dfrac{x}{x^2+x}$ (iii) $\dfrac{x}{\cos x}$ (iv) $\dfrac{\sqrt{9-x^2}}{2-x}$.

12. "a, b, c, d are four unequal real numbers and $f(a)=f(b)$, $f(c)=f(d)$. So $f(x)$ is a constant function." Is the statement correct ?

13. Find the domain of definition and range of the function $f(x)=\sin\log_e\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)$. [I. I. T. 1985]

14. (i) If $f(x)=\frac{1}{2}(a^x+a^{-x})$ [$a>0$], show that

$$f(x+y)+f(x-y)=2f(x)\,f(y).$$

(ii) If $f(x)=\log_e{}^x$ and $\phi(x)=\log_e{}^x$, show that $f\{\phi(x)\}=\phi\{f(x)\}$.

15. Of the following statements which are correct ?

If $y=f(x)=\dfrac{x+2}{x-1}$, then

(A) $x=f(y)$; (B) $f(1)=3$; (C) If $x<1$, y increases with x ; (D) $f(x)$ is a rational function of x. [I. I. T. 1984]

16 If $f(x)=\log\dfrac{1-x}{1+x}$, show that $f(a)+f(b)=f\left(\dfrac{a+b}{1+ab}\right)$.

17. Are the two functions x and $\dfrac{x^2}{x}$ the same ? Give reasons for the answer. [H. S. 1979]

18. "If $\underset{x\to a}{Lt}\,f(x)$ exists, then $\underset{x\to a}{Lt}\,f(x)=f(a)$." Is the statement valid ? Support your answer with reason or example.

19. "If $f(a)$ does not exist, $\underset{x\to a}{Lt}\,f(x)$ may exist". Support the statement with an example.

20. Comment on the validity of the following statements with reasons :

(i) $\underset{x\to a}{Lt}\dfrac{x^2-a^2}{x-a}=\underset{x\to a}{Lt}\dfrac{x^2}{x-a}-\underset{x\to a}{Lt}\dfrac{a^2}{x-a}$

(ii) $\underset{x\to a}{Lt}\dfrac{x^2-a^2}{x-a}=\underset{x\to a}{Lt}(x^2-a^2)\cdot\underset{x\to a}{Lt}\dfrac{1}{x-a}$.

21. $\underset{x\to 1}{Lt}\{x+\sqrt{x^2-1}\}=\underset{x\to 1}{Lt}(x)+\underset{x\to 1}{Lt}\sqrt{x^2-1}$

But as $\underset{x\to 1}{Lt}\sqrt{x^2-1}$ does not exist, so $\underset{x\to 1}{Lt}\{x+\sqrt{x^2-1}\}$ does not exist. Is the above statemene corred ?

22. $\underset{x\to 3}{Lt}\sqrt{3-x}=0$. Is the result correct ? Give reasons in support of your answer.

23. Are the two results,

(i) $\underset{x\to 0}{Lt}\dfrac{\sin x^\circ}{x}=1$ (ii) $\underset{x\to 0}{Lt}\dfrac{\log_{10}(1+x)}{x}=1$ correct ?

Give reasons in support of your answer.

24. Evaluate the following limits :

(i) $\underset{x\to 0}{Lt}\dfrac{\sin 2x}{x}$ (ii) $\underset{x\to 0}{Lt}\dfrac{\log(1+3x)}{x}$ [Joint Entrance, 1984]

(iii) $\underset{x\to a}{Lt}\dfrac{x^5-a^5}{x^4-a^4}$ (iv) $\underset{x\to 0}{Lt}\dfrac{e^{2x}-1}{x}$.

25. (i) If the function $f(x)$ is continuous at $x=a$ then $f(a)$ exists.

(ii) If $\underset{x\to a}{Lt}f(x)$ exists, then $f(x)$ is continuous at $x=a$.

(iii) If $f(a)$ is defined, then $f(x)$ is continuous at $x=a$.

(iv) If the function $f(x)$ is continuous at $x=a$ then $\underset{h\to 0}{Lt}f(a+h)=f(a)$.

(v) If $f(x)$ is continuous at $x=a$, then $\underset{x\to a-}{Lt}f(x)=f(a)$.

(vi) If $\underset{x\to a+}{Lt}f(x)=f(a)$, then the function $f(x)$ is continuous at $x=a$.

Comment on the validity of the above six statements.

26. "The function $\dfrac{|x-2|}{x-2}$ is continuous everywhere".

Show that the above statement is not true.

27. "If $\underset{x\to a}{Lt}f(x)=\underset{x\to a}{Lt}g(x)$ then $f(x)$ and $g(x)$ are identical".

Is the statement correct ? Support your answer with reasons.

28. $f(x)=\dfrac{\sin[x]}{[x]}$ when $[x]\neq 0$.

$=0$ when $[x]=0$.

where $[x]$ is the greatest integer less than or, equal to x.

$\underset{x \to 0}{Lt} f(x) =$ (A) 1 (B) 0 (C) -1 (D) None of these. Which is the correct answer ?

29. $f(x) = x$ when $x \geqslant 1$
$\quad = 2 - x$ when $0 < x < 1$
$\quad = x^2$ when $x \leqslant 0$

State whether $f(x)$ is continuous at $x=0$ and $x=1$ or not.

30. State whether the following functions are continuous at the specified points or not :

(i) $f(x) = \dfrac{\sin x}{x}$, at $x=0$

(ii) $f(x) = 5$, at $x=5$

(iii) $f(x) = \dfrac{x}{x}$, at $x=0$.

31. "If a function is continuous at $x=a$, then it is differentiable at $x=a$." Is the above statement correct ? Support your answer with reasons or examples.

32. Is the converse of the above statement (Ex. 31) true ? Answer with reasons.

33. $f(x)$ is an even function. Show that $f'(0)=0$.

34. $f'(x)=0$ for all values of x. Show that $f(x)$ is a constant function.

35. $\dfrac{d}{dx}\{f(x)\} = \dfrac{d}{dx}\{g(x)\}$. Will $f(x)$ be equal to $g(x)$? Give reasons for your answer. [H. S. 1981]

36. Are the results (i) $\dfrac{d}{dx}(e^x) = x.e^{x-1}$ (ii) $\dfrac{d}{dx}(\sin x^2) = \cos x^2$ correct ?

37. Determine (i) $\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$ (ii) $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$

(iii) $\dfrac{d}{dx}\left(e^{\log x}\right)$ (iv) $\dfrac{d}{dx}(\cos x^2)$ (v) $\dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)$

(vi) $\dfrac{d}{dx}(\log x^5)$.

38. $f(x) = x(\sqrt{x} - \sqrt{x+1})$

which of the following are correct ?

(A) $f(x)$ is continuous but not differentiable at $x=0$.

(B) $f(x)$ is differentiable at $x=0$.

(*C*) $f(x)$ is not differentiable at $x=0$.
(*D*) None of these. [I. I. T. 1985]

39. Show that if $f(x)=\log_x(\log_e x)$ then $f'(x)=\frac{1}{e}$

[I. I. T. 1985]

40. The function $f(x)$ is differentiable at $x=a$ and the function $g(x)$ is not differentiable at the point. Is it possible that the function $f(x)+g(x)$, $f(x)g(x)$ or $\frac{f(x)}{g(x)}$ are differentiable at $x=a$?

41. (i) $y=\sin x$. Find the value of $\frac{d^2y}{dx^2}+y$.

(ii) If $y=c_1e^x+c_2e^{-x}$, show that $\frac{d^2y}{dx^2}+y=0$.

(iii) If $y=4\cos 5x$, prove that $\frac{d^2y}{dx^2}=-25y$. [H. S. '78]

42. (i) If $f(x)$ is continuous at $x=a$, then $f''(x)$ exists at $x=a$.

(ii) If $f'(x)$ possesses a finite value at $x=a$, then $f''(x)$ will possess a finite value at $x=a$.

(iii) If $f''(x)$ possesses a finite value at $x=a$, then the function $f(x)$ is continuous at $x=a$.

(iv) Continuity of the function $f'(x)$ at $x=a$ is necessary for the existence of a finite value of $f(x)$ at $x=a$.

Which of the above statements are correct ?

43. Is there any function, which is equal to its own derivative or integral ?

44. (i) $c=\int e^x \cos x\, dx$ and $s=\int e^x \sin x\, dx$. Show that

$$c+s=e^x \sin x.$$

(ii) Evaluate $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$.

45. State the values of the following integrals :

(i) $\int\frac{dx}{x^{\frac{2}{3}}}$ (ii) $\int\frac{dx}{\sqrt{x}}$ (iii) $\int e^{x^2} x\,dx$.

(iv) $\int\frac{dx}{a-bx}$ (v) $\int\frac{x^2dx}{1+x^2}$ (vi) $\int e^{-\frac{1}{x}}\cdot\frac{1}{x^2}\,dx$.

46. Is the following statement correct ?

"If $\int_0^1 f(x)\,dx=\int_0^1 \phi(x)\,dx$, then $f(x)=\phi(x)$". Examine by a suitable example.

47. Does the formula $\int a^x dx = \frac{a^x}{\log_e^a} + c$ where $a>0$ and c is a constant hold when $a=1$? [*cf.* H.S. 1984]

48. If $f'(x)=\log x$ and $f(1)=-5$, then
$f(x)=$ (a) $x\log x+1$ (b) $x\log x-1$ (c) $x(\log x-1)-4$.
(d) $(\log x-1)-5$. (e) None of these.
Which is the correct answer ?

49. $\int_0^{\frac{\pi}{4}} \sec\theta\,(\sec\theta-\tan\theta)\,d\theta$

$=$(i) $2+\sqrt{2}$ (ii) $2-\sqrt{2}$ (iii) $2+\frac{1}{2}\sqrt{2}$ · (iv) $2-\frac{1}{2}\sqrt{2}$
(v) None of these. Which is the correct answer ?

50. $\int_0^1 \frac{dx}{(x+2)\sqrt{x+1}}$

$=$(i) $\frac{\pi}{2\sqrt{2}}$ (ii) $\frac{1}{4}(\tan^{-1}2-\pi)$ (iii) $\frac{1}{2}\tan^{-1}\sqrt{2}$
(iv) $2\tan^{-1}\sqrt{2}-\frac{\pi}{2}$ (v) None of these. Which is the correct answer ?

51. "$\frac{1}{x^2-a^2}=-\frac{1}{a^2-x^2}$ $\therefore \int\frac{dx}{x^2-a^2}=-\int\frac{dx}{a^2-x^2}$
So integration of $\int\frac{dx}{x^2-a^2}$ and $\int\frac{dx}{a^2-x^2}$ separately is not necessary". Comment on the above statement.

52. State whether the following statements are correct or not :

(i) $\int_a^b f(x)dx=\int_a^b g(x)\,dx$ (ii) $\int_a^b f(x)dx=\int_a^b f(z)\,dz$.

53. "If $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$, then $f(x)=f(a-x)$" is the above statement correct ?

54. The value of $\int_{-1}^{1} |x|\,dx$ is

(i) 1 (ii) -1 (iii) $\frac{1}{2}$ (iv) 0 (v) none of these. Which is the correct answer ?

55. $\int_0^4 |x-1|\,dx=$(i) 5 (ii) 7 (iii) 12 (iv) 20

(v) None of these. Which is correct ?

56. Show that $\int_a^x f(x)\,dx$ is an indefinite integral.

57. If $f(x)$ be an odd function, show that $\int_{-a}^{a} f(x)\,dx=0$.

58. If $f(x)$ be an even function, show that

$$\int_{-a}^{a} f(x)\,dx=2\int_{0}^{a} f(x)\,dx.$$

59. Comment with reasons on the validity of the following statements :

(i) Ae^{a+x} contains two arbitrary constants of integration. So $y=Ae^{a+x}$ is the general solution of a second order differential equation.

(ii) Every differential equation possesses one and only one general solution.

60. "The equation $\dfrac{d^2y}{dx^2}-a\left(\dfrac{dy}{dx}\right)^2=0$ is a differential equation of the second order and second degree". Comment on the validity of the statement.

61. $\dfrac{d^2y}{dx^2}=x^2\left(\dfrac{dy}{dx}\right)^2$ is a differential equation of the

(i) first order and second degree.

(ii) second order and second degree.

(iii) second order and first degree.

—which is the correct answer ?

62. Solve the differential equation $\dfrac{dy}{dx}=\dfrac{2}{y}$, under the condition that $y=0$ when $x=0$.

63. Eliminate

(i) The arbitrary constant 'm' of the equation $y=e^{mx}$.

(ii) The arbitrary constants 'm and c' of the equation $y=mx+c$.

64. "Though the equation $ax+by+c=0$ $(b\neq 0)$ contains three arbitrary constants, by eliminating the three constants one shall get a differential equation of the second order"—Support the statement with reasons.

65. "The general solution of the differential equation $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$ is $y=c_1e^{3x}+c_2e^{3x}$." Is the above statement correct ?

66. Prove that the rate of change of the ordinate of a point of the curve $y=ae^{4x}$ with respect to the abscissa is proportional to the ordinate.

67. Find the average rate of change of $y=x^2$ between $x=1$ and 2 and also rate of change at $x=1$.

68. Show that the rate of change of the area of a circular figure with respect to the radius is equal to the length of the circumference of the circle.

69. Show that $\log_e^{10{\cdot}01}=2{\cdot}304$ (given : $\log_e^{10}=2{\cdot}303$)

70. $\log_{10}^{10{\cdot}1}=$ (i) 1·1 (ii) 1·04343 (iii) 1·004343 (iv) None of these. Which is the correct answer ?

[Given : $\log_{10}^{e}=0{\cdot}4343$]

71. The function $f(x)=x^3-6x^2-36x+7$ is increasing in the interval (i) $-2\leqslant x\leqslant 6$. (ii) $x\leqslant -2$ or, $x\geqslant 6$. (iii) None of these. Which is the correct answer ?

72. When $x=1{\cdot}997$, then the approximate value of x^4+4x^2+1 is (i) 32·658 (ii) 32·856 (iii) 32·568 (iv) None of these. Which is the correct answer ?

73. If $x=2{\cdot}00012$, the approximate value of x^3+4x^2+2x+2 is (i) 30 (ii) 30·0036 (iii) 36·003 (iv) none of these. Which is the correct answer ?

74. Displacement s at time t is given by $s=\frac{1}{2}t^2+\sqrt{t}$. Acceleration at that time with proper units is (i) $\frac{1}{32}$ (ii) $\frac{7}{8}$ (iii) $\frac{31}{32}$ (iv) none of these. Which is the correct answer ?

75. Is it possible to draw a tangent to a circle from its centre ?

[H. S. '79]

76. (i) "From the focus of a parabola, (ii) the centre of an ellipse or a hyperbola, tangents cannot be drawn to the corresponding curve". Support the statement with reasons.

77. The slope of the tangent to the curve $y=x^3+x$ at the point (0, 0) is (i) 0 (ii) 1 (iii) ∞ (iv) none of these. Which is the correct answer ?

78. At which point of the curve $y=x+\frac{1}{x}$, $\frac{dy}{dx}$ is 0 ? Which is its geometrical significance ?

79. At which point of the parabola $y=x^2$, the tangent to the curve is parallel to the straight line $y=4x-5$? [C.U. ; B.Sc. 1982]

80. The length of the subnormal to a curve is constant and is equal to $2a$. The curve passes through the origin. Show that its equation is $y^2=4ax$.

81. The length of the subtangent to a curve is constant $(2a)$. If the curve passes through the point $(0, 1)$, show that the equation of the curve is $y=e^{\frac{x}{2a}}$.

82. The tangent to a curve at a point of it is perpendicular to the straight line joining the point with the origin. Show that the curve is a circle.

83. $y=x^3-8$ has
(i) a maximum at $x=2$; (ii) a minimum at $x=2$; (iii) none of these. Which is the correct answer ?

84. The maximum value of $f(x)=2x^3-21x^2+36x-20$ occurs at (i) $x=1$ (ii) $x=3$ (iii) $x=6$ (iv) none of these. Which is correct answer ?

85. "If a function possesses a maximum at a point, then existence of the derivative of the function at the point is not necessary"—Support the statement with an example.

86. Are the following statements correct ?
(i) $f'(x)>0$ for all x and $a>b$. $\therefore$ $f(a)>f(b)$.
(ii) $f'(2)<0$; so, $f'(2\cdot1)<f'(2)$.

87. $\int_0^2 x(x-1)(x-2)dx=0$.
So the area of the curve bounded by the curve $y=x(x-1)(x-2)$, the x-axis and the ordinates $x=0$ and $x=2$ is 0." Comment on the validity of the statement.

88. Correct or modify the following statement with reason.
"The maximum value of a function is always greater than its minimum value".

89. Correct or justify giving reasons :
(i) A stone is let fall from the roof of a building. It reaches the ground in 2 seconds. The height of the building is not less than 20 metres. [H.S. 1980]

(ii) A particle is projected vertically upwards from the surface of the earth with a given velocity. Its time of rise is equal to its time of fall to the surface of the earth. [H. S. 1981]

(iii) The equation of motion of a particle moving along a straight line is given by $x=16t+5t^2$; show that the particle moves with uniform acceleration. [H. S. 1985]

(iv) For a particle moving with uniform acceleratiom and covering a distance s in time. t, the velocities at half time and half distance are equal. [H. S. 1987 ; 1989]

ANSWERS WITH HINTS

1. (i) Let the number be $100x+10y+z$ and $x+y+z$ be divisible by 3.

(ii) Let the three consecutive integers be $x-1$, x and $x+1$.

(iii) See Diff. Calculus Example 1 ex. 5.

2. rational. 3. $\frac{5}{6}$.

4. $x\sqrt{3}+y\sqrt{5}=0$.

$\therefore\ 3x^2+5y^2+2\sqrt{15}xy=0$.

Now $3x^2+5y^2$ is rational and $2\sqrt{15}xy$ is irrational.

$\therefore\ 3x^2+5y^2=0 \quad \therefore\ x=0, \quad y=0.$

5. Not valid. $(2+\sqrt{3})+(5-\sqrt{3})=7$ is rational.

6. Let $c>0$. $\therefore\ f(c)=\dfrac{|c|}{c}=\dfrac{c}{c}=1$;

$$f(-c)=\frac{|-c|}{-c}=\frac{c}{-c}=-1$$

$$\therefore\ |f(c)-f(-c)|=|1-(-1)|=|2|=2$$

when $c<0$, $f(c)=-1$; $f(-c)=1$.

7. (i) Let $y=0$ $\therefore\ f(x+0)=f(x)+f(0)$

or, $f(x)=f(x)+f(0)$ $\therefore\ f(0)=0$.

(ii) Let $x=2, y=0$ $\therefore\ f(x+y)=f(2+0)=f(2)$

$\therefore\ f(2)=f(2)\,.\,f(0)$ $\therefore\ f(0)=1$.

8. Hints : $f(0)=-f(0)$.

9. (i) Hints : $\dfrac{m(m-1)(m-2)\cdots(m+r-1)}{\lfloor r}$

$$=\frac{\underline{|m+r-1}}{\underline{|m-1}\,\lfloor r}={}^{m+r-1}C_r.$$ [m is a natural number]

(ii) Let of the given p positive integers x be even and y be odd. The sum of the x even integers is an even integer. If y be odd, then the sum of the y odd integers is odd and so the sum of the p positive integers will be odd. So y cannot be odd.

10. (i) $x \leqslant 6$ (for otherwise $6-x$ is negative and $\sqrt{6-x}$ is imaginary.

(ii) $-2 \leqslant x \leqslant 2$; (iii) $x \geqslant 3$ or, $x \leqslant -3$.

(iv) $1 \leqslant x \leqslant 2$ [For, if $x<1$, $\sqrt{x-1}$ is imaginary and if $x>2$, $\sqrt{x-1}>1$.]

11. (i) All real numbers other than 2.

(ii) All real numbers other than 0 or -1.

(iii) All real numbers other than $(2n+1)\frac{\pi}{2}$ $[n=0, \pm 1, \pm 2, \cdots]$

(iv) $-3 \leqslant x \leqslant 3$ and $x \neq 2$.

12. Incorrect. Let $f(x)=x^2$; $a=2$; $b=-2$; $c=1$; $d=-1$.

13. If $\log_e \dfrac{\sqrt{4-x^2}}{1-x}$ be defined, $\dfrac{\sqrt{4-x^2}}{1-x}>0$ and $x \neq 1$.

$\therefore$ $-2 \leqslant x \leqslant 2$ and $x \neq 1$. Again $x=1$ is included in $-2 \leqslant x \leqslant 2$.

So the required domain of definition is $-2 \leqslant x \leqslant 2$.

Again $\left|\sin \log_e \dfrac{\sqrt{4-x^2}}{1-x}\right| \leqslant 1$. So, if $y = \sin \log_e \dfrac{\sqrt{4-x^2}}{1-x}$ then the range of y is $|y| \leqslant 1$.

14. (i) $f(x+y)+f(x-y)=\frac{1}{2}(a^{x+y}+a^{-x-y})+\frac{1}{2}(a^{x-y}+a^{-x+y})$

$$=\tfrac{1}{2}(a^x \cdot a^y+a^{-x} \cdot a^{-y}+a^x \cdot a^{-y}+a^{-x} \cdot a^y)$$
$$=\tfrac{1}{2}\{a^x(a^y+a^{-y})+a^{-x}(a^y+a^{-y})\}$$
$$=\tfrac{1}{2}(a^x+a^{-x})(a^y+a^{-y})=2 \cdot \tfrac{1}{2}(a^x+a^{-x}) \cdot \tfrac{1}{2}(a^y+a^{-y})$$
$$=2f(x) \cdot f(y).$$

15. (A) and (D) are correct.

16. $f(a)+f(b)=\log\dfrac{1-a}{1+a}+\log\dfrac{1-b}{1+b}=\log\left(\dfrac{1-a}{1+a}\cdot\dfrac{1-b}{1+b}\right)$

$$=\log\frac{1-a-b+ab}{1+a+b+ab}=\log\frac{1+ab-(a+b)}{1+ab+(a+b)}$$

$$=\log\frac{1-\dfrac{a+b}{1+ab}}{1+\dfrac{a+b}{1+ab}}.$$

17. Sec Differential Calculus Ex. 18, Examples 2.

18. Not valid. See Diff. Cal. Chapter Four "Continuity".

19. Take $\underset{x\to 0}{Lt}\ \frac{x}{x}=1$ as an example.

20. See Diff. Cal. § 3·7, Ex. 4 and Ex. 8.

21. The statement is not valid. For $\underset{x\to 1}{Lt}\ \{x+\sqrt{x^2-1}\}$ $\neq \underset{x\to 1}{Lt}\ x+\underset{x\to 1}{Lt}\ \sqrt{x^2-1}$.

22. Not true.

23. (i) Incorrect ; correct answer $\frac{\pi}{180}$,
(ii) Incorrect ; correct answer $\log_{10}{}^{e}$.

24. (i) 2. (ii) 3. (iii) $\frac{5}{4}a$ (iv) 2.

25. (i) Valid. (ii) Not always true and so not valid. (iii) Not always true and so not valid. (iv) Valid. (v) Valid. (vi) Not always true and so not valid.

26. Discontinuous at $x=2$. Similar to Ex. 16 (ii) Examples 3 Diff. Cal.

27. Incorrect ; $\underset{x\to 2}{Lt}\ x^2=4$ and $\underset{x\to 2}{Lt}\ (6-x)=4$ But x^2 and $6-x$ are two different functions.

28. (D) Here $\underset{x\to 0+}{Lt} f(x)=0$; $\underset{x\to 0-}{Lt}\ f(x)=\sin 1\neq 0$.

29. Discontinuous at $x=0$; Continuous at $x=1$.

30. (i) and (iii) discontinuous ; (ii) continuous.

31. Not true. 32. True.

33. Let $f(-x)=f(x)$ $\therefore$ $\frac{d}{dx}\{f(-x)\}=\frac{d}{dx}\{f(x)\}$
or, $-f'(-x)=f'(x)$ or, $f'(-x)=-f'(x)$ $\therefore$ $f'(0)=-f'(0)$.

34. $f'(x)=0$ means $\frac{dy}{dx}=0$ when $y=f(x)$. So the gradient of the tangent to the curve $y=f(x)$ at every point (x, y) is 0 i.e., the curve is a straight line parallel to the x-axis. So $y=f(x)=c$. (a constant).

35. $\frac{d}{dx}(x^2)=2x=\frac{d}{dx}(x^2-2)$ But $x^2\neq x^2-2$.

36. Both false.

37. (i) $-\frac{1}{x^2}$, (ii) $-\frac{2}{x^3}$ (iii) 1. (iv) $-2x\sin x^2$ (v) $-\operatorname{cosec}^2 x$
(vi) $\frac{5}{x}$

38. (c) $\underset{x \to 0-}{Lt} f(x)$ does not exist ; for as $x<0$, then $\sqrt{x}$ is imaginary.

39. $f(x)=\log_x(\log_e{}^x)=\log_e(\log e^x)\log_x{}^e=\dfrac{\log_e(\log_e{}^x)}{\log_e{}^x}$

$$\therefore \; f'(x)=\frac{\log_e{}^x \cdot \dfrac{1}{\log_e{}^x}\cdot\dfrac{1}{x}-\dfrac{1}{x}\cdot\log_e(\log_e{}^x)}{(\log_e{}^x)^2}=\frac{1-\log_e(\log_e{}^x)}{x(\log_e{}^x)^2}$$

$$\therefore \; f'(e)=\frac{1-\log_e\log(e^e)}{e(\log_e{}^e)^2}=\frac{1-\log_e{}^1}{e\,.\,1^2}=\frac{1-0}{e\,.\,1}=\frac{1}{e}.$$

40. In some cases. For example, if $f(x)=\dfrac{x}{x-a}$ and $g(x)=\dfrac{a}{x-a}$. None of $f(x)$ and $g(x)$ is differentiable at $x=a$. But $f(x)-g(x)=1$ is differentiable at $x=a$.

41. (i) 0. (iii) $y=4\cos 5x$; $y_1=-20\sin 5x$;
$y_2=-100\cos 5x=-25.4\cos 5x=-25y$.

42. (i) incorrect ; (ii) incorrect ; (iii) correct ; (iv) correct.

43. e^x.

44. (i) See Integral Calculus Examples 3 Ex. 5(b).

(ii) $\dfrac{e^x}{x}$. See Integral Calculus Examples 3. Ex. 5(a) (i).

45. (i) $-\dfrac{1}{x}+c$ (ii) $2\sqrt{x}+c$ (iii) $\frac{1}{2}e^{x^2}+c$

(iv) $-\dfrac{1}{b}\log(a-bx)+c$ (v) $x-\tan^{-1}x+c$ (vi) $e^{-\frac{1}{x}}+c$.

46. Not correct ; Take $f(x)=1$ and $\phi(x)=2x$.

47. Not correct ; as $\log_e{}^1=0$.

48. (c). 49. (ii) 50. (iv). 51. See Integral Calculus Note of § 2·7.

52. (i) Incorrect ; (ii) Correct. See § 5·5 Property (2) of definite integrals.

53. Incorrect ; $\displaystyle\int_0^{\frac{\pi}{2}}\sin x\,dx=1.=\int_0^{\frac{\pi}{2}}\sin\left(\tfrac{\pi}{2}-x\right)dx$ 6

But $\sin x \neq \sin\left(\frac{\pi}{2}-x\right)$.

54. (i) Correct ; hints : $\displaystyle\int_{-1}^{1}|x|\,dx=\int_{-1}^{0}(-x)dx+\int_{0}^{1}x\,dx.$

55. (i) Correct ; hints : Let $x-1=z$ $\therefore$ $\displaystyle\int_0^4|x-1|dx=\int_{-1}^{3}|z|dz.$

56. Let $\int_a^x f(x)dx=\phi(x)+c$.

$\therefore$ $\int_0^x f(x)dx=\Big[\phi(x)+c\Big]_a^x=\phi(x)-\phi(a)$ which is indefinite.

57. See Integral Calculus § 5·5 cor (ii).

58. See Integral Calculus § 5·5 cor (i).

59. (i) Not true ; $Ae^{a+x}=A.e^a.e^x=Be^x$. (ii) True.

60. Incorrect. Second order and first degree. 61. (iii).

62. $\dfrac{dy}{dx}=\dfrac{2}{y}$ or, $ydy=2x\,dx$ or, $\int ydy=2\int dx$ or, $\dfrac{y^2}{2}=2x+c$

when $x=0$, then $y=0$ $\therefore$ $c=0$ $\therefore$ $\dfrac{y^2}{2}=2x$ or, $y^2=4x$.

63. (i) $y=e^{\frac{x}{y}\frac{dy}{dx}}$ (ii) $\dfrac{d^2y}{dx^2}=0$.

64. Here the arbitrary constants a, b, c are not independent.

66. $\dfrac{dy}{dx}=a4e^{4x}=4\,.\,y$.

67. Average rate of change$=\dfrac{3^2-1^2}{3-1}=\dfrac{9-1}{2}=4$.

rate of change at $x=1$ is

$$\left[\frac{dy}{dx}\right]_{x=1}=[\,2x\,]_{x=1}=2.$$

70. (iii). 71. (ii). 72. (ii). 73. (ii). 74. (iii).

75. (i) No ; 76. In no case. 77. (ii).

78. $x=\pm1$. The tangents to the curve $y=x+\frac{1}{x}$ at the point (1, 2) and $(-1, -2)$ are parallel to the x-axis.

79. (2, 4). 83. (iii). 84. (i). 85. $|\,x\,|=0$ is minimum at $x=0$; but derivative of $|\,x\,|$ at $x=0$ does not exist.

86. None is always true. 87. Not true.

88. The statement is not always true. The maximum value of $x+\dfrac{1}{x}$ is less than its minimum value.

89. (i) Incorrect ; will be less than 20 metres.

(ii) Correct, (iii) Correct, (iv) Not correct.